Towards Mixed Gröbner Basis Algorithms: the Multihomogeneous and Sparse Case

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Computing Gröbner basis over $K[x_1, \ldots, x_n]$

Consider $(f_1, \ldots, f_r) \in K[x_1, \ldots, x_n]$

Gröbner basis

- Consider the ideal of $f_1, \ldots, f_r$ in $\mathbb{K}[x_1, \ldots, x_n]$.
- There is a (finite) Gröbner basis.

- Membership (Normal forms)
- Solving (Elimination)
- etc...

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Computation

- If the homogenization of $f_1, \ldots, f_r$ (over $\mathbb{K}[x_0, x_1, \ldots, x_n]$) is a *regular sequence*
  - Avoid redundant computations (F5)
  - Complexity bounds (Castelnuovo-Mumford Regularity)

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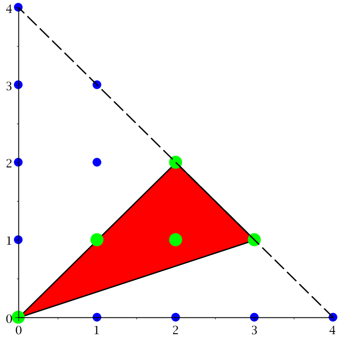
For sparse systems, the homogenization is **NOT** a *regular sequence*.

Sparse systems

- Support of $f = \sum_\alpha c_\alpha \boldsymbol{X}^\alpha \longrightarrow$ Monomials in f, $\{\alpha : c_\alpha \neq 0\}$.
- Sparse system $\longrightarrow$ The supports of the polynomials are "small".

$$1 + xy + x^2y + x^2y^2 + x^3y = \mathbf{1} + \mathbf{0}\cdot x + \mathbf{0}\cdot y + \mathbf{0}\cdot x^2 + \boldsymbol{xy} + \mathbf{0}\cdot y^2 + \mathbf{0}\cdot x^3 + \boldsymbol{x^2y} + \mathbf{0}\cdot xy^2 + \mathbf{0}\cdot y^3 + \mathbf{0}\cdot x^4 + \boldsymbol{x^3y} + \boldsymbol{x^2y^2} + \mathbf{0}\cdot xy^3 + \mathbf{0}\cdot y^4$$

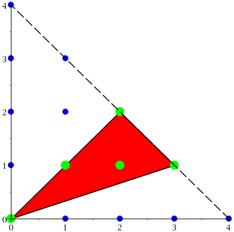


Sparse systems

- Support of $f = \sum_\alpha c_\alpha \boldsymbol{X}^\alpha \longrightarrow$ Monomials in f, $\{\alpha : c_\alpha \neq 0\}$.
- Sparse system $\longrightarrow$ The supports of the polynomials are "small".

- Unmixed sparse system $\longrightarrow$ The polynomials have the same support.
  Mixed sparse system $\longrightarrow$ Different supports.

$1 + xy + x^2y + x^2y^2 + x^3y$



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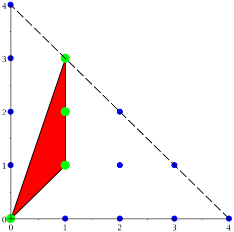
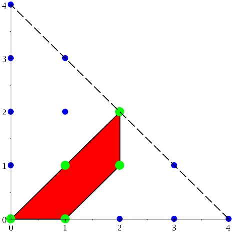


$1 + x + xy + x^2y + x^2y^2$



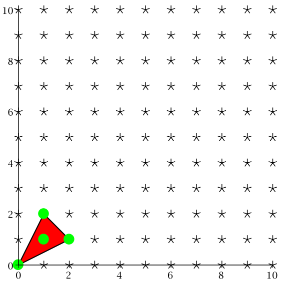
Previous work (Non-exhaustive!)

