Volume of alcoved polyhedra and Mahler conjecture

M.J. de la Puente
F. Matemáticas, U. Complutense (UCM), Madrid (Spain)
mpuente@ucm.es

(joint work with **P.L. Clavería**)

ISSAC 2018, CUNY, New York (USA)

- $P \subset \mathbb{R}^n$ **body** if P is compact **convex**, non–empty interior

- $P \subset \mathbb{R}^n$ **body** if P is compact **convex**, non–empty interior
- P **centrally symmetric** if $P = -P$

- $P \subset \mathbb{R}^n$ **body** if P is compact **convex**, non–empty interior
- P **centrally symmetric** if $P = -P$
- **polar** of P is $P^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall y \in P\}$

- $P \subset \mathbb{R}^n$ **body** if P is compact **convex**, non–empty interior
- P **centrally symmetric** if $P = -P$
- **polar** of P is $P^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall y \in P\}$
- **polytope** is convex hull in $\mathbb{R}^n$ of finite set

- $P \subset \mathbb{R}^n$ **body** if P is compact **convex**, non–empty interior
- P **centrally symmetric** if $P = -P$
- **polar** of P is $P^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall y \in P\}$
- **polytope** is convex hull in $\mathbb{R}^n$ of finite set $p_1, p_2, \ldots, p_r \in \mathbb{R}^n$,

- $P \subset \mathbb{R}^n$ **body** if P is compact **convex**, non–empty interior
- P **centrally symmetric** if $P = -P$
- **polar** of P is $P^\circ := \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1, \forall y \in P\}$
- **polytope** is convex hull in $\mathbb{R}^n$ of finite set $p_1, p_2, \ldots, p_r \in \mathbb{R}^n$, $P = \text{conv}(p_1, p_2, \ldots, p_r)$

$P = \mathrm{conv}(p_1, p_2, \ldots, p_r),$

$P = \operatorname{conv}(p_1, p_2, \ldots, p_r),$
$P^\circ =$

$P = \mathrm{conv}(p_1, p_2, \ldots, p_r),$
$P^\circ = \{x \in \mathbb{R}^n : \langle x, p_k \rangle \leq 1,$

$P = \mathrm{conv}(p_1, p_2, \ldots, p_r)$,
$P^\circ = \{x \in \mathbb{R}^n : \langle x, p_k \rangle \leq 1, \forall k = 1, 2, \ldots, r\}$,

$P = \mathrm{conv}(p_1, p_2, \ldots, p_r)$,
$P^\circ = \{x \in \mathbb{R}^n : \langle x, p_k \rangle \leq 1, \forall k = 1, 2, \ldots, r\}$,

Example:

$P = \mathrm{conv}(p_1, p_2, \ldots, p_r)$,
$P^\circ = \{x \in \mathbb{R}^n : \langle x, p_k \rangle \leq 1, \forall k = 1, 2, \ldots, r\}$,

Example:

$p_1 = (1, 1)$,

$P = \text{conv}(p_1, p_2, \ldots, p_r)$,
$P^\circ = \{x \in \mathbb{R}^n : \langle x, p_k \rangle \leq 1, \forall k = 1, 2, \ldots, r\}$,

Example:

$p_1 = (1, 1), p_2 = (1, -1)$,

$P = \mathrm{conv}(p_1, p_2, \dots, p_r)$,
$P^\circ = \{x \in \mathbb{R}^n : \langle x, p_k \rangle \leq 1, \forall k = 1, 2, \dots, r\}$,

Example:

$p_1 = (1,1), p_2 = (1,-1), p_3 = -p_1$,

$P = \mathrm{conv}(p_1, p_2, \dots, p_r)$,
$P^\circ = \{x \in \mathbb{R}^n : \langle x, p_k \rangle \le 1, \forall k = 1, 2, \dots, r\}$,

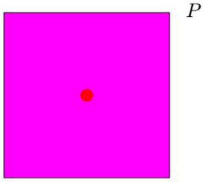
Example:

$p_1 = (1, 1), p_2 = (1, -1), p_3 = -p_1, p_4 = -p_2 \in \mathbb{R}^2$,

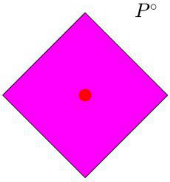
$P = \mathrm{conv}(p_1, p_2, \ldots, p_r)$,
$P^\circ = \{x \in \mathbb{R}^n : \langle x, p_k \rangle \leq 1, \forall k = 1, 2, \ldots, r\}$,

Example:

$p_1 = (1, 1), p_2 = (1, -1), p_3 = -p_1, p_4 = -p_2 \in \mathbb{R}^2$,



$-1 \leq x_1 \leq 1$
$-1 \leq x_2 \leq 1$



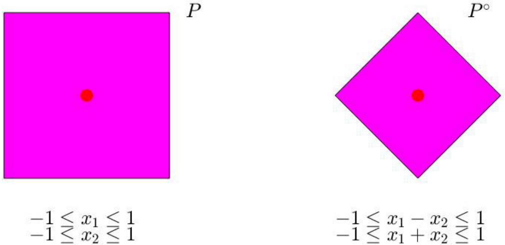
$-1 \leq x_1 - x_2 \leq 1$
$-1 \leq x_1 + x_2 \leq 1$

$P = \text{conv}(p_1, p_2, \ldots, p_r)$,

$P^\circ = \{x \in \mathbb{R}^n : \langle x, p_k \rangle \leq 1, \forall k = 1, 2, \ldots, r\}$,

Example:

$p_1 = (1,1), p_2 = (1,-1), p_3 = -p_1, p_4 = -p_2 \in \mathbb{R}^2$,

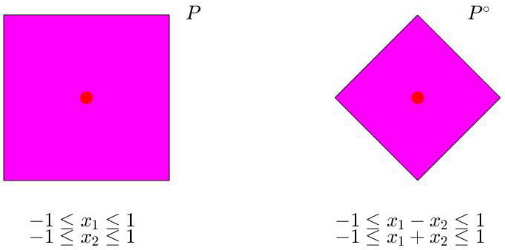


$\langle x, p_1 \rangle = x_1 + x_2$,

$P = \mathrm{conv}(p_1, p_2, \dots, p_r)$,
$P^\circ = \{x \in \mathbb{R}^n : \langle x, p_k \rangle \leq 1, \forall k = 1, 2, \dots, r\}$,

Example:

$p_1 = (1, 1), p_2 = (1, -1), p_3 = -p_1, p_4 = -p_2 \in \mathbb{R}^2$,



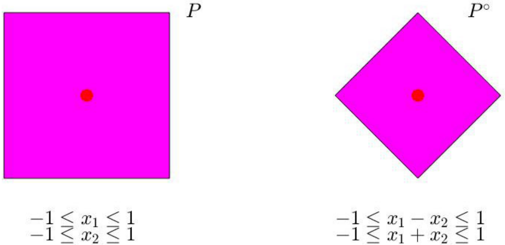
$\langle x, p_1 \rangle = x_1 + x_2, \langle x, p_2 \rangle = x_1 - x_2$,

$P = \mathrm{conv}(p_1, p_2, \ldots, p_r)$,

$P^\circ = \{x \in \mathbb{R}^n : \langle x, p_k \rangle \le 1, \forall k = 1, 2, \ldots, r\}$,

Example:

$p_1 = (1, 1), p_2 = (1, -1), p_3 = -p_1, p_4 = -p_2 \in \mathbb{R}^2$,

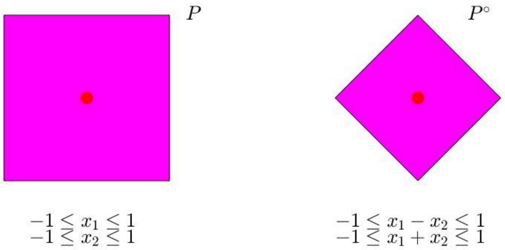


$\langle x, p_1 \rangle = x_1 + x_2, \langle x, p_2 \rangle = x_1 - x_2, \langle x, p_3 \rangle = -x_1 - x_2$,

$P = \mathrm{conv}(p_1, p_2, \ldots, p_r)$,
$P^\circ = \{x \in \mathbb{R}^n : \langle x, p_k \rangle \leq 1, \forall k = 1, 2, \ldots, r\}$,

Example:

$p_1 = (1, 1), p_2 = (1, -1), p_3 = -p_1, p_4 = -p_2 \in \mathbb{R}^2$,



$\langle x, p_1 \rangle = x_1 + x_2, \langle x, p_2 \rangle = x_1 - x_2, \langle x, p_3 \rangle = -x_1 - x_2,$
$\langle x, p_4 \rangle = -x_1 + x_2$

$P \subset \mathbb{R}^n$ **centrally symmetric** **convex** body,

$P \subset \mathbb{R}^n$ **centrally symmetric** **convex** body, Euclidean norm,

$P \subset \mathbb{R}^n$ **centrally symmetric** **convex** body, Euclidean norm, usual volume,

$P \subset \mathbb{R}^n$ **centrally symmetric** **convex** body, Euclidean norm, usual volume,

Key obs: When $\mathrm{vol}(P)$ increases, then $\mathrm{vol}(P^\circ)$ decreases

$P \subset \mathbb{R}^n$ **centrally symmetric convex** body, Euclidean norm, usual volume,

Key obs: When $\mathrm{vol}(P)$ increases, then $\mathrm{vol}(P^\circ)$ decreases

How large is $\mathrm{vol}(P)\,\mathrm{vol}(P^\circ)$?

$P \subset \mathbb{R}^n$ **centrally symmetric** **convex** body, Euclidean norm, usual volume,

Key obs: When $\mathrm{vol}(P)$ increases, then $\mathrm{vol}(P^\circ)$ decreases

How large is $\mathrm{vol}(P)\,\mathrm{vol}(P^\circ)$?

Unit cube C_n (edge$=2$),

$P \subset \mathbb{R}^n$ **centrally symmetric convex** body, Euclidean norm, usual volume,

Key obs: When $\mathrm{vol}(P)$ increases, then $\mathrm{vol}(P^\circ)$ decreases

How large is $\mathrm{vol}(P)\,\mathrm{vol}(P^\circ)$?

