Desingularization of First Order Linear Difference Systems with Rational Function Coefficients

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First Order Systems

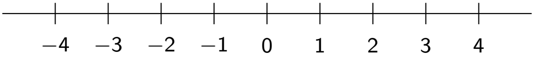
$$\begin{pmatrix} f(z+1) \\ g(z+1) \end{pmatrix} = \begin{pmatrix} \frac{3(z+2)}{z} & \frac{z+6}{z} \\ \frac{-2(z+2)+1}{z} & \frac{-3}{z} \end{pmatrix} \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}$$

What is a Difference System?

Definition

$$[A] \qquad Y(z+1) = A(z)Y(z)$$

- ▹ Y: d-dimensional column vector.
- ▹ A: invertible matrix of size $d \times d$ with entries in $\mathbb{K}(z)$, $\mathbb{K} \leq \mathbb{C}$.



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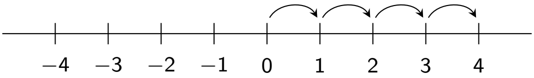

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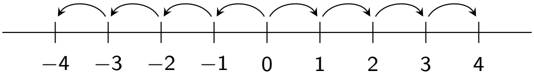

What is a Solution?

$$[A] \qquad Y(z+1) = A(z)Y(z)$$

Meromorphic Functions

$f : \mathbb{C} \setminus S \to \mathbb{C}^d$, where S is a set of isolated points.

Number Sequences

$s : \mathbb{Z} \to \mathbb{C}^d$, for all z where $A(z-1)$ is defined.

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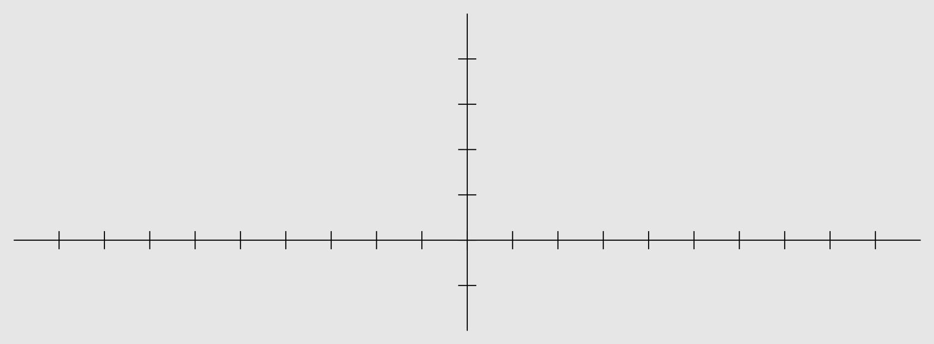
What is a Solution?

Theorem

The set of meromorphic solutions of $[A]$ is a vector space of dimension d over the field of 1-periodic meromorphic functions. (Norlund 1924)

Singularities in Meromorphic Solutions



$$\color{red}{Y(z+1)} = \frac{1}{z-(-3+2i)} A(z) \color{red}{Y(z)}$$

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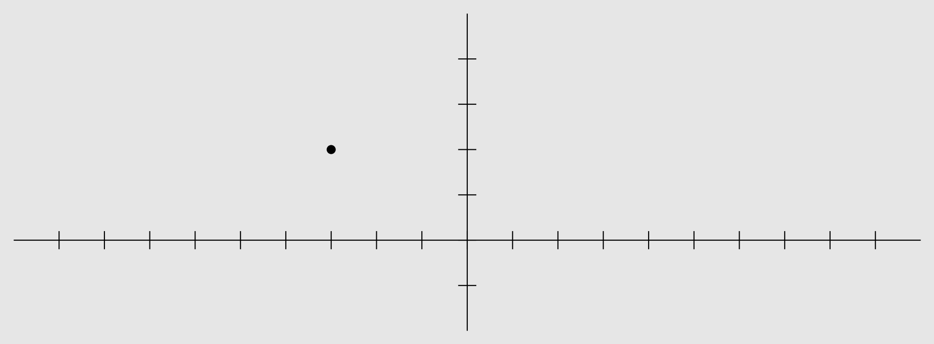

