Sparse Polynomial Interpolation With Arbitrary Orthogonal Polynomial Bases

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Outline

1. Chebyshev Polynomials

2. Problem Statement

3. Chebyshev Bases (With A Known Sparsity t)

4. Deterministic Early Termination

Chebyshev Polynomials

Let K be a field

Chebyshev Polynomials of degree n

Tchebyshev-1: $T_n(x)$ Chebyshev-2: $U_n(x)$

Chebyshev-3: $V_n(x)$ Chebyshev-4: $W_n(x)$

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$T_0(x) = 1$ $T_1(x) = x$

$U_0(x) = 1$ $U_1(x) = 2x$

$V_0(x) = 1$ $V_1(x) = 2x - 1$

$W_0(x) = 1$ $W_1(x) = 2x + 1$

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$$T_0(x) = 1 \qquad T_1(x) = x \qquad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

$$U_0(x) = 1 \qquad U_1(x) = 2x \qquad U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$$

$$V_0(x) = 1 \qquad V_1(x) = 2x - 1 \qquad V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x)$$

$$W_0(x) = 1 \qquad W_1(x) = 2x + 1 \qquad W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x)$$

for $n \geq 2$

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1. If t (number of terms) is given, using $2t$ evaluations write $f(x)$ as

   i. $$f(x) = \sum_{j=1}^{t} c_j T_{\delta_j}(x)$$ (Chebyshev-1 Basis)

   ii. $$f(x) = \sum_{j=1}^{t} c_j U_{\delta_j}(x)$$ (Chebyshev-2 Basis)

   iii. $$f(x) = \sum_{j=1}^{t} c_j V_{\delta_j}(x)$$ (Chebyshev-3 Basis)

   iv. $$f(x) = \sum_{j=1}^{t} c_j W_{\delta_j}(x)$$ (Chebyshev-4 Basis)

   where $c_j \neq 0$ and $0 \leq \delta_1 < \cdots < \delta_t$

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   where $c_j \neq 0$ and $0 \leq \delta_1 < \cdots < \delta_t$

2. If $B \geq t$ is given, interpolate $f(x)$ with **exactly** $t + B$ evaluations

Problem Statement: Previous results

[Prony, 1795]

→ Interpolation in power basis

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→ Interpolation in Chebyshev-1 polynomials

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→ Interpolation in Chebyshev-2 polynomials

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→ Interpolation in Chebyshev-1 polynomials

[Kaltofen & Lee, 2003]

→ Recovers **unknown** t from given a degree bound for $f(x)$

[Potts & Tasche, 2014]

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Chebyshev Bases (With A Known Sparsity t)

Reduction to power bases:

Chebyshev-1: $T_n(T_m(y)) = T_{mn}(y) = T_m(T_n(y)), \forall m, n \in \mathbb{Z}_{\geq 0}$

$$T_n\left(\frac{y + \frac{1}{y}}{2}\right) = \frac{y^n + \frac{1}{y^n}}{2}, \forall n \geq 0$$

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Chebyshev-2: $\Big(y - \frac{1}{y}\Big) U_n\Big(\frac{y+\frac{1}{y}}{2}\Big) = y^{n+1} - \frac{1}{y^{n+1}}, \forall n \geq 0$ (17 Years)

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Chebyshev-3: $\Big(y+\frac{1}{y}\Big)V_n\Big(\frac{y^2+\frac{1}{y^2}}{2}\Big) = y^{2n+1} + \frac{1}{y^{2n+1}}, \forall n \geq 0$

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$$= \sum_{j=1}^{t} c_j \left(y^{\delta_j + 1} - \frac{1}{y^{\delta_j + 1}}\right) \in \mathsf{K}\left[y, \frac{1}{y}\right]$$

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$a_i = g(\omega^i) = \left(\omega^i - \frac{1}{\omega^i}\right) f\left(\frac{\omega^i + \frac{1}{\omega^i}}{2}\right) = -g\left(\frac{1}{\omega^i}\right) = -a_{-i}$ for $\omega \in \mathsf{K}$

