An Approach for Certifying Homotopy Continuation Paths: Univariate Case

Michael Burr
Joint Work with Juan Xu and Chee Yap

Clemson University
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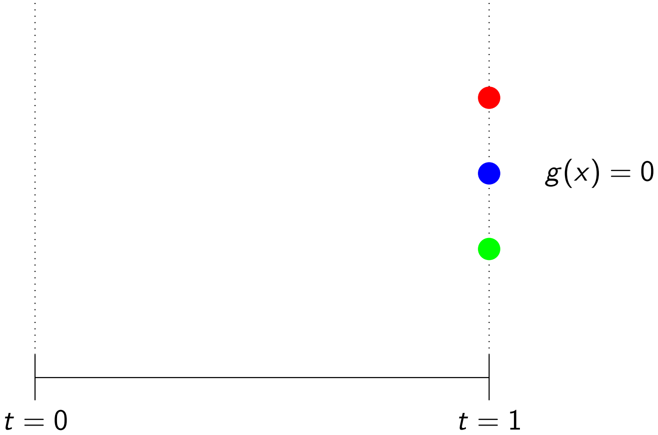
Basic Homotopy Continuation:



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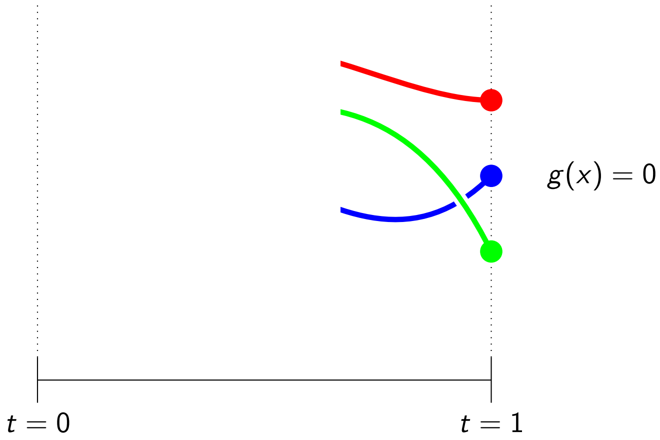

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2. Deform the system and track solutions

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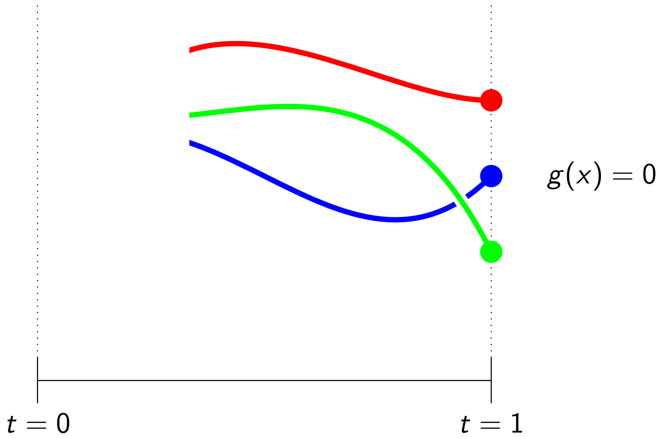

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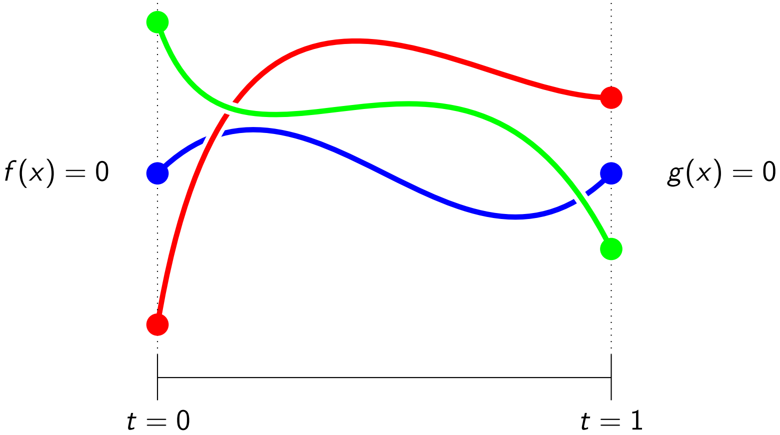

1. Start with a start system with known solutions
2. Deform the system and track solutions to find solutions to target system

Univariate Homotopy Continuation

Given: Target Polynomial $f \in \mathbb{Q}[x]$.
Choose: Initial Polynomial $g \in \mathbb{Q}[x]$.
Complex number $\gamma \in \mathbb{C}$.
Algorithm: Start with approximations for roots of g.
Track roots from $t = 1$ to $t = 0$ of
$H(x, t) = \gamma t g(x) + (1 - t) f(x)$.