- Toric varieties
  [Demazure, 1970], [Hochster, 1971], [Satake, 1973], [Kempf, Knudsen, Mumford & Saint-Donat, 1973], [Miyake & Oda, 1975], [Ehlers, 1975], [Bernstein, 1975], [Kusnirenko, ∼1975] [Khovanskii, 1977], . . .
  . . . [Oda, 1988] . . . [Fulton, 1993] . . . [Cox, Little & Schenck, 2011]

- Sparse resultant
  [Gelfand, Kapranov & Zelevinsky, 1990], [Kapranov, Sturmfels & Zelevinsky, 1992], [Sturmfels, 1993], [Pedersen & Sturmfels, 1993], [Gelfand, Kapranov & Zelevinsky, 1994], [Canny & Emiris, 1995], [D´Andrea, 2002], [D´Andrea & Sombra, 2013]

- Sparse GB
  [Sturmfels, 1991], [Faugère, Spaenlehauer & Svartz, 2014]

Semigroup algebras: $\mathbb{K}[S]$ and $\mathbb{K}[S^h]$



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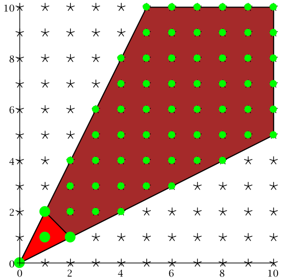

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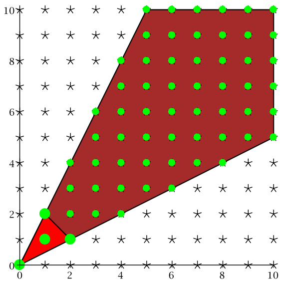
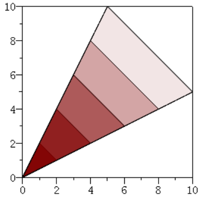


$\mathbb{K}[S]$

$\longrightarrow$



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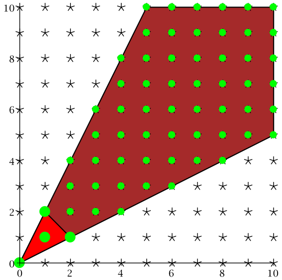

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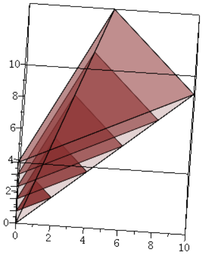

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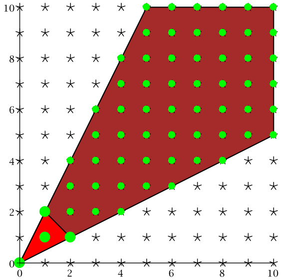

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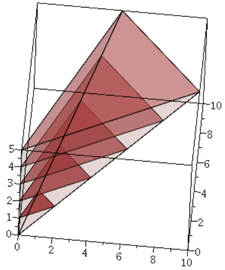

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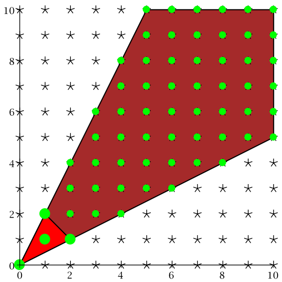

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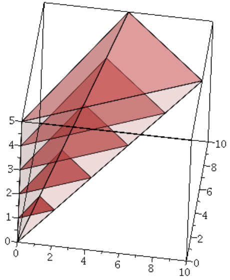

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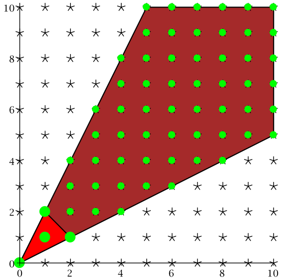

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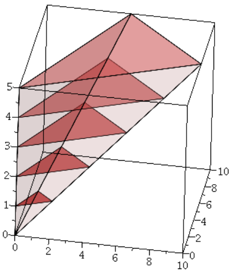

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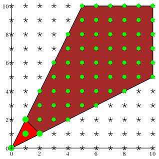

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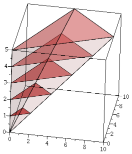

$\mathbb{K}[S^h]$

Compute GB over $\mathbb{K}[S]$

- There is a Gröbner basis.
- ✓ If the homogenization of $f_1, \ldots, f_r$ (over $\mathbb{K}[S^h]$) is a *regular sequence*
  - No redundant computations (F5)
  - Complexity bounds (C-M Regularity)

[Faugère, Spaenlehauer & Svartz, 2014]

Generic **unmixed** systems $\rightarrow$ homogenization $f_1, \ldots, f_r$ is *regular*.