Free evaluation: $a_0 = 0$

Prony's algorithm [Prony, 1795] can reconstruct $g(y)$

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Chebyshev-3 and Chebyshev-4 Bases can be done in a similar way

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Chebyshev-4: $\left(y-\frac{1}{y}\right)W_n\left(\frac{y^2+\frac{1}{y^2}}{2}\right)=y^{2n+1}-\frac{1}{y^{2n+1}}$

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Chebyshev-3: $\left(y+\frac{1}{y}\right)V_n\left(\frac{y^2+\frac{1}{y^2}}{2}\right) = T_{2n+1}\left(\frac{y+\frac{1}{y}}{2}\right)$ $\rightarrow$ Chebyshev-1

Chebyshev-4: $W_n\left(\frac{y^2+\frac{1}{y^2}}{2}\right) = U_{2n}\left(\frac{y+\frac{1}{y}}{2}\right)$ $\rightarrow$ Chebyshev-2

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Chebyshev-1 Basis: Write $f(x)$ as $f(x) = \sum_{j=1}^{t} c_j T_{\delta_j}(x)$

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$$= \sum_{j=1}^{t} \frac{c_j}{2}\left(y^{\delta_j} + \frac{1}{y^{\delta_j}}\right) \in \mathsf{K}\left[y, \frac{1}{y}\right]$$

$a_i = g(\omega^i) = f\left(\frac{\omega^i + \frac{1}{\omega^i}}{2}\right) = g\left(\frac{1}{\omega^i}\right) = a_{-i}$ for $\omega \in \mathsf{K}$

Prony's algorithm [Prony, 1795] can reconstruct $g(y)$

[Arnold & Kaltofen, 2015] uses $2t+1$ evaluations if $\delta_1 \neq 0$

ESSAI EXPÉRIMENTAL

ET ANALYTIQUE

Sur les lois de la Dilatabilité des fluides élastiques et sur celles de la Force expansive de la vapeur de l'eau et de la vapeur de l'alkool, à différentes températures.

Par R. PRONY.

J'ai donné, *n.°* *19* de mes leçons d'analyse, une solution de la premièr partie du problème qu'on emploie très-souvent, principalement comm méthode de correction; Lagrange a publié, sur le même objet, un trè beau mémoire (*), où il envisage la question plus généralement qu'on n l'avait encore fait. Les élèves qui posséderont la théorie exposée *n.os* *19*, *20 et 21* de mes leçons, pourront, sans difficulté, entreprendre l'étu de cet ouvrage, et tireront un grand profit du temps qu'ils y aur consacré.

(*) Voyez les Mém. de l'Acad. des Sciences, *année 1772*.

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$$\frac{\omega^{\delta_{i_1}} + \frac{1}{\omega^{\delta_{i_1}}}}{2} \neq \frac{\omega^{\delta_{i_2}} + \frac{1}{\omega^{\delta_{i_2}}}}{2} \iff \left| \left\{ \omega^{\delta_{i_1}}, \omega^{\delta_{i_2}}, \frac{1}{\omega^{\delta_{i_1}}}, \frac{1}{\omega^{\delta_{i_2}}} \right\} \right| \geq 3$$

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Let $\left| \left\{ \frac{1}{\omega^{\delta_t}}, \ldots, \frac{1}{\omega^{\delta_1}}, \omega^{\delta_1}, \ldots, \omega^{\delta_t} \right\} \right| = 2t$ or $=2t-1$ with $\delta_1=0$

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Let $\left| \left\{ \frac{1}{\omega^{\delta_t}}, \ldots, \frac{1}{\omega^{\delta_1}}, \omega^{\delta_1}, \ldots, \omega^{\delta_t} \right\} \right| = 2t$ or $=2t-1$ with $\delta_1 = 0$

We can save that extra evaluation in [Arnold & Kaltofen, 2015] using the **symmetry** of the term locator polynomial

$$\Lambda(z) = \prod_{j=1}^{t} \left(\left(z - \omega^{\delta_j}\right) \left(z - \frac{1}{\omega^{\delta_j}}\right) \right) = z^{2t} + \lambda_1 z^{2t-1} + \cdots + \lambda_1 z + 1$$