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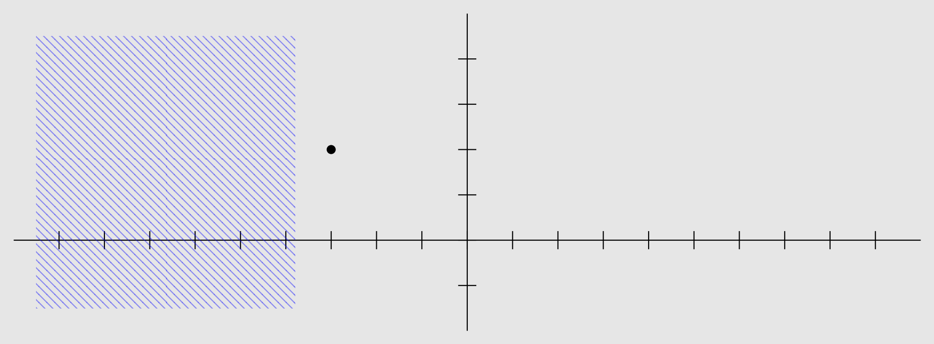

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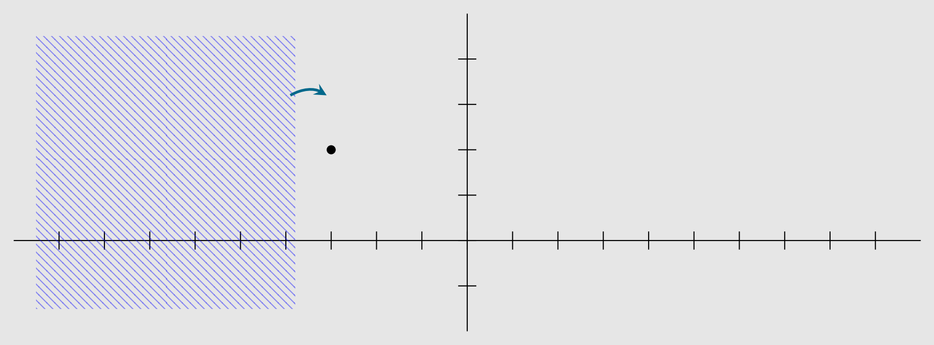

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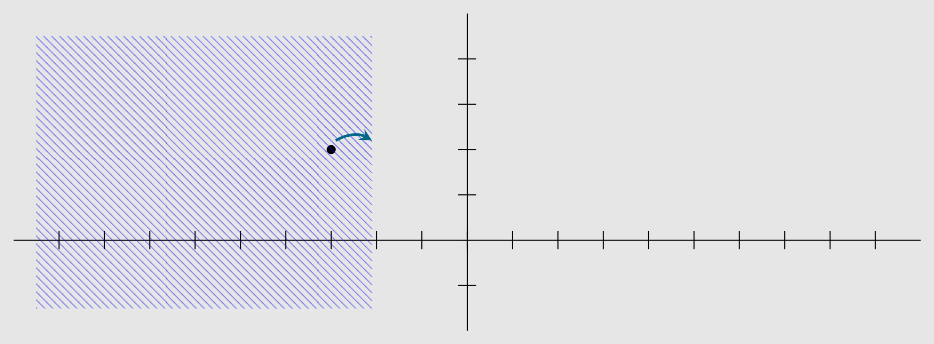

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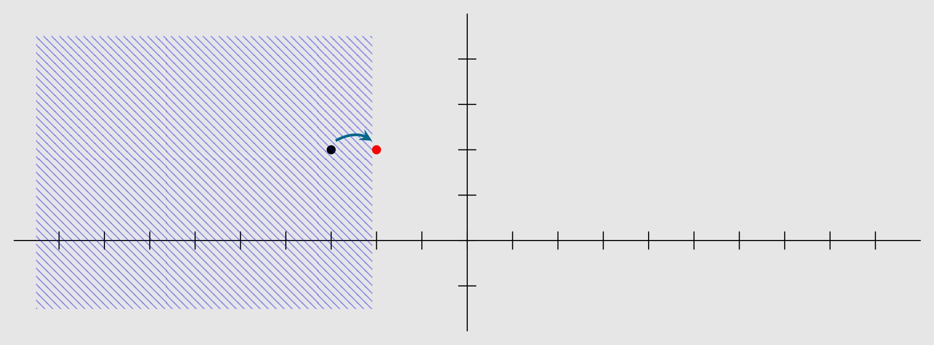