Generic **mixed** systems $\rightarrow$ homogenization of $f_1, \ldots, f_r$ is **NOT** a *regular sequence*.

General approach for affine regular sequences

- For every k, given $GB(hom(\langle f_1, \ldots, f_{k-1} \rangle))$, compute $GB(\boldsymbol{J_k^h})$ where

$$\boldsymbol{J_k^h} := \boxed{hom(\langle f_1, \ldots, f_{k-1} \rangle)} + \boxed{\langle hom(f_k) \rangle}.$$

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For GRevLex orders: $GB(\boldsymbol{J_k^h}) \longrightarrow \begin{cases} \boldsymbol{GB(\langle f_1, \ldots, f_k \rangle)}, \text{ and} \\ GB(hom(\langle f_1, \ldots, f_k \rangle)) \end{cases}.$

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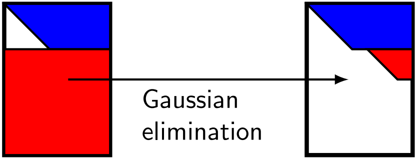
- **Lazard's approach to** $GB(\boldsymbol{J_k^h})$, for each $d \leq reg(\boldsymbol{J_k^h})$,

  Compute a triangular basis of vector space $(\boldsymbol{J_k^h})_d$.

$(\boldsymbol{J_k^h})_d$



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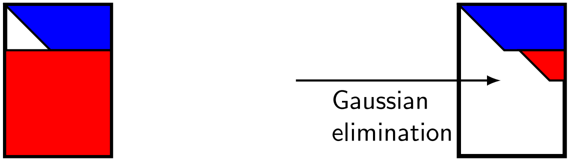

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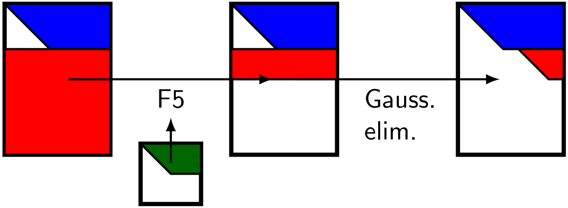
- Lazard's approach to $GB(\boldsymbol{J_k^h})$, for each d, compute triangular basis of $(\boldsymbol{J_k^h})_d$.
- **Affine F5 criterion**, if $(f_1, \ldots, f_r)$ is an *affine regular sequence*,

Reductions to zero in $(\boldsymbol{J_k^h})_d \longleftrightarrow$ polynomials in $\boxed{(\boldsymbol{J_{k-1}^h})_{d-deg(f_k)}}$.

$(J_k^h)_d$



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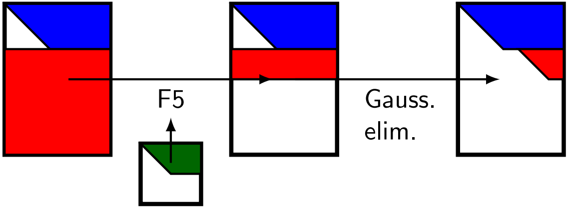


We want to do the same over $\mathbb{K}[S]$, we need GRevLex orders.