Unit cube C_n (edge$=2$), C_n° is **unit cross–polytope**

$P \subset \mathbb{R}^n$ centrally symmetric **convex** body, Euclidean norm, usual volume,

Key obs: When $\mathrm{vol}(P)$ increases, then $\mathrm{vol}(P^\circ)$ decreases

How large is $\mathrm{vol}(P)\,\mathrm{vol}(P^\circ)$?

Unit cube C_n (edge$=2$), C_n° is **unit cross–polytope**

Unit ball B_n

$P \subset \mathbb{R}^n$ **centrally symmetric** **convex** body, Euclidean norm, usual volume,

Key obs: When $\mathrm{vol}(P)$ increases, then $\mathrm{vol}(P^\circ)$ decreases

How large is $\mathrm{vol}(P)\,\mathrm{vol}(P^\circ)$?

Unit cube C_n (edge$=2$), C_n° is **unit cross–polytope**

Unit ball $B_n = B_n^\circ$

$P \subset \mathbb{R}^n$ **centrally symmetric convex** body, Euclidean norm, usual volume,

Key obs: When $\text{vol}(P)$ increases, then $\text{vol}(P^\circ)$ decreases

How large is $\text{vol}(P)\,\text{vol}(P^\circ)$?

Unit cube C_n (edge$=2$), C_n° is **unit cross–polytope**

Unit ball $B_n = B_n^\circ$

Blaschke–Santaló inequality (proved 1949):

$P \subset \mathbb{R}^n$ **centrally symmetric convex** body, Euclidean norm, usual volume,

Key obs: When $\mathrm{vol}(P)$ increases, then $\mathrm{vol}(P^\circ)$ decreases

How large is $\mathrm{vol}(P)\,\mathrm{vol}(P^\circ)$?

Unit cube C_n (edge$=2$), C_n° is **unit cross–polytope**

Unit ball $B_n = B_n^\circ$

Blaschke–Santaló inequality (proved 1949):

$$\mathrm{vol}(P)\,\mathrm{vol}(P^\circ) \leq \mathrm{vol}(B_n)\,\mathrm{vol}(B_n^\circ)$$

$P \subset \mathbb{R}^n$ centrally symmetric **convex** body, Euclidean norm, usual volume,

Key obs: When $\mathrm{vol}(P)$ increases, then $\mathrm{vol}(P^\circ)$ decreases

How large is $\mathrm{vol}(P)\,\mathrm{vol}(P^\circ)$?

Unit cube C_n (edge$=2$), C_n° is **unit cross–polytope**

Unit ball $B_n = B_n^\circ$

Blaschke–Santaló inequality (proved 1949):

$$\mathrm{vol}(P)\,\mathrm{vol}(P^\circ) \le \mathrm{vol}(B_n)\,\mathrm{vol}(B_n^\circ)$$

Mahler conjecture (open since 1939):

$P \subset \mathbb{R}^n$ centrally symmetric **convex** body, Euclidean norm, usual volume,

Key obs: When $\mathrm{vol}(P)$ increases, then $\mathrm{vol}(P^\circ)$ decreases

How large is $\mathrm{vol}(P)\,\mathrm{vol}(P^\circ)$?

Unit cube C_n (edge$=2$), C_n° is **unit cross–polytope**

Unit ball $B_n = B_n^\circ$

Blaschke–Santaló inequality (proved 1949):

$$\mathrm{vol}(P)\,\mathrm{vol}(P^\circ) \leq \mathrm{vol}(B_n)\,\mathrm{vol}(B_n^\circ)$$

Mahler conjecture (open since 1939):

$$\mathrm{vol}(C_n)\,\mathrm{vol}(C_n^\circ) \leq \mathrm{vol}(P)\,\mathrm{vol}(P^\circ)$$

- $P \subset \mathbb{R}^n$ centrally symmetric **convex** body

- $P \subset \mathbb{R}^n$ centrally symmetric **convex** body
- **Unit cube** $\mathrm{vol}(C_n) = 2^n$,

- $P \subset \mathbb{R}^n$ centrally symmetric **convex** body
- **Unit cube** $\mathrm{vol}(C_n) = 2^n$, $\quad\quad \mathrm{vol}(C_n^\circ) = \frac{2^n}{n!}$

- $P \subset \mathbb{R}^n$ centrally symmetric **convex** body
- **Unit cube** $\mathrm{vol}(C_n) = 2^n$, $\quad \mathrm{vol}(C_n^\circ) = \frac{2^n}{n!}$
- **Unit ball** $\mathrm{vol}(B_n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$

- $P \subset \mathbb{R}^n$ centrally symmetric **convex** body
- **Unit cube** $\operatorname{vol}(C_n) = 2^n$, $\qquad \operatorname{vol}(C_n^\circ) = \frac{2^n}{n!}$
- **Unit ball** $\operatorname{vol}(B_n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$

$$\boxed{\frac{4^n}{n!} \overset{?}{\leq} \operatorname{vol}(P)\operatorname{vol}(P^\circ) \leq \frac{\pi^n}{\Gamma(\frac{n}{2}+1)^2}}$$

- $P \subset \mathbb{R}^n$ centrally symmetric **convex** body
- **Unit cube** $\mathrm{vol}(C_n) = 2^n$, $\qquad \mathrm{vol}(C_n^\circ) = \frac{2^n}{n!}$
- **Unit ball** $\mathrm{vol}(B_n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$

$$\boxed{\frac{4^n}{n!} \overset{?}{\leq} \mathrm{vol}(P)\,\mathrm{vol}(P^\circ) \leq \frac{\pi^n}{\Gamma(\frac{n}{2}+1)^2}}$$

is **Mahler conj.**

- $P \subset \mathbb{R}^n$ centrally symmetric **convex** body
- **Unit cube** $\mathrm{vol}(C_n) = 2^n, \qquad \mathrm{vol}(C_n^\circ) = \frac{2^n}{n!}$
- **Unit ball** $\mathrm{vol}(B_n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$

$$\boxed{\frac{4^n}{n!} \overset{?}{\leq} \mathrm{vol}(P)\,\mathrm{vol}(P^\circ) \leq \frac{\pi^n}{\Gamma(\frac{n}{2}+1)^2}}$$

is **Mahler conj.** and **Blaschke–Santaló thm.**

To compute the **volume** of a polytope

To compute the **volume** of a polytope is a $\#P$–**problem**

To compute the **volume** of a polytope is a $\#P$–**problem** (Dyer and Frieze 1988, L. Khachiyan 1989)

Alcoved polytope:

Alcoved polytope: equations of facets of $P \subset \mathbb{R}^n$ are of TWO types:

Alcoved polytope: equations of facets of $P \subset \mathbb{R}^n$ are of TWO types:

- $x_i = a_i$

Alcoved polytope: equations of facets of $P \subset \mathbb{R}^n$ are of TWO types:

- $x_i = a_i$
- $x_i - x_j = a_{ij}$,

Alcoved polytope: equations of facets of $P \subset \mathbb{R}^n$ are of TWO types:

- $x_i = a_i$
- $x_i - x_j = a_{ij}$, with $a_i, a_{ij} \in \mathbb{R}$

Alcoved polytope: equations of facets of $P \subset \mathbb{R}^n$ are of TWO types:

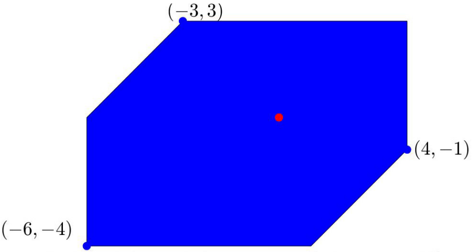
- $x_i = a_i$
- $x_i - x_j = a_{ij}$, with $a_i, a_{ij} \in \mathbb{R}$

$\Rightarrow$ **arrange** coeffs. in **matrix** $(n+1) \times (n+1) \Rightarrow$ **do linear algebra**

Alcoved polytope: equations of facets of $P \subset \mathbb{R}^n$ are of TWO types:

- $x_i = a_i$
- $x_i - x_j = a_{ij}$, with $a_i, a_{ij} \in \mathbb{R}$

$\Rightarrow$ **arrange** coeffs. in **matrix** $(n+1) \times (n+1)$ $\Rightarrow$ **do linear algebra**

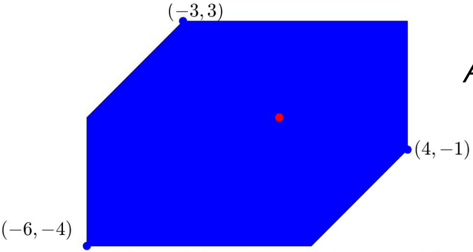


$-6 \leq x_1 \leq 4, \quad -4 \leq x_2 \leq 3, \quad -6 \leq x_1 - x_2 \leq 5$

Alcoved polytope: equations of facets of $P \subset \mathbb{R}^n$ are of TWO types:

- $x_i = a_i$
- $x_i - x_j = a_{ij}$, with $a_i, a_{ij} \in \mathbb{R}$

$\Rightarrow$ **arrange** coeffs. in **matrix** $(n+1) \times (n+1)$ $\Rightarrow$ **do linear algebra**



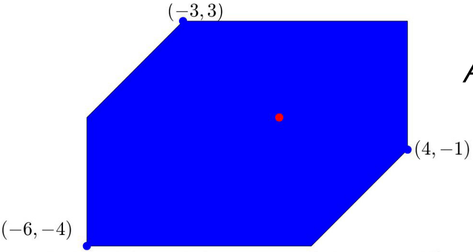
$-6 \le x_1 \le 4, \quad -4 \le x_2 \le 3, \quad -6 \le x_1 - x_2 \le 5$

$$A(P) = \begin{bmatrix} 0 & -6 & -6 \\ -5 & 0 & -4 \\ -4 & -3 & 0 \end{bmatrix}$$

Alcoved polytope: equations of facets of $P \subset \mathbb{R}^n$ are of TWO types:

- $x_i = a_i$
- $x_i - x_j = a_{ij}$, with $a_i, a_{ij} \in \mathbb{R}$

$\Rightarrow$ **arrange** coeffs. in **matrix** $(n+1) \times (n+1) \Rightarrow$ **do linear algebra**