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We focus on the case of nonsingular bounded paths.

Tracking Framework

Predictor: From approximate root x_i at time t_i:
"guess" approximate root x_{i+1} at t_{i+1}.

Corrector: From approximate root x_i at time t_i:
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Potential Errors

- Path jumping
  - Predictor suggests approximation near different solution path
- Singularities
  - We assume f is square-free.
  - No singularities along path when γ is random, a.s.
  - No divergence to infinity when γ is random in the univariate case, a.s.

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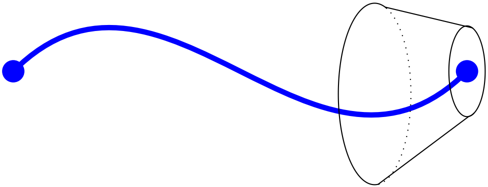

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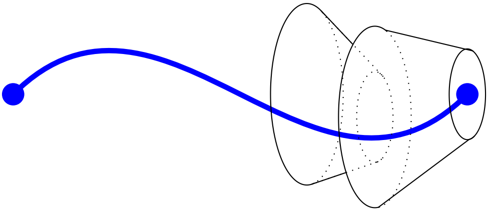

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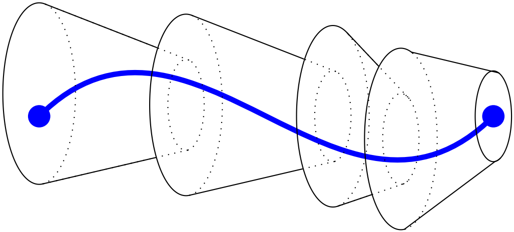

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- Interval arithmetic curve tracking
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2. Large steps and tubes
   - Alpha theory-based certification uses very small steps and radii.
   - The alpha convergence region is very small.
3. Applicable to general homotopies
   - Can be applied to most homotopies that are used.
   - Specifically designed for homotopy continuation.

Certified Homotopy

Predictor: From well-isolated root x_i at time t_i:
"guess" well-isolated root x_{i+1} at t_{i+1}.

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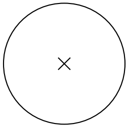
Well-isolated root: Distance to closest root is at most a third of distance to second closest root.

Isolating disk: Maintain a disk that is guaranteed to have a single root.

Disk Schematic:



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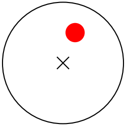

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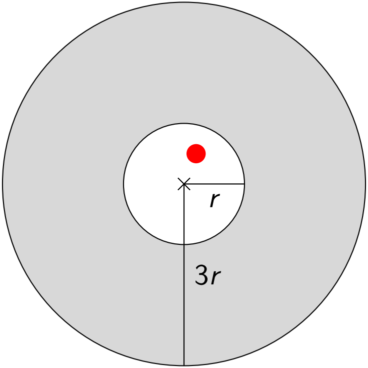

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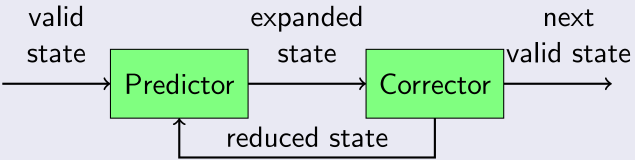
- The disk is guaranteed to contain a root.
- There are no additional roots within a disk three times as large.