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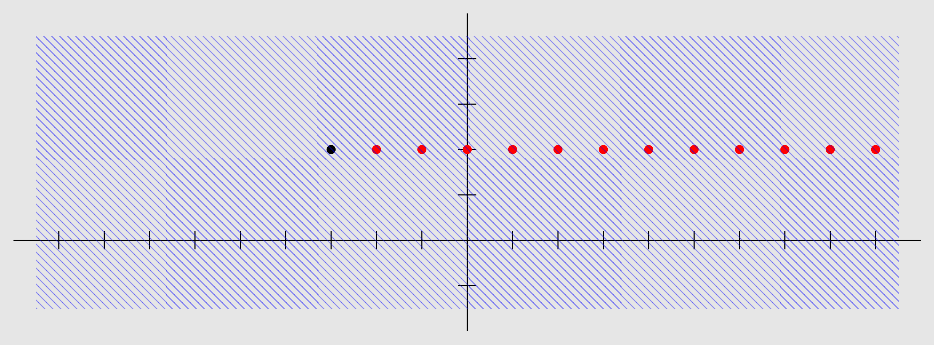

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An Example for a Fundamental System

$$Y(z+1) = \begin{pmatrix} 0 & 1 \\ \frac{-2(z+1)}{z-2} & \frac{3(z-1)}{z-2} \end{pmatrix} Y(z)$$

$$Y(z) = \begin{pmatrix} 2^z & z^3 + 5z + 6 \\ 2^{z+1} & z^3 + 3z^2 + 8z + 12 \end{pmatrix}$$

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Question

Which poles in A correspond to poles in solutions?

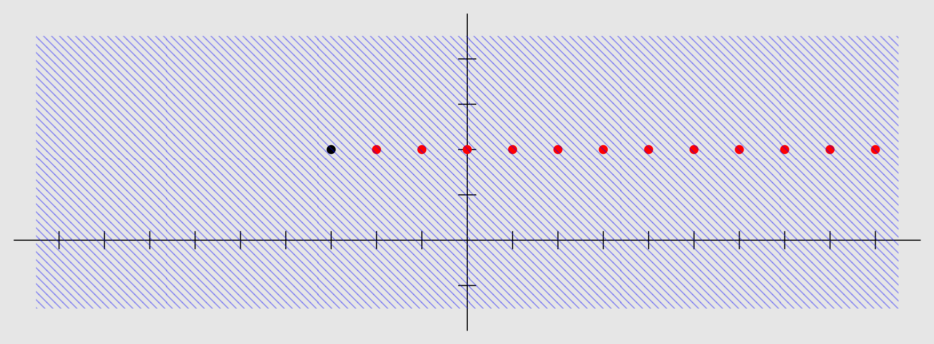
Removable and Apparent Singularities

Definition

Let $\zeta \in \mathbb{C}$ be a pole of $A(z)$. If any solution of $[A]$ which is holomorphic in some left half-plane can be analytically continued to a meromorphic solution which is

- holomorphic at $\zeta + 1$, then ζ is called a removable singularity.
- holomorphic at each point of $\zeta + \mathbb{N}^+$, then ζ is called an apparent singularity.