$-6 \leq x_1 \leq 4, \quad -4 \leq x_2 \leq 3, \quad -6 \leq x_1 - x_2 \leq 5$

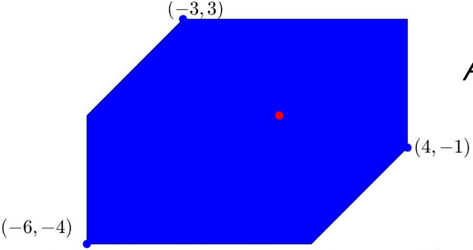
$$A(P) = \begin{bmatrix} 0 & -6 & -6 \\ -5 & 0 & -4 \\ -4 & -3 & 0 \end{bmatrix}$$

- $a_{i,n+1} \leq x_i \leq -a_{n+1,i}$

Alcoved polytope: equations of facets of $P \subset \mathbb{R}^n$ are of TWO types:

- $x_i = a_i$
- $x_i - x_j = a_{ij}$, with $a_i, a_{ij} \in \mathbb{R}$

$\Rightarrow$ **arrange** coeffs. in **matrix** $(n+1) \times (n+1) \Rightarrow$ **do linear algebra**



$-6 \le x_1 \le 4, \quad -4 \le x_2 \le 3, \quad -6 \le x_1 - x_2 \le 5$

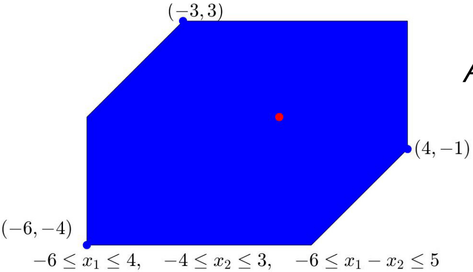
$$A(P) = \begin{bmatrix} 0 & -6 & -6 \\ -5 & 0 & -4 \\ -4 & -3 & 0 \end{bmatrix}$$

- $a_{i,n+1} \le x_i \le -a_{n+1,i}$
- $a_{i,j} \le x_i - x_j \le -a_{j,i}$

Alcoved polytope: equations of facets of $P \subset \mathbb{R}^n$ are of TWO types:

- $x_i = a_i$
- $x_i - x_j = a_{ij}$, with $a_i, a_{ij} \in \mathbb{R}$

$\Rightarrow$ **arrange** coeffs. in **matrix** $(n+1)\times(n+1) \Rightarrow$ **do linear algebra**



$$A(P) = \begin{bmatrix} 0 & -6 & -6 \\ -5 & 0 & -4 \\ -4 & -3 & 0 \end{bmatrix}$$

- $a_{i,n+1} \le x_i \le -a_{n+1,i}$
- $a_{i,j} \le x_i - x_j \le -a_{j,i}$
- $a_{ii} = 0$

Which matrices $[a_{ij}] \in M_{n+1}$ yield **alcoved polytopes** $P \subset \mathbb{R}^n$?

Which matrices $[a_{ij}] \in M_{n+1}$ yield **alcoved polytopes** $P \subset \mathbb{R}^n$?

Thm. (Sergeev 2009, de la Puente 2013)

Which matrices $[a_{ij}] \in M_{n+1}$ yield **alcoved polytopes** $P \subset \mathbb{R}^n$?

Thm. (Sergeev 2009, de la Puente 2013) Every **normal idempotent matrix** gives rise to an **alcoved polytope**

Which matrices $[a_{ij}] \in M_{n+1}$ yield **alcoved polytopes** $P \subset \mathbb{R}^n$?

Thm. (Sergeev 2009, de la Puente 2013) Every **normal idempotent matrix** gives rise to an **alcoved polytope**

- $A = [a_{ij}]$ **normal matrix:** $a_{ii} = 0$ and $a_{ij} \leq 0$, all i, j

Which matrices $[a_{ij}] \in M_{n+1}$ yield **alcoved polytopes** $P \subset \mathbb{R}^n$?

Thm. (Sergeev 2009, de la Puente 2013) Every **normal idempotent matrix** gives rise to an **alcoved polytope**

- $A = [a_{ij}]$ **normal matrix:** $a_{ii} = 0$ and $a_{ij} \leq 0$, all i, j
- A **idempotent (wrt tropical product):** $A \odot A = A$

Which matrices $[a_{ij}] \in M_{n+1}$ yield **alcoved polytopes** $P \subset \mathbb{R}^n$?

Thm. (Sergeev 2009, de la Puente 2013) Every **normal idempotent matrix** gives rise to an **alcoved polytope**

- $A = [a_{ij}]$ **normal matrix:** $a_{ii} = 0$ and $a_{ij} \leq 0$, all i, j
- A **idempotent (wrt tropical product):** $A \odot A = A$

$\oplus = \max$

Which matrices $[a_{ij}] \in M_{n+1}$ yield **alcoved polytopes** $P \subset \mathbb{R}^n$?

Thm. (Sergeev 2009, de la Puente 2013) Every **normal idempotent matrix** gives rise to an **alcoved polytope**

- $A = [a_{ij}]$ **normal matrix:** $a_{ii} = 0$ and $a_{ij} \leq 0$, all i, j
- A **idempotent (wrt tropical product):** $A \odot A = A$

$\oplus = \max$ is **trop. sum**

Which matrices $[a_{ij}] \in M_{n+1}$ yield **alcoved polytopes** $P \subset \mathbb{R}^n$?

Thm. (Sergeev 2009, de la Puente 2013) Every **normal idempotent matrix** gives rise to an **alcoved polytope**

- $A = [a_{ij}]$ **normal matrix:** $a_{ii} = 0$ and $a_{ij} \leq 0$, all i, j
- A **idempotent (wrt tropical product):** $A \odot A = A$

$\oplus = \max$ is **trop. sum** and $\odot = +$

Which matrices $[a_{ij}] \in M_{n+1}$ yield **alcoved polytopes** $P \subset \mathbb{R}^n$?

Thm. (Sergeev 2009, de la Puente 2013) Every **normal idempotent matrix** gives rise to an **alcoved polytope**

- $A = [a_{ij}]$ **normal matrix:** $a_{ii} = 0$ and $a_{ij} \leq 0$, all i, j
- A **idempotent (wrt tropical product):** $A \odot A = A$

$\oplus = \max$ is **trop. sum** and $\odot = +$ is **trop. product**

$\oplus = \max$ **tropical sum**, $\odot = +$ **trop. product**

$\oplus = \max$ **tropical sum**, $\odot = +$ **trop. product**

$$A = \begin{bmatrix} 0 & -6 & -6 \\ -5 & 0 & -4 \\ -4 & -3 & 0 \end{bmatrix} \quad A \oplus B = \begin{bmatrix} 0 & -2 & -6 \\ 0 & 0 & 0 \\ -4 & -3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -2 & -7 \\ 0 & 0 & 0 \\ -5 & -5 & 0 \end{bmatrix} \quad A \odot B =$$

$$\begin{bmatrix} 0 & -2 & -6 \\ \max\{-5, \underline{0}, -9\} & 0 & 0 \\ \max\{-4, \underline{-3}, -5\} & -3 & 0 \end{bmatrix}$$

- A, B are **normal**
- A **idempotent**,

$\oplus = \max$ **tropical sum**, $\odot = +$ **trop. product**

$$A = \begin{bmatrix} 0 & -6 & -6 \\ -5 & 0 & -4 \\ -4 & -3 & 0 \end{bmatrix} \quad A \oplus B = \begin{bmatrix} 0 & -2 & -6 \\ 0 & 0 & 0 \\ -4 & -3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -2 & -7 \\ 0 & 0 & 0 \\ -5 & -5 & 0 \end{bmatrix} \quad A \odot B =$$

$$\begin{bmatrix} 0 & -2 & -6 \\ \max\{-5, \underline{0}, -9\} & 0 & 0 \\ \max\{-4, \underline{-3}, -5\} & -3 & 0 \end{bmatrix}$$

- A, B are **normal**
- A **idempotent**, $B \odot B = \begin{bmatrix} 0 & -2 & -2 \\ 0 & 0 & 0 \\ -5 & -5 & 0 \end{bmatrix}$,

$\bigoplus = \max$ **tropical sum**, $\bigodot = +$ **trop. product**

$$A = \begin{bmatrix} 0 & -6 & -6 \\ -5 & 0 & -4 \\ -4 & -3 & 0 \end{bmatrix} \quad A \bigoplus B = \begin{bmatrix} 0 & -2 & -6 \\ 0 & 0 & 0 \\ -4 & -3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -2 & -7 \\ 0 & 0 & 0 \\ -5 & -5 & 0 \end{bmatrix} \quad A \bigodot B =$$

$$\begin{bmatrix} 0 & -2 & -6 \\ \max\{-5, \underline{0}, -9\} & 0 & 0 \\ \max\{-4, \underline{-3}, -5\} & -3 & 0 \end{bmatrix}$$

- A, B are **normal**
- A **idempotent**, $B \bigodot B = \begin{bmatrix} 0 & -2 & -2 \\ 0 & 0 & 0 \\ -5 & -5 & 0 \end{bmatrix}$, B not idempotent

$\oplus = \max$ **tropical sum**, $\odot = +$ **trop. product**

$$A = \begin{bmatrix} 0 & -6 & -6 \\ -5 & 0 & -4 \\ -4 & -3 & 0 \end{bmatrix} \quad A \oplus B = \begin{bmatrix} 0 & -2 & -6 \\ 0 & 0 & 0 \\ -4 & -3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -2 & -7 \\ 0 & 0 & 0 \\ -5 & -5 & 0 \end{bmatrix} \quad A \odot B =$$

$$\begin{bmatrix} 0 & -2 & -6 \\ \max\{-5, \underline{0}, -9\} & 0 & 0 \\ \max\{-4, \underline{-3}, -5\} & -3 & 0 \end{bmatrix}$$

- A, B are **normal**
- A **idempotent**, $B \odot B = \begin{bmatrix} 0 & -2 & -2 \\ 0 & 0 & 0 \\ -5 & -5 & 0 \end{bmatrix}$, B not idempotent
- $P(A)$ **alcoved**,