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$$\begin{bmatrix} a_{-2t+1} & \cdots & a_{-t+1} & \cdots & a_0 \\ a_{-2t+2} & \cdots & a_{-t+2} & \cdots & a_1 \\ \vdots & & \vdots & & \vdots \\ a_0 & \cdots & a_{t-1} & \cdots & a_{2t-1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \lambda_1 \\ \vdots \\ \lambda_{2t-1} = \lambda_1 \end{bmatrix} = - \begin{bmatrix} a_1 \\ \vdots \\ a_{2t-1} \\ \alpha \end{bmatrix}$$

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$$\longrightarrow \underbrace{\begin{bmatrix} 2a_{t-1} & \cdots & a_1 + a_{2t-3} & a_0 + a_{2t-2} \\ 2a_{t-2} & \cdots & a_0 + a_{2t-4} & a_1 + a_{2t-3} \\ \vdots & & \vdots & \vdots \\ 2a_0 & \cdots & 2a_{t-2} & 2a_{t-1} \end{bmatrix}}_{\bar{H} \quad \text{(“fold” of the coefficient matrix)}} \cdot \begin{bmatrix} \lambda_t/2 \\ \lambda_{t-1} \\ \vdots \\ \lambda_1 \end{bmatrix} = - \begin{bmatrix} a_1 + a_{2t-1} \\ a_2 + a_{2t-2} \\ \vdots \\ 2a_t \end{bmatrix}$$

$\bar{H}$ is nonsingular

Deterministic Early Termination

Let $B \geq t$ be a bound for the **unknown** sparsity t

Chebyshev-1 Basis: Write $f(x)$ as $f(x) = \sum_{j=1}^{t} c_j T_{\delta_j}(x)$

We can interpolate $f(x)$ with **exactly** $t + B$ evaluations

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Define

$$g(y) \stackrel{\text{def}}{=} f\Big(\frac{y+\frac{1}{y}}{2}\Big) = \sum_{j=1}^{t} c_j T_{\delta_j}\Big(\frac{y+\frac{1}{y}}{2}\Big)$$
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$$\mathscr{H}_\infty = \begin{bmatrix} \cdots & a_{-2t+1} & \cdots & a_{-2} & a_{-1} & a_0 \\ \cdots & a_{-2t+2} & \cdots & a_{-1} & a_0 & a_1 \\ \cdots & a_{-2t+3} & \cdots & a_0 & a_1 & a_2 \\ \iddots & \vdots & \iddots & \vdots & \vdots & \vdots \\ \cdots & a_0 & \cdots & a_{2t-3} & a_{2t-2} & a_{2t-1} \\ \iddots & \vdots & \iddots & \vdots & \vdots & \vdots \end{bmatrix}$$

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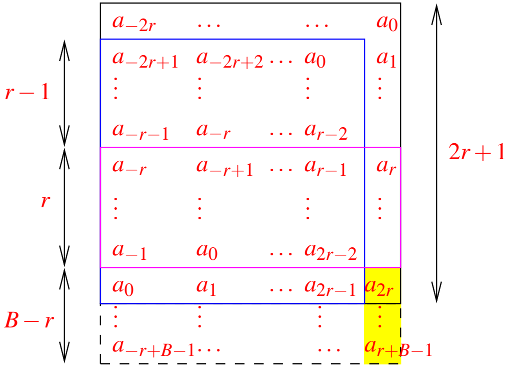
We can use Berlekamp/Massey Alg with $O(t+B)$ sequence elements

We know only a **soft**-quadratic time Toeplitz solver that locates nonsingular $2t \times 2t$ submatrix from $t+B$ sequence elements [Brent, Gustavson and Yun, 1980; Chan and Hansen, 1992; Sayed and Kailath, 1995]

Deterministic Early Termination



Largest nonsingular square submatrix of $\mathscr{H}_\infty$ reveals $t = r$
Any later nonsingular submatrix will have $t > B$
Our algorithm stops after **exactly** $t + B$ evaluations
If we use less than $t + B$ evaluations, our algorithm may recover a different polynomial