Shift-Minimal Singularities



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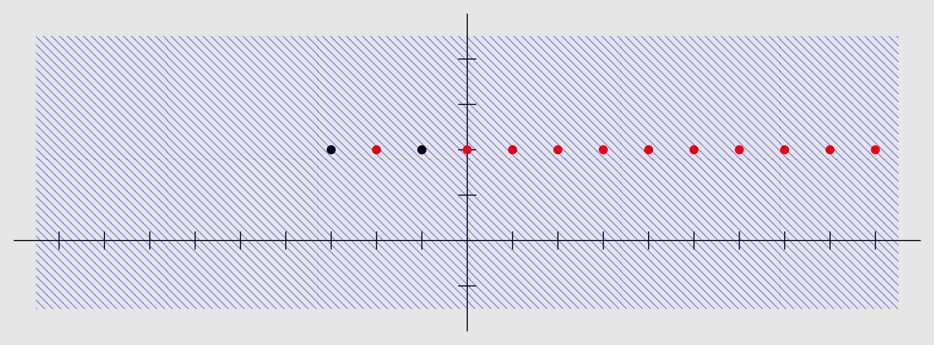

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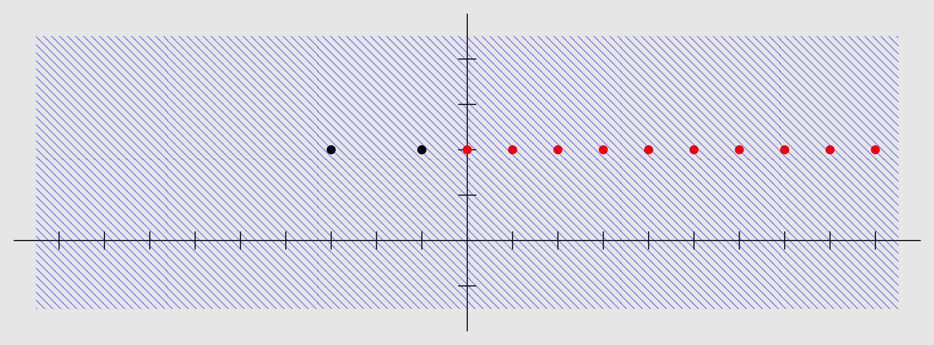

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An Algebraic Formulation

Definition

Let $q \in \mathbb{K}[z]$ be irreducible. Then q is a shift-minimal pole of A if $q \mid den(A)$ and for all $j \in \mathbb{N}^+$, $q(z+j) \nmid den(A)$.

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Definition

Let q be a shift-minimal pole of A. We say A is (partially) desingularizable at q if there exists a transformation $T \in \mathrm{GL}_d(\mathbb{K}(z))$ with polynomial entries such that

- ▷ $\mathrm{ord}_q(T[A]) > \mathrm{ord}_q(A)$,
- ▷ $\mathrm{ord}_p(T[A]) \geq \mathrm{ord}_p(A)$ for all irred. $p \in \mathbb{K}[z]$.

We call T a desingularizing transformation.

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Removable is Desingularizable

desingularizable $\Leftrightarrow$ removable / apparent

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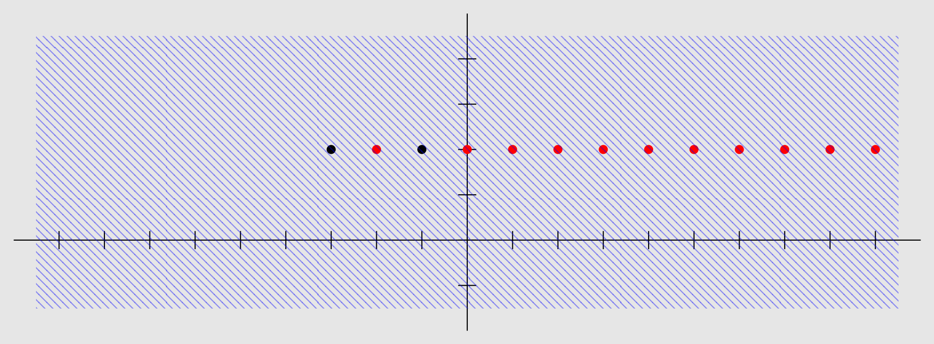
Theorem

Let $\zeta \in \mathbb{C}$ be such that there is a shift-minimal, irreducible q with $q(\zeta) = 0$. Then ζ is a removable singularity of $[A]$ iff $[A]$ is desingularizable at q.

Removable or Apparent?