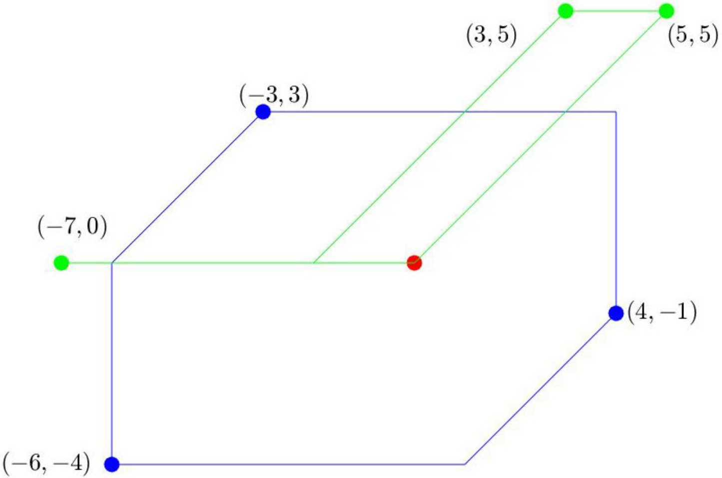
$\oplus = \max$ **tropical sum**, $\odot = +$ **trop. product**

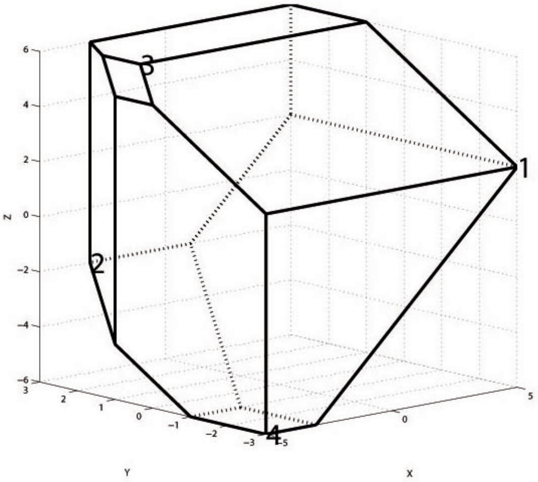
$$A = \begin{bmatrix} 0 & -6 & -6 \\ -5 & 0 & -4 \\ -4 & -3 & 0 \end{bmatrix} \quad A \oplus B = \begin{bmatrix} 0 & -2 & -6 \\ 0 & 0 & 0 \\ -4 & -3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -2 & -7 \\ 0 & 0 & 0 \\ -5 & -5 & 0 \end{bmatrix} \quad A \odot B = \begin{bmatrix} 0 & -2 & -6 \\ \max\{-5, \underline{0}, -9\} & 0 & 0 \\ \max\{-4, \underline{-3}, -5\} & -3 & 0 \end{bmatrix}$$

- A, B are **normal**
- A **idempotent**, $B \odot B = \begin{bmatrix} 0 & -2 & -2 \\ 0 & 0 & 0 \\ -5 & -5 & 0 \end{bmatrix}$, B not idempotent
- $P(A)$ **alcoved**, $P(B)$ not alcoved

$P(A)$ **alcoved** (hence **convex**), $P(B)$ not alcoved



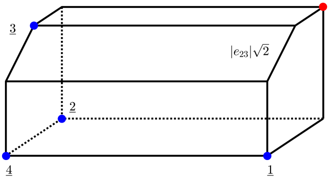


dodecahedron with f–vector $(v, e, f) = (20, 30, 12)$

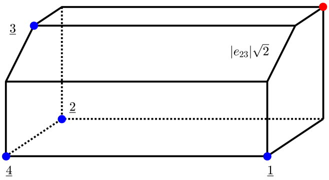
For $n = 4$ get **volume** of **alcoved** $P \subset \mathbb{R}^3$

For $n = 4$ get **volume** of **alcoved** $P \subset \mathbb{R}^3$ in terms of matrix entries a_{ij} of $A \in M_4$?

For $n = 4$ get **volume** of **alcoved** $P \subset \mathbb{R}^3$ in terms of matrix entries a_{ij} of $A \in M_4$?

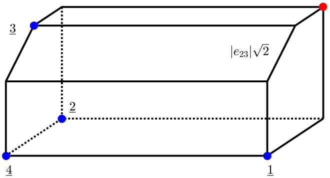


For $n = 4$ get **volume** of **alcoved** $P \subset \mathbb{R}^3$ in terms of matrix entries a_{ij} of $A \in M_4$?



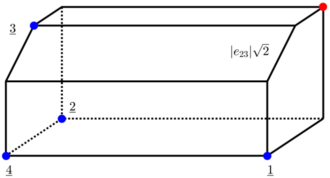
- **normal idempotent**
  $A = \mathbf{B} - \mathbf{E}$

For $n = 4$ get **volume** of **alcoved** $P \subset \mathbb{R}^3$ in terms of matrix entries a_{ij} of $A \in M_4$?



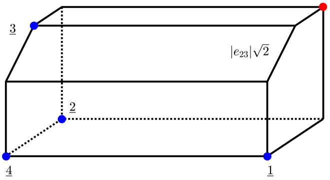
- **normal idempotent**
  $A = \mathbf{B} - \mathbf{E}$
- define **bounding box B**

For $n = 4$ get **volume** of **alcoved** $P \subset \mathbb{R}^3$ in terms of matrix entries a_{ij} of $A \in M_4$?



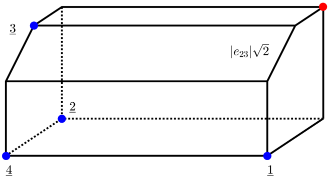
- **normal idempotent**
  $A = \mathbf{B} - \mathbf{E}$
- define **bounding box B**
- def. six **cants**

For $n = 4$ get **volume** of **alcoved** $P \subset \mathbb{R}^3$ in terms of matrix entries a_{ij} of $A \in M_4$?



- **normal idempotent** $A = \mathbf{B} - \mathbf{E}$
- define **bounding box B**
- def. six **cants** arranged in **cycle**

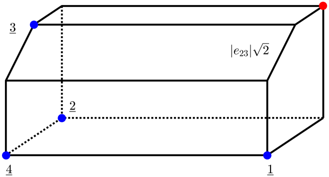
For $n = 4$ get **volume** of **alcoved** $P \subset \mathbb{R}^3$ in terms of matrix entries a_{ij} of $A \in M_4$?



- **normal idempotent**
  $A = \mathbf{B} - \mathbf{E}$
- define **bounding box B**
- def. six **cants** arranged in **cycle**

Easy: $\mathrm{vol}(P) = \mathrm{vol}(\mathrm{box}) - \sum_{j=1}^{6} \mathrm{vol}\, P_j + \sum_j \mathrm{vol}(P_j \cap P_{j+1})$

For $n = 4$ get **volume** of **alcoved** $P \subset \mathbb{R}^3$ in terms of matrix entries a_{ij} of $A \in M_4$?

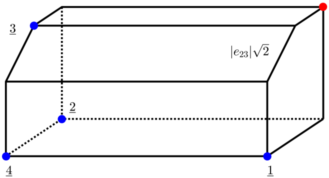


- **normal idempotent**
  $A = \mathbf{B} - \mathbf{E}$
- define **bounding box B**
- def. six **cants** arranged in **cycle**

Easy: $\mathrm{vol}(P) = \mathrm{vol}(\mathrm{box}) - \sum_{j=1}^{6} \mathrm{vol}\, P_j + \sum_j \mathrm{vol}(P_j \cap P_{j+1})$

Thm.:

For $n = 4$ get **volume** of **alcoved** $P \subset \mathbb{R}^3$ in terms of matrix entries a_{ij} of $A \in M_4$?

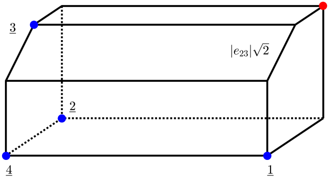


- **normal idempotent** $A = \mathsf{B} - \mathsf{E}$
- define **bounding box B**
- def. six **cants** arranged in **cycle**

Easy: $\text{vol}(P) = \text{vol(box)} - \sum_{j=1}^{6} \text{vol}\, P_j + \sum_j \text{vol}(P_j \cap P_{j+1})$

$$\textbf{Thm.:}\ \text{vol}(P) = \ell_1 \ell_2 \ell_3 - \sum_{j=1}^{6} \frac{c_j^2 \ell_j}{2} + \sum_{j=1}^{6} \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6}$$

For $n = 4$ get **volume** of **alcoved** $P \subset \mathbb{R}^3$ in terms of matrix entries a_{ij} of $A \in M_4$?



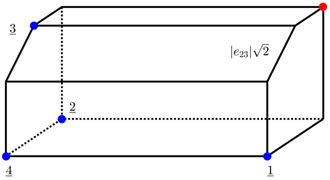
- **normal idempotent**
  $A = \mathbf{B} - \mathbf{E}$
- define **bounding box B**
- def. six **cants** arranged in **cycle**

Easy: $\mathrm{vol}(P) = \mathrm{vol}(\mathrm{box}) - \sum_{j=1}^{6} \mathrm{vol}\, P_j + \sum_j \mathrm{vol}(P_j \cap P_{j+1})$

$$\textbf{Thm.:}\ \mathrm{vol}(P) = \ell_1\ell_2\ell_3 - \sum_{j=1}^{6} \frac{c_j^2 \ell_j}{2} + \sum_{j=1}^{6} \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6}$$

- ℓ_1, ℓ_2, ℓ_3 are edge–lengths of **bounding box B**

For $n = 4$ get **volume** of **alcoved** $P \subset \mathbb{R}^3$ in terms of matrix entries a_{ij} of $A \in M_4$?



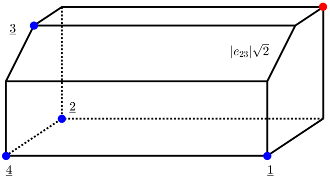
- **normal idempotent**
  $A = \mathbf{B} - \mathbf{E}$
- define **bounding box B**
- def. six **cants** arranged in **cycle**

Easy: $\mathrm{vol}(P) = \mathrm{vol}(\mathrm{box}) - \sum_{j=1}^{6} \mathrm{vol}\, P_j + \sum_j \mathrm{vol}(P_j \cap P_{j+1})$

Thm.: $\mathrm{vol}(P) = \ell_1 \ell_2 \ell_3 - \sum_{j=1}^{6} \frac{c_j^2 \ell_j}{2} + \sum_{j=1}^{6} \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6}$

- ℓ_1, ℓ_2, ℓ_3 are edge–lengths of **bounding box B**
- $m_j := \min\{|c_j|, |c_{j+1}|\}$,

For $n = 4$ get **volume** of **alcoved** $P \subset \mathbb{R}^3$ in terms of matrix entries a_{ij} of $A \in M_4$?