Sparse Degree

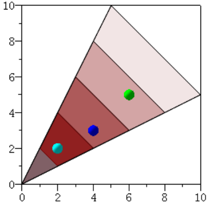
$sp(X^\alpha) =$ minimal $\boldsymbol{s}$ s.t. $X^{\alpha,\boldsymbol{s}} \in \mathbb{K}[S^h]$.
For $f := \sum_\alpha c_\alpha X^\alpha$,

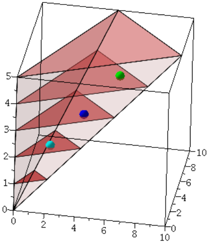
$$sp(f) = max(\{sp(X^\alpha) : c_\alpha \neq 0\}).$$

Example

$$sp(x^2\, y^2 + x^4\, y^3 + x^6\, y^5) = 4$$

$$\begin{cases} sp(x^2\, y^2) = 2, \\ sp(x^4\, y^3) = 3, \\ sp(x^6\, y^5) = 4 \end{cases}$$





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Sparse order

An order $\prec$ is compatible with the sparse degree $\iff$ $(\forall \alpha, \beta \in S)$ if $sp(X^\alpha) < sp(X^\beta)$, then $X^\alpha \prec X^\beta$.

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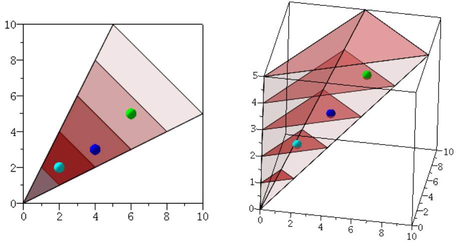
$$LM_\prec(x^2\,y^2 + x^4\,y^3 + x^6\,y^5) = x^6\,y^5$$

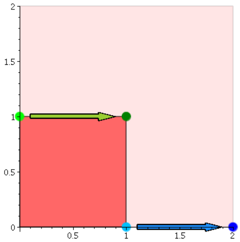


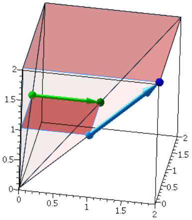
Sparse orders

- Not "behave well" with multiplication.

$$X^\alpha \cdot LM_\prec(f) \neq LM_\prec(X^\alpha \cdot f)$$

- Not monomial orders.
- The division might not terminate.





$$\begin{cases} LM_\prec(y + x) = y \\ LM_\prec(x \cdot (y + x)) = x^2 \neq x \cdot LM_\prec(x + y) \end{cases}$$

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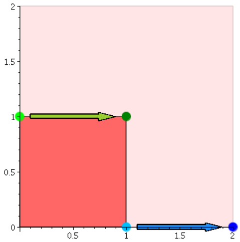

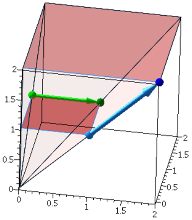

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X^{α} **divides** X^{β}, $X^{\alpha} || X^{\beta}$, iff $\begin{cases} \frac{X^{\beta}}{X^{\alpha}} \in \mathbb{K}[S] \\ sp(X^{\alpha}) + sp(\frac{X^{\beta}}{X^{\alpha}}) = sp(X^{\beta}). \end{cases}$

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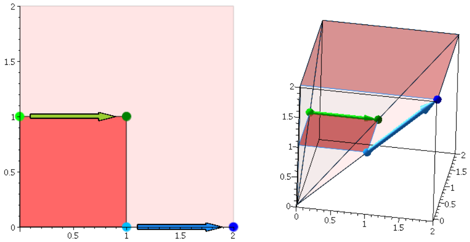

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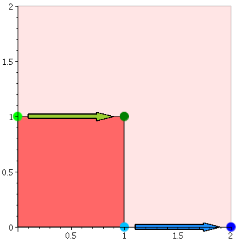

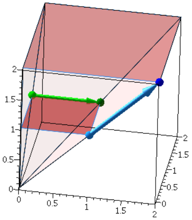

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The division algorithm terminates,

If $LM_{\prec}(f) || X^{\alpha}$, then $LM_{\prec}\left(\frac{X^{\alpha}}{LM_{\prec}(f)} \cdot f\right) = X^{\alpha}$.