Main Subroutine



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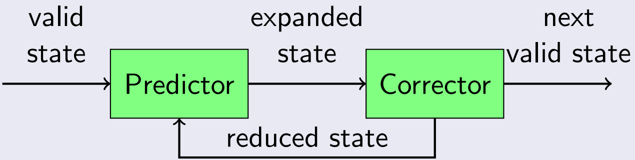
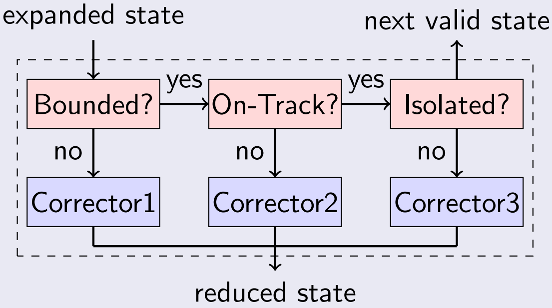
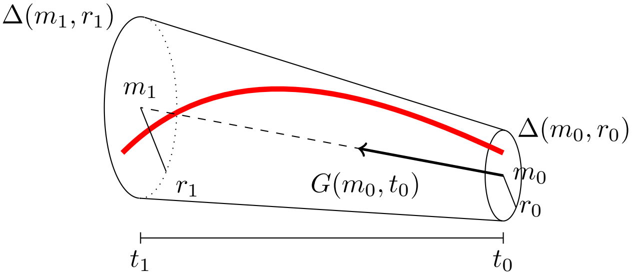
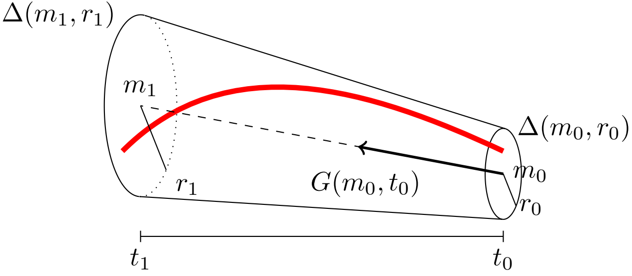


Corrector Module



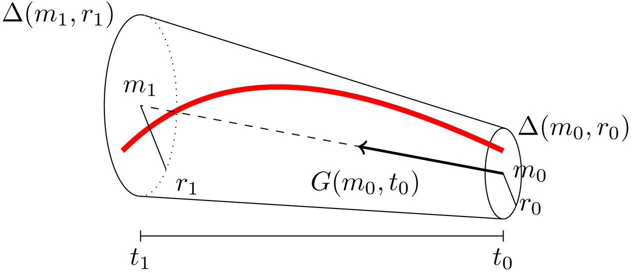
Tests





Three tests:

- Bounded: No singularities within frustum



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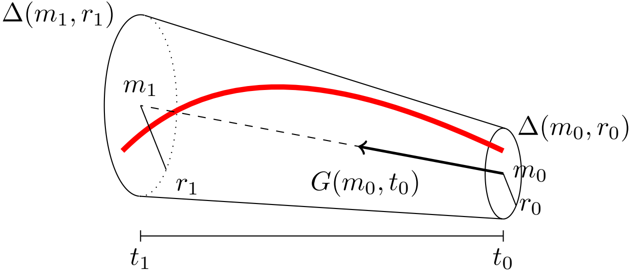

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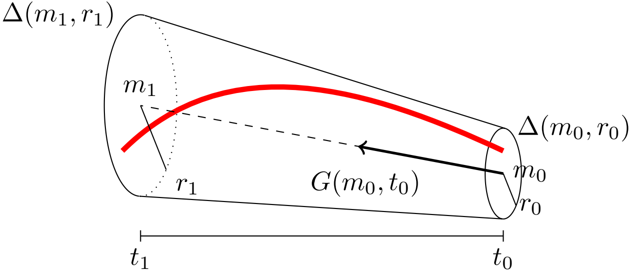

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Every step is certified, so entire path and solutions are certified.

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No singularities within frustum (T)

$$0 \notin \frac{\partial H}{\partial x}(T).$$

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On-Track

Curve does not leave frustum

Estimates position of curve using maximum and minimum values of

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On-Track

Curve does not leave frustum

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Isolated

Left disk of frustum has unique root.

Uses Pellet test and Graeffe iteration:

$$T_1(m, r, F) : |F_1(m)|r > \sum_{i \neq 1} |F_i(m)|r^i$$

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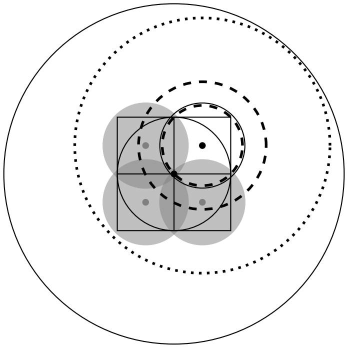

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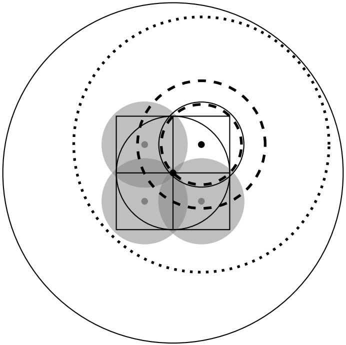

- Uses Pellet test and Graeffe iteration to find smaller disk.
- Succeeds on at least one disk.
- Uses heuristics to decide whether to shrink step size or disk.