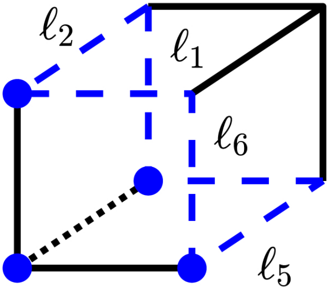


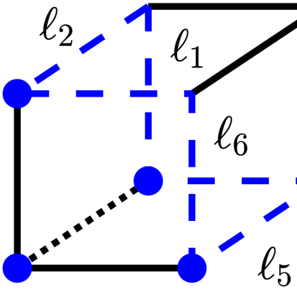
- **normal idempotent**
  $A = \mathbf{B} - \mathbf{E}$
- define **bounding box B**
- def. six **cants** arranged in **cycle**

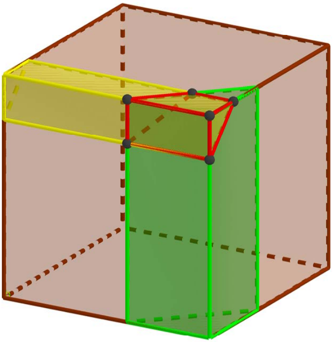
Easy: $\mathrm{vol}(P) = \mathrm{vol}(\mathrm{box}) - \sum_{j=1}^{6} \mathrm{vol}\, P_j + \sum_j \mathrm{vol}(P_j \cap P_{j+1})$

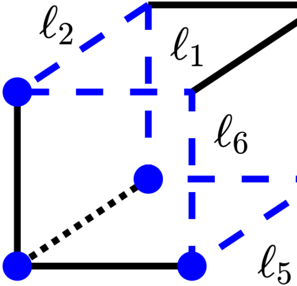
$$\textbf{Thm.:}\ \mathrm{vol}(P) = \ell_1\ell_2\ell_3 - \sum_{j=1}^{6} \frac{c_j^2 \ell_j}{2} + \sum_{j=1}^{6} \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6}$$

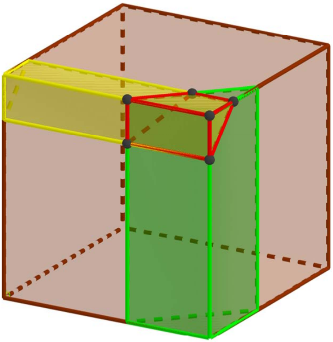
- ℓ_1, ℓ_2, ℓ_3 are edge–lengths of **bounding box B**
- $m_j := \min\{|c_j|, |c_{j+1}|\}$, $M_j := \max\{|c_j|, |c_{j+1}|\}$, c_j is **cant parameter**, $j = 1, 2, \ldots, 6$


ℓ_2
ℓ_1
ℓ_6
ℓ_5


ℓ_2
ℓ_1
ℓ_6
ℓ_5




ℓ_2
ℓ_1
ℓ_6
ℓ_5



$$\operatorname{vol}(P) = \ell_1\ell_2\ell_3 - \sum_{j=1}^{6} \frac{c_j^2 \ell_j}{2} + \sum_{j=1}^{6} \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6}$$

$$\boxed{\mathrm{vol}(P) = \ell_1\ell_2\ell_3 - \sum_{j=1}^{6} \frac{c_j^2 \ell_j}{2} + \sum_{j=1}^{6} \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6}}$$

- $\ell_k = a_{k4},\ k = 1, 2, 3$

$$\text{vol}(P) = \ell_1\ell_2\ell_3 - \sum_{j=1}^{6} \frac{c_j^2 \ell_j}{2} + \sum_{j=1}^{6} \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6}$$

- $\ell_k = a_{k4},\ k = 1, 2, 3$
- each c_j **cant parameter** is the difference of two entries in A

$$\mathrm{vol}(P) = \ell_1\ell_2\ell_3 - \sum_{j=1}^{6} \frac{c_j^2 \ell_j}{2} + \sum_{j=1}^{6} \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6}$$

- $\ell_k = a_{k4}, \ k = 1, 2, 3$
- each c_j **cant parameter** is the difference of two entries in A
- $m_j := \min\{|c_j|, |c_{j+1}|\}$,

$$\boxed{\mathrm{vol}(P) = \ell_1\ell_2\ell_3 - \sum_{j=1}^{6} \frac{c_j^2 \ell_j}{2} + \sum_{j=1}^{6} \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6}}$$

- $\ell_k = a_{k4}$, $k = 1, 2, 3$
- each c_j **cant parameter** is the difference of two entries in A
- $m_j := \min\{|c_j|, |c_{j+1}|\}$, $M_j := \max\{|c_j|, |c_{j+1}|\}$,

$$\text{vol}(P) = \ell_1\ell_2\ell_3 - \sum_{j=1}^{6} \frac{c_j^2 \ell_j}{2} + \sum_{j=1}^{6} \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6}$$

- $\ell_k = a_{k4},\ k = 1, 2, 3$
- each c_j **cant parameter** is the difference of two entries in A
- $m_j := \min\{|c_j|, |c_{j+1}|\},\ M_j := \max\{|c_j|, |c_{j+1}|\},$

vol(P) is **deg 3 rational homog polynomial in** a_{ij}

$$\boxed{\mathrm{vol}(P) = \ell_1\ell_2\ell_3 - \sum_{j=1}^{6} \frac{c_j^2 \ell_j}{2} + \sum_{j=1}^{6} \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6}}$$

- $\ell_k = a_{k4},\ k = 1, 2, 3$
- each c_j **cant parameter** is the difference of two entries in A
- $m_j := \min\{|c_j|, |c_{j+1}|\},\ M_j := \max\{|c_j|, |c_{j+1}|\},$

vol(P) is **deg 3 rational homog polynomial in** a_{ij}

- $A = \mathbf{B} - \mathbf{E}$

$$\boxed{\mathrm{vol}(P) = \ell_1\ell_2\ell_3 - \sum_{j=1}^{6} \frac{c_j^2 \ell_j}{2} + \sum_{j=1}^{6} \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6}}$$

- $\ell_k = a_{k4},\ k = 1, 2, 3$
- each c_j **cant parameter** is the difference of two entries in A
- $m_j := \min\{|c_j|, |c_{j+1}|\},\ M_j := \max\{|c_j|, |c_{j+1}|\}$,

vol(P) is **deg 3 rational homog polynomial in** a_{ij}

- $A = \mathbf{B} - \mathbf{E}$
- ℓ_k are the dimensions of **bounding box B**

$$\mathrm{vol}(P) = \ell_1\ell_2\ell_3 - \sum_{j=1}^{6} \frac{c_j^2 \ell_j}{2} + \sum_{j=1}^{6} \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6}$$

- $\ell_k = a_{k4}, \ k = 1, 2, 3$
- each c_j **cant parameter** is the difference of two entries in A
- $m_j := \min\{|c_j|, |c_{j+1}|\}, \ M_j := \max\{|c_j|, |c_{j+1}|\}$,

vol(P) is **deg 3 rational homog polynomial in** a_{ij}

- $A = \mathbf{B} - \mathbf{E}$
- ℓ_k are the dimensions of **bounding box B**
- the entries of **E** are the c_j's or zero

Mahler conjecture for 3–dim alcoved polytopes

Mahler conjecture for 3–dim alcoved polytopes

- **Thm.** (de la Puente, 2012)

Mahler conjecture for 3–dim alcoved polytopes

- **Thm.** (de la Puente, 2012) A symmetric matrix $\Leftrightarrow P$ centrally symmetric

Mahler conjecture for 3–dim alcoved polytopes

- **Thm.** (de la Puente, 2012) A symmetric matrix $\Leftrightarrow P$ centrally symmetric
- Reduction by affine map to case

Mahler conjecture for 3–dim alcoved polytopes

- **Thm.** (de la Puente, 2012) A **symmetric matrix** $\Leftrightarrow P$ **centrally symmetric**
- Reduction by affine map to case P is **unit cube** (edge$=2$),

Mahler conjecture for 3–dim alcoved polytopes

- **Thm.** (de la Puente, 2012) A symmetric matrix $\Leftrightarrow P$ centrally symmetric
- Reduction by affine map to case P is unit cube (edge$= 2$), and cant parameters are x, y, z

Mahler conjecture for 3–dim alcoved polytopes

- **Thm.** (de la Puente, 2012) A symmetric matrix $\Leftrightarrow P$ centrally symmetric
- Reduction by affine map to case P is unit cube (edge$=2$), and cant parameters are x, y, z with $\boxed{-1 \leq z \leq y \leq x \leq 0}$

Mahler conjecture for 3–dim alcoved polytopes

- **Thm.** (de la Puente, 2012) A symmetric matrix $\Leftrightarrow P$ centrally symmetric
- Reduction by affine map to case P is unit cube (edge$=2$), and cant parameters are x, y, z with $\boxed{-1 \le z \le y \le x \le 0}$
- $\mathrm{vol}(P) = 8 - x^2(y+z) - y^2 z + \frac{1}{3}(2x^3 + y^3) - 2(x^2 + y^2 + z^2)$

Mahler conjecture for 3–dim alcoved polytopes

- **Thm.** (de la Puente, 2012) A symmetric matrix $\Leftrightarrow P$ centrally symmetric
- Reduction by affine map to case P is unit cube (edge= 2), and cant parameters are x, y, z with $\boxed{-1 \leq z \leq y \leq x \leq 0}$
- $\mathrm{vol}(P) =$
  $8 - x^2(y+z) - y^2 z + \frac{1}{3}\left(2x^3 + y^3\right) - 2\left(x^2 + y^2 + z^2\right)$
- $\mathrm{vol}(P^\circ) =$
  $\frac{2/3}{(2+x)(2+y)} + \frac{2/3}{(2+y)(2+z)} + \frac{2/3}{(2+z)(2+x)} + \frac{1/3}{2+y} + \frac{2/3}{2+z} + \frac{1}{3}$