Sparse Gröbner basis

$$X^\alpha \textbf{ divides } X^\beta,\ X^\alpha || X^\beta, \text{ iff } \begin{cases} \frac{X^\beta}{X^\alpha} \in \mathbb{K}[S] \\ sp(X^\alpha) + sp(\frac{X^\beta}{X^\alpha}) = sp(X^\beta). \end{cases}$$

G is **sparse Gröbner basis** (sGB) of an ideal $I \subset \mathbb{K}[S]$ wrt $\prec$ iff

$$\langle G \rangle = I \text{ and } (\forall f \in I)(\exists g \in G)\ LM_\prec(g) || LM_\prec(f).$$

Main theorems

- Sparse Gröbner basis $\rightarrow$ finite and unique (reduced).
- Division algorithm wrt sGB(I) $\rightarrow$ Normal form in $\mathbb{K}[S]/I$.
- Sparse version of GRevLex order. Dense systems $\rightarrow$ sGB $=$ GB wrt GRevLex.
- Algorithm to compute sGB. *Regular* mixed system $\rightarrow$ no reductions to zero.

Sparse Gröbner basis

$$X^\alpha \textbf{ divides } X^\beta,\ X^\alpha || X^\beta, \text{ iff } \begin{cases} \frac{X^\beta}{X^\alpha} \in \mathbb{K}[S] \\ sp(X^\alpha) + sp(\frac{X^\beta}{X^\alpha}) = sp(X^\beta). \end{cases}$$

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$$\langle G \rangle = I \text{ and } (\forall f \in I)(\exists g \in G)\ LM_\prec(g) || LM_\prec(f).$$

Main theorems

- Sparse Gröbner basis $\rightarrow$ finite and unique (reduced).
- Division algorithm wrt sGB(I) $\rightarrow$ Normal form in $\mathbb{K}[S]/I$.
- Sparse version of GRevLex order. Dense systems $\rightarrow$ sGB $=$ GB wrt GRevLex.
- Algorithm to compute sGB. *Regular* mixed system $\rightarrow$ no reductions to zero.

We are working on bounds for the regularity...

Mixed Multihomogeneous Systems

The complexity of solving multihomogeneous systems

- Multihomogeneous systems $\rightarrow$ polynomials in

$$\mathbb{K}[x_{1,0}, x_{1,1}, \ldots, x_{1,n_1}] \otimes \mathbb{K}[x_{2,0}, \ldots, x_{2,n_2}] \otimes \cdots \otimes \mathbb{K}[x_{s,0}, \ldots, x_{s,n_s}].$$

- Square system $=$ $(n_1 + \cdots + n_s)$ equations $\rightarrow$ Generically, finite number of solutions over $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$.
- Multigraded algebra $\rightarrow$ Vectors of degrees wrt blocks of variables.

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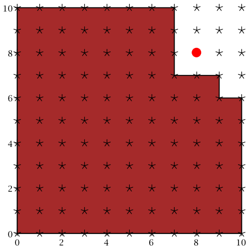

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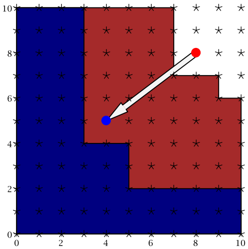

(New) Multigraded F5 Criterion

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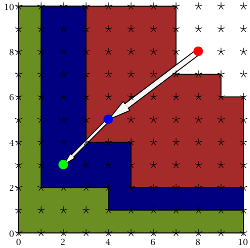

The complexity of solving multihomogeneous systems

Results

- Algorithm to solve, over $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, 0-dimensional square **mixed multihomogeneous** systems, which performs **no reduction to zero**.

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Multihomogeneous Macaulay bound → Generalization of Macaulay bound

Macaulay bound	Multihomogeneous Macaulay bound
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- Consider $f_1, f_2, f_3 \in \mathbb{K}[x_{1,0}, x_{1,1}] \otimes \mathbb{K}[x_{2,0}, x_{2,1}, x_{2,2}]$ such that $multideg(f_1) = (1,2)$, $multideg(f_2) = (2,2)$ & $multideg(f_3) = (0,3)$.
- Then $(n_1, n_2) = (1,2)$. Generically, finite number of solutions over $\mathbb{P}^1 \times \mathbb{P}^2$,

  Multihomogeneous Macaulay Bound $= (3,7) - (1,2) + (1,1) = \mathbf{(3,6)}$.

Summing-up

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- Direct critical pairs algorithm.
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Thank you!