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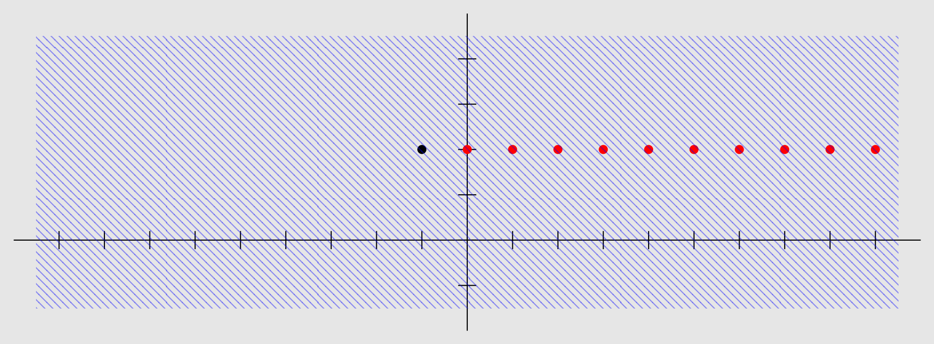

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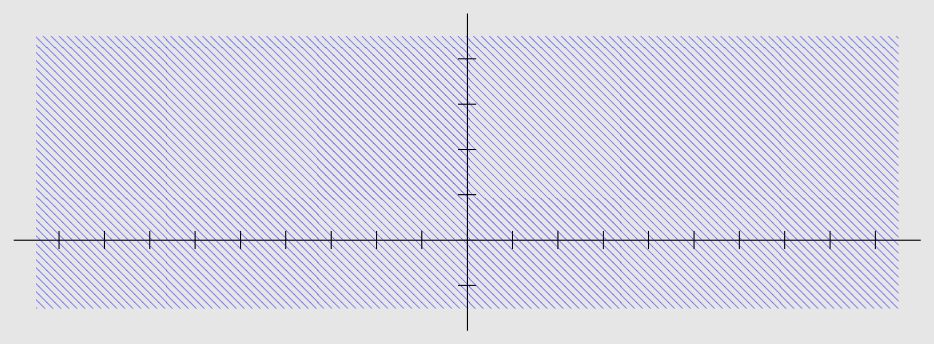

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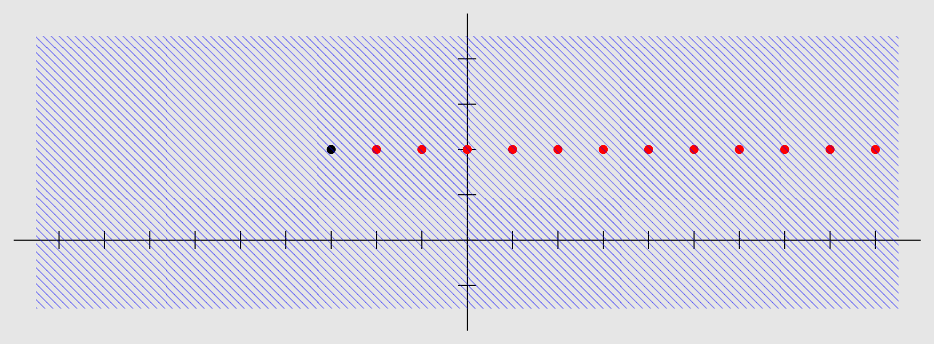

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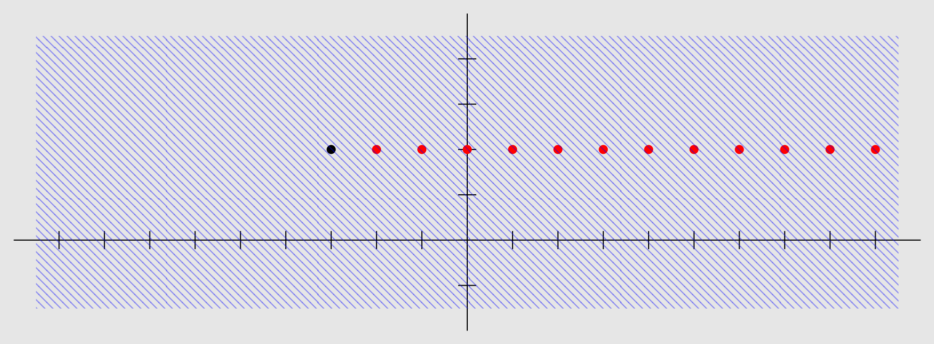

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We assume $\mathrm{den}(A) = q$ with q irred.