Multiply,

Multiply, clear denominators,

Multiply, clear denominators, pass terms to LHS, get deg 6 polynomial

$MC = 2x^4yz - 3x^3y^2z - 3x^3yz^2 + xy^4z - 3xy^3z^2 + 8x^4y + 6x^4z - 12x^3y^2$
$-23x^3yz - 9x^3z^2 - 6x^2y^2z - 6x^2yz^2 + 4xy^4 - 15xy^3z - 9xy^2z^2 - 6xyz^3 + 2y^4z - 6y^3z^2 + 24x^4 - 40x^3y - 38x^3z - 30x^2y^2$
$-66x^2yz - 24x^2z^2 - 12xy^3 - 54xy^2z - 24xyz^2 - 18xz^3 + 10y^4 - 34y^3z - 24y^2z^2 - 12yz^3 - 8x^3 - 156x^2y - 144x^2z - 72xy^2$
$-72xyz - 72xz^2 - 28y^3 - 144y^2z - 60yz^2 - 48z^3 - 192x^2 - 96xy - 120xz - 192y^2 - 144yz - 192z^2 - 96x - 144y - 192z$

Multiply, clear denominators, pass terms to LHS, get deg 6 polynomial

$$MC = 2x^4yz - 3x^3y^2z - 3x^3yz^2 + xy^4z - 3xy^3z^2 + 8x^4y + 6x^4z - 12x^3y^2$$
$$-23x^3yz - 9x^3z^2 - 6x^2y^2z - 6x^2yz^2 + 4xy^4 - 15xy^3z - 9xy^2z^2 - 6xyz^3 + 2y^4z - 6y^3z^2 + 24x^4 - 40x^3y - 38x^3z - 30x^2y^2$$
$$-66x^2yz - 24x^2z^2 - 12xy^3 - 54xy^2z - 24xyz^2 - 18xz^3 + 10y^4 - 34y^3z - 24y^2z^2 - 12yz^3 - 8x^3 - 156x^2y - 144x^2z - 72xy^2$$
$$-72xyz - 72xz^2 - 28y^3 - 144y^2z - 60yz^2 - 48z^3 - 192x^2 - 96xy - 120xz - 192y^2 - 144yz - 192z^2 - 96x - 144y - 192z$$

$MC|_{\mathcal{S}} \geq 0$, $\mathcal{S}$ simplex $-1 \leq z \leq y \leq x \leq 0$ **Mahler c.**

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$
- How?

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$
- How? Express MC as a **linear comb.**, with **non–negative** coeffs., of **products** of polys w_j, non–negative on $\mathcal{S}$.

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$
- How? Express MC as a **linear comb.**, with **non–negative** coeffs., of **products** of polys w_j, non–negative on $\mathcal{S}$. Which w_j?

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$
- How? Express MC as a **linear comb.**, with **non–negative** coeffs., of **products** of polys w_j, non–negative on $\mathcal{S}$. Which w_j?

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$
- How? Express MC as a **linear comb.**, with **non–negative** coeffs., of **products** of polys w_j, non–negative on $\mathcal{S}$. Which w_j?

$w_1 = 1 + z,$

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$
- How? Express MC as a **linear comb.**, with **non–negative** coeffs., of **products** of polys w_j, non–negative on $\mathcal{S}$. Which w_j?

$$w_1 = 1 + z, \quad w_2 = y - z,$$

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$
- How? Express MC as a **linear comb.**, with **non–negative** coeffs., of **products** of polys w_j, non–negative on $\mathcal{S}$. Which w_j?

$$w_1 = 1 + z, \quad w_2 = y - z, \quad w_3 = x - y,$$

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$
- How? Express MC as a **linear comb.**, with **non–negative** coeffs., of **products** of polys w_j, non–negative on $\mathcal{S}$. Which w_j?

$$w_1 = 1 + z, \quad w_2 = y - z, \quad w_3 = x - y, \quad w_4 = -x$$

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$
- How? Express MC as a **linear comb.**, with **non–negative** coeffs., of **products** of polys w_j, non–negative on $\mathcal{S}$. Which w_j?

$$w_1 = 1 + z, \quad w_2 = y - z, \quad w_3 = x - y, \quad w_4 = -x$$

giving **equations of the $\mathcal{S}$ facets**

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$
- How? Express MC as a **linear comb.**, with **non–negative** coeffs., of **products** of polys w_j, non–negative on $\mathcal{S}$. Which w_j?

$$w_1 = 1 + z, \quad w_2 = y - z, \quad w_3 = x - y, \quad w_4 = -x$$

giving **equations of the $\mathcal{S}$ facets**

$$\boxed{1 = w_1 + w_2 + w_3 + w_4} \text{ (relation)}$$

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$
- How? Express MC as a **linear comb.**, with **non–negative** coeffs., of **products** of polys w_j, non–negative on $\mathcal{S}$. Which w_j?

$$w_1 = 1 + z, \quad w_2 = y - z, \quad w_3 = x - y, \quad w_4 = -x$$

giving **equations of the $\mathcal{S}$ facets**

$$\boxed{1 = w_1 + w_2 + w_3 + w_4} \text{ (relation)}$$

Thus, compute
$MC_j = 1^{6-j} MC_j$

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$
- How? Express MC as a **linear comb.**, with **non–negative** coeffs., of **products** of polys w_j, non–negative on $\mathcal{S}$. Which w_j?

$$w_1 = 1 + z, \quad w_2 = y - z, \quad w_3 = x - y, \quad w_4 = -x$$

giving **equations of the $\mathcal{S}$ facets**

$$\boxed{1 = w_1 + w_2 + w_3 + w_4} \text{ (relation)}$$

Thus, compute
$MC_j = 1^{6-j} MC_j = (w_1 + w_2 + w_3 + w_4)^{6-j} MC_j,$

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$
- How? Express MC as a **linear comb.**, with **non–negative** coeffs., of **products** of polys w_j, non–negative on $\mathcal{S}$. Which w_j?

$$w_1 = 1 + z, \quad w_2 = y - z, \quad w_3 = x - y, \quad w_4 = -x$$

giving **equations of the $\mathcal{S}$ facets**

$$\boxed{1 = w_1 + w_2 + w_3 + w_4} \text{ (relation)}$$

Thus, compute
$MC_j = 1^{6-j} MC_j = (w_1 + w_2 + w_3 + w_4)^{6-j} MC_j$, deg 6 homogeneous in w_1, w_2, w_3, w_4,

- Prove MC is non–negative on semi–algebraic set $\mathcal{S}$
- How? Express MC as a **linear comb.**, with **non–negative** coeffs., of **products** of polys w_j, non–negative on $\mathcal{S}$. Which w_j?

$$w_1 = 1 + z, \quad w_2 = y - z, \quad w_3 = x - y, \quad w_4 = -x$$

giving **equations of the $\mathcal{S}$ facets**

$$\boxed{1 = w_1 + w_2 + w_3 + w_4} \text{ (relation)}$$

Thus, compute
$MC_j = 1^{6-j} MC_j = (w_1 + w_2 + w_3 + w_4)^{6-j} MC_j$, deg 6 homogeneous in w_1, w_2, w_3, w_4, add from $j = 1$ to 6

$$MC = 192w_1^5w_2 + 336w_1^5w_3 + 432w_1^5w_4 + 768w_1^4w_2^2 + 2112w_1^4w_2w_3 + 2472w_1^4w_2w_4 + 1152w_1^4w_3^2 + 2568w_1^4w_3w_4$$

$$+1224w_1^4w_4^2 + 1200w_1^3w_2^3 + 4524w_1^3w_2^2w_3 + 5076w_1^3w_2^2w_4 + 4824w_1^3w_2w_3^2 + 10440w_1^3w_2w_3w_4 + 4992w_1^3w_2w_4^2$$

$$+1528w_1^3w_3^3 + 4896w_1^3w_3^2w_4 + 4740w_1^3w_3w_4^2 + 1380w_1^3w_4^3 + 912w_1^2w_2^4 + 4392w_1^2w_2^3w_3 + 4830w_1^2w_2^3w_4 + 6960w_1^2w_2^2w_3^2$$

$$+14850w_1^2w_2^2w_3w_4 + 7146w_1^2w_2^2w_4^2 + 4442w_1^2w_2w_3^3 + 14034w_1^2w_2w_3^2w_4 + 13656w_1^2w_2w_3w_4^2 + 4050w_1^2w_2w_4^3$$

$$+972w_1^2w_3^4 + 4092w_1^2w_3^3w_4 + 6072w_1^2w_3^2w_4^2 + 3702w_1^2w_3w_4^3 + 774w_1^2w_4^4 + 336w_1w_2^5 + 1980w_1w_2^4w_3 + 2160w_1w_2^4w_4$$

$$+4176w_1w_2^3w_3^2 + 8862w_1w_2^3w_3w_4 + 4302w_1w_2^3w_4^2 + 4030w_1w_2^2w_3^3 + 12657w_1w_2^2w_3^2w_4 + 12414w_1w_2^2w_3w_4^2$$

$$+3744w_1w_2^2w_4^3 + 1790w_1w_2w_3^4 + 7475w_1w_2w_3^3w_4 + 11163w_1w_2w_3^2w_4^2 + 6918w_1w_2w_3w_4^3 + 1482w_1w_2w_4^4 + 292w_1w_3^5$$

$$+1534w_1w_3^4w_4 + 3120w_1w_3^3w_4^2 + 2988w_1w_3^2w_4^3 + 1326w_1w_3w_4^4 + 216w_1w_4^5 + 48w_2^6 + 336w_2^5w_3 + 366w_2^5w_4 + 888w_2^4w_3^2$$

$$+1884w_2^4w_3w_4 + 924w_2^4w_4^2 + 1152w_2^3w_3^3 + 3615w_2^3w_3^2w_4 + 3582w_2^3w_3w_4^2 + 1098w_2^3w_4^3 + 776w_2^2w_3^4 + 3233w_2^2w_3^3w_4$$

$$+4875w_2^2w_3^2w_4^2 + 3072w_2^2w_3w_4^3 + 672w_2^2w_4^4 + 256w_2w_3^5 + 1340w_2w_3^4w_4 + 2752w_2w_3^3w_4^2 + 2682w_2w_3^2w_4^3 + 1218w_2w_3w_4^4$$

$$+204w_2w_4^5 + 32w_3^6 + 204w_3^5w_4 + 540w_3^4w_4^2 + 728w_3^3w_4^3 + 516w_3^2w_4^4 + 180w_3w_4^5 + 24w_4^6.$$