Deterministic Early Termination: Example

$$g(y) = \frac{32768}{5281339833}\left(\frac{1}{y^6} + y^6\right) - \frac{1024}{2540327}\left(\frac{1}{y^5} + y^5\right) + \frac{64}{7227}\left(\frac{1}{y^4} + y^4\right) - \frac{744}{8687}\left(\frac{1}{y^3} + y^3\right) + \frac{62}{153}\left(\frac{1}{y^2} + y^2\right) + \frac{254}{189}$$

$$\mathscr{T} = \begin{bmatrix} g(2) & g(2^2) & g(2^3) & \dots & g(2^{11}) & \dots \\ g(2^2) & g(2) & g(2^2) & \dots & g(2^{10}) & \dots \\ g(2^3) & g(2^2) & g(2) & \dots & g(2^9) & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ g(2^{11}) & g(2^{10}) & g(2^9) & \dots & g(2) & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}$$

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Leading principal submatrices of $\mathscr{T}$ have ranks 1, 2, 2, 2, 2, 2, 4, 6, 8, 10, 11, 11, 11, …

Deterministic Early Termination: Example

$$g(y) = \frac{32768}{5281339833}\left(\frac{1}{y^6}+y^6\right) - \frac{1024}{2540327}\left(\frac{1}{y^5}+y^5\right) + \frac{64}{7227}\left(\frac{1}{y^4}+y^4\right) - \frac{744}{8687}\left(\frac{1}{y^3}+y^3\right) + \frac{62}{153}\left(\frac{1}{y^2}+y^2\right) + \frac{254}{189}$$

$$\mathscr{T} = \begin{bmatrix} g(2) & g(2^2) & g(2^3) & \dots & g(2^{11}) & \dots \\ g(2^2) & g(2) & g(2^2) & \dots & g(2^{10}) & \dots \\ g(2^3) & g(2^2) & g(2) & \dots & g(2^9) & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ g(2^{11}) & g(2^{10}) & g(2^9) & \dots & g(2) & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}$$

Leading principal submatrices of $\mathscr{T}$ have ranks 1, 2, 2, 2, 2, 2, 4, 6, 8, 10, 11, 11, 11, …

Early evaluations interpolate $y + \frac{1}{y}$

Parametrized Recursive Bases

Let K be a field and $u, v, w \in \mathsf{K}, u \neq 0, v \neq 0$

$$V_0^{[u,v,w]}(x) = 1$$
$$V_1^{[u,v,w]}(x) = ux + w$$
$$V_n^{[u,v,w]}(x) = vxV_{n-1}^{[u,v,w]}(x) - V_{n-2}^{[u,v,w]}(x), \forall n \geq 2$$

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Reduction to power basis:

$$\left(x - \frac{1}{x}\right) V_n^{[u,v,w]}\left(\frac{x + \frac{1}{x}}{v}\right) = \frac{u}{v}\left(x^{n+1} - \frac{1}{x^{n+1}}\right) + w\left(x^n - \frac{1}{x^n}\right) + \left(\frac{u}{v} - 1\right)\left(x^{n-1} - \frac{1}{x^{n-1}}\right)$$

We can compute u, v, w that optimize the sparsity in polynomial time

Chebyshev Term Degrees

Chebyshev term degrees in [Lakshman & Saunders, 1995]:

If

$$\zeta = T_\delta(\beta) \quad \text{where} \quad \beta = \frac{\omega + \frac{1}{\omega}}{2}$$

are given, we can compute the **Chebyshev term degree** δ without precomputing the order of $\omega \in \mathbb{F}_p$

Open Questions

1. Other orthogonal polynomials?

   Are there reduction formulas?

2. Mixed bases?

   Example: How to interpolate

$$f_1(x) = 2\,T_{45}(x) - 91\,U_{131}(x)$$
$$f_2(x) = 37\,V_{100}^{[1,2,3]}(x) - 99\,V_{200}^{[11,12,13]}(x)$$

Thank You !