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Goal

Construct T step by step as a composition of easy to understand transformations.

Column Reduced Form

$$A = q^{-1}(A_0 + qA_1 + q^2A_2 + \dots)$$

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Lemma

There exists a unimodular polynomial transformation S such that $S[A]$ is of the form

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with polynomial matrices B_1 and B_2 and the number of columns of B_1 is equal to the rank of A_0.

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$$\begin{pmatrix} 0 & 1 \\ \frac{-2(z+1)}{z-2} & \frac{3(z-1)}{z-2} \end{pmatrix} \xrightarrow{S = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} \frac{z+1}{z-2} & 0 \\ \frac{-2(z+1)}{z-2} & 2 \end{pmatrix}$$

Dispersion

Lemma

Let q be removable from A. Then there exists a positive integer ℓ such that

$$q(z+\ell) \mid \operatorname{num}(\det(A)).$$

Definition

We call the largest such ℓ the dispersion of A (at q).

$$\begin{pmatrix} \frac{z+2}{z} & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \frac{z+1}{z-2} & 0 \\ \frac{-2(z+1)}{z-2} & 2 \end{pmatrix}$$

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2 – 3

Lowering The Dispersion

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$$\Big\downarrow \; T_2(z)T_3(z) = \begin{pmatrix} z-1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

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Shearing Transformation

Lemma

Let A be desingularizable at q with dispersion ℓ and in column reduced form

$$\Big(\underbrace{\frac{1}{q}B_1}_{r \text{ columns}} \quad B_2 \Big),$$

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$$D = \operatorname{diag}(\underbrace{q, \ldots, q}_{r \text{ times}}, 1, \ldots, 1)$$

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Then $D[A]$ is either desingularized or desingularizable with dispersion $< \ell$.

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$$D = \operatorname{diag}(\underbrace{q, \ldots, q}_{r \text{ times}}, 1, \ldots, 1)$$

Then $D[A]$ is either desingularized or desingularizable with dispersion $< \ell$. Furthermore, any desingularizing transformation T for A can be written as $T = D\tilde{T}$, where $\tilde{T}$ is invertible and has polynomial entries.

The Algorithm

desingularize(A,q)

1. Bring A into column reduced form.
2. Reduce dispersion with appropriate shearing transformation.
3. Repeat until A is desingularized or the dispersion is 0.

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Theorem

The algorithm gives the smallest possible desingularizing transformation T. Any other desingularizing transformation $\tilde{T}$ is of the form

$$\tilde{T} = T \cdot \bar{T},$$

with invertible $\bar{T}$ having polynomial entries.

But There is More

- Works for general automorphisms (with computable and finite dispersion).
- An alternative desingularization algorithm that shifts the roots instead of the poles.
- Rank reduction if desingularization is not possible.
- A sufficient and necessary condition for desingularizability.

Theorem

Let $q \in \mathbb{K}[z]$ be a shift-minimal pole of $[A]$. Let $\tilde{A} = q^n A$, so that the order of q in $\tilde{A}$ is zero. Then A is (partially) desingularizable at q iff there exists a positive integer k such that the entries of

$$\tilde{A}(z)\tilde{A}(z-1)\dots\tilde{A}(z-k).$$

are divisible by q.

Previous and Related Work

Desingularization of Scalar Equations

- ▷ Abramov, van Hoeij 1999
- ▷ Tsai 2000
- ▷ Abramov, B., van Hoeij 2006
- ▷ Chen, J., Kauers, Singer 2013
- ▷ Chen, Kauers, Singer 2015
- ▷ Zhang 2016
- ▷ Chen, Kauers, Li, Zhang 2017
- ▷ Koutschan, Zhang 2018

Desingularization of Systems

- ▷ B. 2010
- ▷ B., Maddah 2015

Scalar Equations vs. Systems