$$MC = 192w_1^5w_2 + 336w_1^5w_3 + 432w_1^5w_4 + 768w_1^4w_2^2 + 2112w_1^4w_2w_3 + 2472w_1^4w_2w_4 + 1152w_1^4w_3^2 + 2568w_1^4w_3w_4$$

$$+1224w_1^4w_4^2 + 1200w_1^3w_2^3 + 4524w_1^3w_2^2w_3 + 5076w_1^3w_2^2w_4 + 4824w_1^3w_2w_3^2 + 10440w_1^3w_2w_3w_4 + 4992w_1^3w_2w_4^2$$

$$+1528w_1^3w_3^3 + 4896w_1^3w_3^2w_4 + 4740w_1^3w_3w_4^2 + 1380w_1^3w_4^3 + 912w_1^2w_2^4 + 4392w_1^2w_2^3w_3 + 4830w_1^2w_2^3w_4 + 6960w_1^2w_2^2w_3^2$$

$$+14850w_1^2w_2^2w_3w_4 + 7146w_1^2w_2^2w_4^2 + 4442w_1^2w_2w_3^3 + 14034w_1^2w_2w_3^2w_4 + 13656w_1^2w_2w_3w_4^2 + 4050w_1^2w_2w_4^3$$

$$+972w_1^2w_3^4 + 4092w_1^2w_3^3w_4 + 6072w_1^2w_3^2w_4^2 + 3702w_1^2w_3w_4^3 + 774w_1^2w_4^4 + 336w_1w_2^5 + 1980w_1w_2^4w_3 + 2160w_1w_2^4w_4$$

$$+4176w_1w_2^3w_3^2 + 8862w_1w_2^3w_3w_4 + 4302w_1w_2^3w_4^2 + 4030w_1w_2^2w_3^3 + 12657w_1w_2^2w_3^2w_4 + 12414w_1w_2^2w_3w_4^2$$

$$+3744w_1w_2^2w_4^3 + 1790w_1w_2w_3^4 + 7475w_1w_2w_3^3w_4 + 11163w_1w_2w_3^2w_4^2 + 6918w_1w_2w_3w_4^3 + 1482w_1w_2w_4^4 + 292w_1w_3^5$$

$$+1534w_1w_3^4w_4 + 3120w_1w_3^3w_4^2 + 2988w_1w_3^2w_4^3 + 1326w_1w_3w_4^4 + 216w_1w_4^5 + 48w_2^6 + 336w_2^5w_3 + 366w_2^5w_4 + 888w_2^4w_3^2$$

$$+1884w_2^4w_3w_4 + 924w_2^4w_4^2 + 1152w_2^3w_3^3 + 3615w_2^3w_3^2w_4 + 3582w_2^3w_3w_4^2 + 1098w_2^3w_4^3 + 776w_2^2w_3^4 + 3233w_2^2w_3^3w_4$$

$$+4875w_2^2w_3^2w_4^2 + 3072w_2^2w_3w_4^3 + 672w_2^2w_4^4 + 256w_2w_3^5 + 1340w_2w_3^4w_4 + 2752w_2w_3^3w_4^2 + 2682w_2w_3^2w_4^3 + 1218w_2w_3w_4^4$$

$$+204w_2w_4^5 + 32w_3^6 + 204w_3^5w_4 + 540w_3^4w_4^2 + 728w_3^3w_4^3 + 516w_3^2w_4^4 + 180w_3w_4^5 + 24w_4^6.$$

All coeffs. are non–negative

$$MC = 192w_1^5w_2 + 336w_1^5w_3 + 432w_1^5w_4 + 768w_1^4w_2^2 + 2112w_1^4w_2w_3 + 2472w_1^4w_2w_4 + 1152w_1^4w_3^2 + 2568w_1^4w_3w_4$$
$$+1224w_1^4w_4^2 + 1200w_1^3w_2^3 + 4524w_1^3w_2^2w_3 + 5076w_1^3w_2^2w_4 + 4824w_1^3w_2w_3^2 + 10440w_1^3w_2w_3w_4 + 4992w_1^3w_2w_4^2$$
$$+1528w_1^3w_3^3 + 4896w_1^3w_3^2w_4 + 4740w_1^3w_3w_4^2 + 1380w_1^3w_4^3 + 912w_1^2w_2^4 + 4392w_1^2w_2^3w_3 + 4830w_1^2w_2^3w_4 + 6960w_1^2w_2^2w_3^2$$
$$+14850w_1^2w_2^2w_3w_4 + 7146w_1^2w_2^2w_4^2 + 4442w_1^2w_2w_3^3 + 14034w_1^2w_2w_3^2w_4 + 13656w_1^2w_2w_3w_4^2 + 4050w_1^2w_2w_4^3$$
$$+972w_1^2w_3^4 + 4092w_1^2w_3^3w_4 + 6072w_1^2w_3^2w_4^2 + 3702w_1^2w_3w_4^3 + 774w_1^2w_4^4 + 336w_1w_2^5 + 1980w_1w_2^4w_3 + 2160w_1w_2^4w_4$$
$$+4176w_1w_2^3w_3^2 + 8862w_1w_2^3w_3w_4 + 4302w_1w_2^3w_4^2 + 4030w_1w_2^2w_3^3 + 12657w_1w_2^2w_3^2w_4 + 12414w_1w_2^2w_3w_4^2$$
$$+3744w_1w_2^2w_4^3 + 1790w_1w_2w_3^4 + 7475w_1w_2w_3^3w_4 + 11163w_1w_2w_3^2w_4^2 + 6918w_1w_2w_3w_4^3 + 1482w_1w_2w_4^4 + 292w_1w_3^5$$
$$+1534w_1w_3^4w_4 + 3120w_1w_3^3w_4^2 + 2988w_1w_3^2w_4^3 + 1326w_1w_3w_4^4 + 216w_1w_4^5 + 48w_2^6 + 336w_2^5w_3 + 366w_2^5w_4 + 888w_2^4w_3^2$$
$$+1884w_2^4w_3w_4 + 924w_2^4w_4^2 + 1152w_2^3w_3^3 + 3615w_2^3w_3^2w_4 + 3582w_2^3w_3w_4^2 + 1098w_2^3w_4^3 + 776w_2^2w_3^4 + 3233w_2^2w_3^3w_4$$
$$+4875w_2^2w_3^2w_4^2 + 3072w_2^2w_3w_4^3 + 672w_2^2w_4^4 + 256w_2w_3^5 + 1340w_2w_3^4w_4 + 2752w_2w_3^3w_4^2 + 2682w_2w_3^2w_4^3 + 1218w_2w_3w_4^4$$
$$+204w_2w_4^5 + 32w_3^6 + 204w_3^5w_4 + 540w_3^4w_4^2 + 728w_3^3w_4^3 + 516w_3^2w_4^4 + 180w_3w_4^5 + 24w_4^6.$$

All coeffs. are non–negative (**certificate** of non–negativeness)

$$
\begin{aligned}
MC = {} & 192w_1^5w_2 + 336w_1^5w_3 + 432w_1^5w_4 + 768w_1^4w_2^2 + 2112w_1^4w_2w_3 + 2472w_1^4w_2w_4 + 1152w_1^4w_3^2 + 2568w_1^4w_3w_4 \\
& +1224w_1^4w_4^2 + 1200w_1^3w_2^3 + 4524w_1^3w_2^2w_3 + 5076w_1^3w_2^2w_4 + 4824w_1^3w_2w_3^2 + 10440w_1^3w_2w_3w_4 + 4992w_1^3w_2w_4^2 \\
& +1528w_1^3w_3^3 + 4896w_1^3w_3^2w_4 + 4740w_1^3w_3w_4^2 + 1380w_1^3w_4^3 + 912w_1^2w_2^4 + 4392w_1^2w_2^3w_3 + 4830w_1^2w_2^3w_4 + 6960w_1^2w_2^2w_3^2 \\
& +14850w_1^2w_2^2w_3w_4 + 7146w_1^2w_2^2w_4^2 + 4442w_1^2w_2w_3^3 + 14034w_1^2w_2w_3^2w_4 + 13656w_1^2w_2w_3w_4^2 + 4050w_1^2w_2w_4^3 \\
& +972w_1^2w_3^4 + 4092w_1^2w_3^3w_4 + 6072w_1^2w_3^2w_4^2 + 3702w_1^2w_3w_4^3 + 774w_1^2w_4^4 + 336w_1w_2^5 + 1980w_1w_2^4w_3 + 2160w_1w_2^4w_4 \\
& +4176w_1w_2^3w_3^2 + 8862w_1w_2^3w_3w_4 + 4302w_1w_2^3w_4^2 + 4030w_1w_2^2w_3^3 + 12657w_1w_2^2w_3^2w_4 + 12414w_1w_2^2w_3w_4^2 \\
& +3744w_1w_2^2w_4^3 + 1790w_1w_2w_3^4 + 7475w_1w_2w_3^3w_4 + 11163w_1w_2w_3^2w_4^2 + 6918w_1w_2w_3w_4^3 + 1482w_1w_2w_4^4 + 292w_1w_3^5 \\
& +1534w_1w_3^4w_4 + 3120w_1w_3^3w_4^2 + 2988w_1w_3^2w_4^3 + 1326w_1w_3w_4^4 + 216w_1w_4^5 + 48w_2^6 + 336w_2^5w_3 + 366w_2^5w_4 + 888w_2^4w_3^2 \\
& +1884w_2^4w_3w_4 + 924w_2^4w_4^2 + 1152w_2^3w_3^3 + 3615w_2^3w_3^2w_4 + 3582w_2^3w_3w_4^2 + 1098w_2^3w_4^3 + 776w_2^2w_3^4 + 3233w_2^2w_3^3w_4 \\
& +4875w_2^2w_3^2w_4^2 + 3072w_2^2w_3w_4^3 + 672w_2^2w_4^4 + 256w_2w_3^5 + 1340w_2w_3^4w_4 + 2752w_2w_3^3w_4^2 + 2682w_2w_3^2w_4^3 + 1218w_2w_3w_4^4 \\
& +204w_2w_4^5 + 32w_3^6 + 204w_3^5w_4 + 540w_3^4w_4^2 + 728w_3^3w_4^3 + 516w_3^2w_4^4 + 180w_3w_4^5 + 24w_4^6.
\end{aligned}
$$

All coeffs. are non–negative (**certificate** of non–negativeness) $\Rightarrow$ **Mahler c.** holds for alcoved 3–dim.