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As the radius and time step decrease to zero, the on-track, isolated, and bounded tests eventually succeed.

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Theorem

The main loop of the algorithm terminates. In other words, there is a lower bound on the step size taken by the algorithm.

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$G(x,t)$ computes tangent lines for all level curves

Predictor

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Idea:

- x_i approximates $x(t_i)$
- $\Delta(x_i, t_i)$ is a well-isolating disk for $x(t_i)$.
- Approximate tangent vector for $x(t_i)$ by $G(x_i, t_i)$.
- "Guess" a well-isolating disk for $x(t_{i+1})$.

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Isolating Disk Guess:

Center:	Value of line through x_i with slope $G(x_i, t_i)$ at time t_{i+1}
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Possible Errors:

- Curve might not remain close to tangent approximation.
- Another solution may enter approximating disk.

Bounded and On-track tests are based on nonvanishing of denominator of $G(x,t)$:

$$G(x,t) = -\frac{\frac{\partial H}{\partial t}(x,t)}{\frac{\partial H}{\partial x}(x,t)}$$

Use interval methods to estimate $\dfrac{\partial H}{\partial x}(x,t)$ and values of $G(x,t)$.

$$\begin{aligned}
\text{Function:} \quad & F : \mathbb{C}^n \to \mathbb{C} \\
\text{Region:} \quad & J \subseteq \mathbb{C}^n \\
\text{Image:} \quad & F(J) = \{F(x) : x \in J\} \\
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- Often easy to implement.
- Overapproximation to image (conservative estimate)
- Converges: $\Box F(J) \to F(x)$ as $J \to x$.

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- If the inequality is true, then the disk with center m and radius r has exactly one root.
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cf. Becker, Sagraloff, Sharma, and Yap, 2016

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- Define: $F^{[0]}(x) = F(x)$ and, recursively,

$$F^{[i]}(x) = (-1)^n (F_e^{[i-1]}(x)^2 - xF_o^{[i-1]}(x)^2).$$

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The Graeffe iteration squares all the roots of F.
After $\lceil \log(1 + \log(n)) \rceil + 4$ iterates, the Pellet test succeeds.

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- Terminate when $t = 0$.

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- Certifies the entire path
- Uses adaptive precision (mpfr/mpfi) when necessary
- Can be run without certification
- Noncertified version is experimentally fast on polynomials with degree in the hundreds.

Number of steps taken ($f(x) = x^2 - 1 - m$).

m	Beltrán-Leykin	Hauenstein *et al.*	Our Approach
10	184	51	12
20	217	67	15
30	237	78	16
40	250	82	18
50	260	88	18
60	269	92	19
80	282	99	21
90	288	103	21
100	292	105	21
1,000	395	162	32
2,000	426	180	36
3,000	446	191	36
10,000	499	220	44
20,000	530	238	48
30,000	547	250	48

Number of steps taken ($f(x) = x^2 - 10^{-k}$).

k	Beltrán-Leykin	Hauenstein *et al.*	Our Approach
1	176	64	9
2	287	68	16
3	390	70	25
4	492	71	33
5	593	71	41
6	695	71	50
7	798	71	58
8	901	71	66
9	1003	71	75
10	1108	71	83

Poly	Paths Ave	Steps Average	Step Size Ave	Radius Small	Time Cert	Time Noncert
wilk15	15	790.3	0.0013	0.0083	6.93	0.77
mign20	20	272.2	0.0037	6.62e-25	81.2	23.6
chrma20	20	574.7	0.0017	0.0031	8.7	0.92
chrma22	21	555	0.0018	0.0029	9.5	1.01
chrmc11	11	279.6	0.0036	0.0057	1.12	0.219
kam3_1	9	975.3	0.0010	7.89e-16	36.6	
cheby20	20	239.4	0.0042	0.00078	4.66	0.374
cheby40	40	482.4	0.0021	0.00021	107	2.7

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- Code is available at SVN repository for https://cs.nyu.edu/exact under progs/homotopyPath