Comprehensive anti-derivatives and parametric continuity

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Abstract

We present a poster and software demonstration regarding the calculation of indefinite integrals, or anti-derivatives, when parameters are present.

1 Extended abstract

The discussion focusses on the specialization problem, and how this should be handled in computer algebra systems, specifically in the context of integration. We define the *comprehensive anti-derivative*, following similar concepts in other areas of computer algebra. This is a piecewise function in the parameters, giving all special cases. We demonstrate a Mathematica package that computes this. We also define continuity in parameters, and demonstrate its benefits and show how this can be achieved. This is also part of the Mathematica package demonstrated.

We shall use the word *specialization* to describe the action of substituting specific values (usually numerical, but not necessarily) into a formula. The *specialization problem* is a label for a cluster of problems associated with formulae and their specializations, the problems ranging from inelegant results to invalid ones. For example, in [2] an example is given in which the evaluation of an integral by specializing a general formula misses a particular case for which a more elegant expression is possible. The focus here, however, is on situations in which specialization leads to invalid or incorrect results. To illustrate the problems, consider

$$I_1 = \int \left(\alpha^{\sigma z} - \alpha^{\lambda z}\right)^2 \, dz = \frac{1}{2\ln\alpha}\left(\frac{\alpha^{2\lambda z}}{\lambda} + \frac{\alpha^{2\sigma z}}{\sigma} - \frac{4\alpha^{(\lambda+\sigma)z}}{\lambda+\sigma}\right) . \tag{1.1}$$

Expressions equivalent to this are returned by Maple, Mathematica and many other systems, such as the Matlab symbolic toolbox. It is easy to see that the specialization $\sigma = 0$ leaves the integrand in (1.1) well defined, but the expression for its integral on the right-hand side is no longer defined. If we pursue this further, we see that there are multiple specializations for which (1.1) fails, *viz.* $\alpha = 0$, $\alpha = 1$, $\lambda = 0$, $\sigma = 0$, $\lambda = -\sigma$, and combinations of these. The question of how or whether to inform computer users of these special cases has been discussed in the CAS literature many times [1]. A list of every special case for (1.1) is as follows.

$$I_1 = \begin{cases} \frac{1}{2\lambda \ln \alpha}\left(\alpha^{2\lambda z} - \alpha^{-2\lambda z} - 4z\lambda \ln \alpha\right) , & \left[\begin{array}{l} \lambda + \sigma = 0 \ , \\ \alpha \neq 0 \ , \ \alpha \neq 1 \ , \ \sigma \neq 0 \ ; \end{array}\right. \\ z + \frac{1}{2\lambda \ln \alpha}\left(\alpha^{\lambda z}(\alpha^{\lambda z} - 4)\right) , & \left[\begin{array}{l} \sigma = 0 \ , \\ \alpha \neq 0 \ , \ \alpha \neq 1 \ , \ \lambda \neq 0 \ ; \end{array}\right. \\ z + \frac{1}{2\sigma \ln \alpha}\left(\alpha^{\sigma z}(\alpha^{\sigma z} - 4)\right) , & \left[\begin{array}{l} \lambda = 0 \ , \\ \alpha \neq 0 \ , \ \alpha \neq 1 \ , \ \sigma \neq 0 \ ; \end{array}\right. \\ 0 & \alpha = 1 \ , \\ 0 & \sigma = \lambda = 0 \ , \\ \text{ComplexInfinity} \ , & \left[\begin{array}{l} \alpha = 0 \ , \\ \Re(\lambda z)\,\Re(\sigma z) < 0 \ ; \end{array}\right. \\ \text{Indeterminate} \ , & \left[\begin{array}{l} \alpha = 0 \ , \\ \Re(\sigma z)\,\Re(\lambda z) \geq 0 \ ; \end{array}\right. \\ \frac{1}{2 \ln \alpha}\left(\frac{\alpha^{2\lambda z}}{\lambda} + \frac{\alpha^{2\sigma z}}{\sigma} - \frac{4\alpha^{(\lambda+\sigma)z}}{\lambda + \sigma}\right) , & \text{otherwise, (generic case)} \ . \end{cases}$$

Expressions such as this will be called *comprehensive antiderivatives*. There are several questions surrounding such expressions. The first is whether comprehensive antiderivatives should be returned to users. A second question is how systems can compute such expressions. The automatic discovery of exceptional cases is not easy. A third question concerns *continuity with respect to parameters*.

We shall discuss why the expression

$$\int x^n \ dx = \frac{x^{n+1}}{n+1} - \frac{1}{n+1}$$

is better than the usual expression, and how we found it.

References

[1] R. M. Corless; D. J. Jeffrey, *Well... it isn't quite that simple. SIGSAM Bulletin* **26**(3), 2–6 (1992).

[2] D. J. Jeffrey; A.D. Rich, Reducing expression size using rule-based integration. *Intelligent Computer Mathematics*, Editor: S. Autexier, LNAI6167, 234–246, Springer (2010)