$$MC = 192w_1^5w_2 + 336w_1^5w_3 + 432w_1^5w_4 + 768w_1^4w_2^2 + 2112w_1^4w_2w_3 + 2472w_1^4w_2w_4 + 1152w_1^4w_3^2 + 2568w_1^4w_3w_4$$

$$+1224w_1^4w_4^2 + 1200w_1^3w_2^3 + 4524w_1^3w_2^2w_3 + 5076w_1^3w_2^2w_4 + 4824w_1^3w_2w_3^2 + 10440w_1^3w_2w_3w_4 + 4992w_1^3w_2w_4^2$$

$$+1528w_1^3w_3^3 + 4896w_1^3w_3^2w_4 + 4740w_1^3w_3w_4^2 + 1380w_1^3w_4^3 + 912w_1^2w_2^4 + 4392w_1^2w_2^3w_3 + 4830w_1^2w_2^3w_4 + 6960w_1^2w_2^2w_3^2$$

$$+14850w_1^2w_2^2w_3w_4 + 7146w_1^2w_2^2w_4^2 + 4442w_1^2w_2w_3^3 + 14034w_1^2w_2w_3^2w_4 + 13656w_1^2w_2w_3w_4^2 + 4050w_1^2w_2w_4^3$$

$$+972w_1^2w_3^4 + 4092w_1^2w_3^3w_4 + 6072w_1^2w_3^2w_4^2 + 3702w_1^2w_3w_4^3 + 774w_1^2w_4^4 + 336w_1w_2^5 + 1980w_1w_2^4w_3 + 2160w_1w_2^4w_4$$

$$+4176w_1w_2^3w_3^2 + 8862w_1w_2^3w_3w_4 + 4302w_1w_2^3w_4^2 + 4030w_1w_2^2w_3^3 + 12657w_1w_2^2w_3^2w_4 + 12414w_1w_2^2w_3w_4^2$$

$$+3744w_1w_2^2w_4^3 + 1790w_1w_2w_3^4 + 7475w_1w_2w_3^3w_4 + 11163w_1w_2w_3^2w_4^2 + 6918w_1w_2w_3w_4^3 + 1482w_1w_2w_4^4 + 292w_1w_3^5$$

$$+1534w_1w_3^4w_4 + 3120w_1w_3^3w_4^2 + 2988w_1w_3^2w_4^3 + 1326w_1w_3w_4^4 + 216w_1w_4^5 + 48w_2^6 + 336w_2^5w_3 + 366w_2^5w_4 + 888w_2^4w_3^2$$

$$+1884w_2^4w_3w_4 + 924w_2^4w_4^2 + 1152w_2^3w_3^3 + 3615w_2^3w_3^2w_4 + 3582w_2^3w_3w_4^2 + 1098w_2^3w_4^3 + 776w_2^2w_3^4 + 3233w_2^2w_3^3w_4$$

$$+4875w_2^2w_3^2w_4^2 + 3072w_2^2w_3w_4^3 + 672w_2^2w_4^4 + 256w_2w_3^5 + 1340w_2w_3^4w_4 + 2752w_2w_3^3w_4^2 + 2682w_2w_3^2w_4^3 + 1218w_2w_3w_4^4$$

$$+204w_2w_4^5 + 32w_3^6 + 204w_3^5w_4 + 540w_3^4w_4^2 + 728w_3^3w_4^3 + 516w_3^2w_4^4 + 180w_3w_4^5 + 24w_4^6.$$

All coeffs. are non–negative (**certificate** of non–negativeness) $\Rightarrow$ **Mahler c.** holds for alcoved 3–dim. and equality only attained by boxes

Summary and future work

Summary and future work

- Volume formula for 3–dim alcoved (rational polynomial)

Summary and future work

- Volume formula for 3–dim alcoved (rational polynomial)
- Mahler conjecture holds for 3–dim alcoved

Summary and future work

- Volume formula for 3–dim alcoved (rational polynomial)
- Mahler conjecture holds for 3–dim alcoved

Future work

Summary and future work

- Volume formula for 3–dim alcoved (rational polynomial)
- Mahler conjecture holds for 3–dim alcoved

Future work

- 3–dim alcoved mixed volume formula

Summary and future work

- Volume formula for 3–dim alcoved (rational polynomial)
- Mahler conjecture holds for 3–dim alcoved

Future work

- 3–dim alcoved mixed volume formula
- Filling 3–space using alcoved

Summary and future work

- Volume formula for 3–dim alcoved (rational polynomial)
- Mahler conjecture holds for 3–dim alcoved

Future work

- 3–dim alcoved mixed volume formula
- Filling 3–space using alcoved
- In higher dimensions find:

Summary and future work

- Volume formula for 3–dim alcoved (rational polynomial)
- Mahler conjecture holds for 3–dim alcoved

Future work

- 3–dim alcoved mixed volume formula
- Filling 3–space using alcoved
- In higher dimensions find: (a) f–vector,

Summary and future work

- Volume formula for 3–dim alcoved (rational polynomial)
- Mahler conjecture holds for 3–dim alcoved

Future work

- 3–dim alcoved mixed volume formula
- Filling 3–space using alcoved
- In higher dimensions find: (a) f–vector, (b) volume

Conclusion

Conclusion

Use NORMAL IDEMPOTENT MATRICES

Conclusion

Use NORMAL IDEMPOTENT MATRICES (operated TROPICALLY)

Conclusion

Use NORMAL IDEMPOTENT MATRICES (operated TROPICALLY) to compute and manipulate ALCOVED POLYTOPES

Conclusion

Use NORMAL IDEMPOTENT MATRICES (operated TROPICALLY) to compute and manipulate ALCOVED POLYTOPES

THANK YOU!

http://www.mat.ucm.es/~mpuente/

To compute the **volume** of a polytope is a $\#P$–**problem**

To compute the **volume** of a polytope is a $\#P$–**problem** (number P or sharp P)

To compute the **volume** of a polytope is a $\#P$–**problem** (number P or sharp P) (Dyer and Frieze 1988, L. Khachiyan 1989)

To compute the **volume** of a polytope is a $\#P$–**problem** (number P or sharp P) (Dyer and Frieze 1988, L. Khachiyan 1989)
What is the **complexity class** $\#P$**?**

To compute the **volume** of a polytope is a $\#P$**–problem** (number P or sharp P) (Dyer and Frieze 1988, L. Khachiyan 1989)

What is the **complexity class** $\#P$**?**

- **An** NP **decision problem**:

To compute the **volume** of a polytope is a $\#P$**–problem** (number P or sharp P) (Dyer and Frieze 1988, L. Khachiyan 1989)

What is the **complexity class** $\#P$**?**

- **An** NP **decision problem**: Are there solutions satisfying...?

To compute the **volume** of a polytope is a $\#P$**–problem** (number P or sharp P) (Dyer and Frieze 1988, L. Khachiyan 1989)

What is the **complexity class** $\#P$**?**

- **An** NP **decision problem**: Are there solutions satisfying...?
- **The related** $\#P$ **problem**:

To compute the **volume** of a polytope is a $\#P$**–problem** (number P or sharp P) (Dyer and Frieze 1988, L. Khachiyan 1989)

What is the **complexity class** $\#P$**?**

- **An** NP **decision problem**: Are there solutions satisfying...?
- **The related** $\#P$ **problem**: How many solutions satisfying... do exist?

To compute the **volume** of a polytope is a $\#P$–**problem** (number P or sharp P) (Dyer and Frieze 1988, L. Khachiyan 1989)
What is the **complexity class** $\#P$**?**

- **An** NP **decision problem**: Are there solutions satisfying...?
- **The related** $\#P$ **problem**: How many solutions satisfying... do exist?

Example:

- In a given list of integers, are there subsets with zero sum?

To compute the **volume** of a polytope is a $\#P$**–problem** (number P or sharp P) (Dyer and Frieze 1988, L. Khachiyan 1989)

What is the **complexity class** $\#P$**?**

- **An** NP **decision problem**: Are there solutions satisfying...?
- **The related** $\#P$ **problem**: How many solutions satisfying... do exist?

Example:

- In a given list of integers, are there subsets with zero sum? is an NP problem

To compute the **volume** of a polytope is a $\#P$**–problem** (number P or sharp P) (Dyer and Frieze 1988, L. Khachiyan 1989)
What is the **complexity class** $\#P$**?**

- **An** NP **decision problem**: Are there solutions satisfying...?
- **The related** $\#P$ **problem**: How many solutions satisfying... do exist?

Example:

- In a given list of integers, are there subsets with zero sum? is an NP problem
- In a given list of integers, how many subsets with zero sum do exist?

To compute the **volume** of a polytope is a $\#P$**–problem** (number P or sharp P) (Dyer and Frieze 1988, L. Khachiyan 1989)

What is the **complexity class** $\#P$**?**

- **An** NP **decision problem**: Are there solutions satisfying...?
- **The related** $\#P$ **problem**: How many solutions satisfying... do exist?

Example:

- In a given list of integers, are there subsets with zero sum? is an NP problem
- In a given list of integers, how many subsets with zero sum do exist? the related $\#P$ problem

To compute the **volume** of a polytope is a $\#P$**–problem** (number P or sharp P) (Dyer and Frieze 1988, L. Khachiyan 1989)

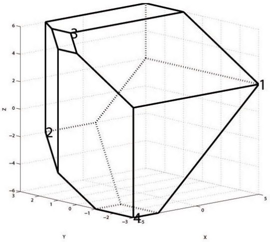
What is the **complexity class** $\#P$**?**

- **An** NP **decision problem**: Are there solutions satisfying...?
- **The related** $\#P$ **problem**: How many solutions satisfying... do exist?

Example:

- In a given list of integers, are there subsets with zero sum? is an NP problem
- In a given list of integers, how many subsets with zero sum do exist? the related $\#P$ problem

51 terms (depending on x, y, z)
125 terms (depending on w_j)
Only missing term in MC is w_1^6. Thus the only real root of MC is given by $w_1 = 1$ and $w_2 = w_3 = w_4 = 0$, equivalently, by $x = y = z = 0$. This shows that equality is only attained by boxes, among centrally symmetric alcoved polyhedra. The conjecture also holds for limits of centrally symmetric alcoved polyhedra.



f–vector
$(v, e, f) \leq (20, 30, 12)$

alcoved dodecahedra

Permanents of matrix