A New Approach for Formal Reduction of Singular Linear Differential Systems Using Eigenrings

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Linear differential system

Let A be a $n \times n$ matrix with coefficients over $\mathcal{C}((x))$

For clarity, $\mathcal{C}$ algebraically closed.

$$[A] : Y' = A(x)Y,$$

where $A(x) = x^{-q-1} \sum_{i=0}^{\infty} x^i A_i = \frac{1}{x^{q+1}}(A_0 + A_1 x + \ldots)$.

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Hukahara[1937]-Turrittin[1955]-Levelt[1975] : A Formal Fundamental Matrix Solution (FFMS) can be written as

$$Y(x) = \phi(x^{1/s}) x^{\Lambda} \exp(Q(x^{-1/s}))$$

- s is the global ramification.
- Q is the *exponential part* of $[A]$.

One strategy to compute an FFMS is to

- First compute the exponential part Q.
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One strategy to compute Q is to

- Compute an "equivalent system" with simple structure : *Formal Reduction*.

Formal Reduction

Gauge Transformation : Change of variable $Y = PZ$, where $P \in GL_n(C((x)))$, leads to a system

$$[B] : Z' = B(x)Z, \qquad B = P[A] := P^{-1}AP - P^{-1}P'$$

Systems $[A]$ and $[B]$ are called equivalent. ($A \underset{C((x))}{\sim} B$)

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- **Maximal decomposition :**

$$[A] \underset{C((x))}{\sim} \left[\begin{pmatrix} B_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & B_\ell \end{pmatrix}\right] = [B_1] \oplus \cdots \oplus [B_\ell]$$

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- More refined decomposition requires field extensions i.e $P \in GL_n(\mathcal{C}((x^{1/s})))$.

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Barkatou[1997]-Pflugel[2000]

Let $A = \frac{1}{x^{q+1}}(A_0 + A_1 x + \ldots)$

1. A_0 has distinct eigenvalues : apply splitting lemma.

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3. Arnold-Wasow Forms - Shearing transformation.

Computing the exponential part Q [Barkatou1997]

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3. Apply Moser rank reduction and iterate.
4. A_0 nilpotent + q is minimal : need to introduce ramification
   - Katz invariant algorithm.
5. Iterate on sub-blocks.

The algorithm finishes when $n = 1$ or $q = 0$.

Example [Barkatou2010]

$$A = \frac{1}{x^4}\begin{pmatrix} 0 & 0 & x & 0 \\ 1 & -x^2 & x^2 & -x^2 \\ 0 & 1 & x^2 & 0 \\ x^2 & x^2 & 0 & -x^2 \end{pmatrix}$$

- Moser irreducible and A_0 is nilpotent.
- The Katz invariant $\kappa = 8/3$ [Barkatou97].
- Ramification $x = t^3$.
- Apply Moser-algorithm to $3t^2A(t^3)$.
- We get an equivalent system, where A_0 has 4 distinct eigenvalues.

The exponential parts can be parameterized as

$$x = t^3, \quad q_1(\frac{1}{t}) = -\frac{3}{8t^8} - \frac{1}{4t^4}, \; q_2(\frac{1}{t}) = \frac{1}{t^3}.$$

- 3 conjugates exponential parts of ramification 3.
- 1 exponential part with non ramification.

Pflugel[2000] : integer degree of the exponential parts can be detected. (using super-irreducible systems Hilali-Wazner[1987], Barkatou-Pflugel[2009])

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Drawback : May introduce unnecessary ramifications.

The new approach

Step 1 Separate different ramifications by computing a maximal decomposition of $[A]$ in $\mathcal{C}((x))$

$$P[A] = [A_1(x)] \oplus \ldots \oplus [A_\ell(x)].$$

Step 2 For each indecomposable $[A_i]$ get the ramification r_i ,i.e, the minimal integer to write the solution.

Step 3 Factorization : to get the smallest system needed (in case of repetition).

Step 4 Recursive call of splitting lemma and Moser-reduction on the ramified system.

Balser-Jurkat-Lutz 1979

There exists $P \in \mathcal{C}((x))^{n\times n}$ such that,

$$P[A] = [A_1] \oplus \ldots \oplus [A_\ell].$$

Each $[A_i]$ has the following properties :

- has one exponential part q_i and its conjugates ($t \to \omega^i t$, $i = 1..r_i - 1$), $t = x^{1/r_i}$.
- $size(A_i) = m_i \times r_i$.

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We want to compute this decomposition !

Back to the example

Construct $P = \begin{bmatrix} 0 & 0 & 1 & x^2 \\ 1 & 0 & 0 & O \\ 0 & 1 & 0 & O \\ x^2 & x^2 & O & 1 \end{bmatrix} + O\left(x^3\right)$, without ramification,

such that

$$P[A] = A_1 \oplus A_2$$

where

$$A_1 := \begin{bmatrix} -x^{-2} - 1 & x^{-2} - 1 & x^{-4} + 3\,x + 2\,x^2 \\ x^{-4} & x^{-2} & 0 \\ 0 & x^{-3} & 0 \end{bmatrix} + O\left(x^3\right)$$

and

$A_2 := \left[\ -x^{-2} + 1 + O\left(x^3\right)\ \right]$, *simple exponential part* $q_2(\frac{1}{t}) = \frac{1}{t}$.

Eigenring

Decomposition/Factorization :
Singer[1996], Van Hoij[1996], Barkatou[1995], Barkatou-Pflugel[1998], Barkatou[2007]

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- **Definition :** The *eigenring* is the set defined by

$$\mathcal{E}_{\mathcal{C}((x))}(A) = \{T \in \mathcal{M}_n(\mathcal{C}((x)))/\,T' = AT - TA\}.$$

- **Computation :** Amounts to find solution in $\mathcal{C}((x))$ of $[A \otimes I_n - I_n \otimes A^T]$.

$\longrightarrow$ Abramov[1999], Barkatou-Pflugel[1999].

First Step - The theorem of decomposition using eigenring Barkatou[2007]

For $T \in \mathcal{E}_{\mathcal{C}((x))}(A)$ such that $\# \operatorname{spec}(T) \geq 2$, there exists $P \in GL_n(\mathcal{C}[[x]])$:

$$P^{-1}TP = \left(\begin{array}{c|c|c} J_1 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & J_\ell \end{array}\right) \text{ then, } P[A] = \left(\begin{array}{c|c|c} B_1 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & B_\ell \end{array}\right)$$

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Maximal decomposition : using generic element (maximal number of distinct eigenvalues).

$$P[A] = [A_1(x)] \oplus ... \oplus [A_\ell(x)]$$

Step 2 : Get the ramification

For each Indecomposable sub-system $[A_i]$ we have the following properties :

1. $[A_i]$ has one "type" of exponential parts $q_i(1/t)$, $q_i(1/\omega_i t)$,..., $q_i(1/\omega_i^{r_i-1} t)$ with $\omega_i^{r_i} = 1$ and $t = x^{1/r_i}$.

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2. $$[A_i] \sim \begin{pmatrix} B_i & \frac{1}{x} I_{r_i} & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \frac{1}{x} I_{r_i} \\ 0 & \cdots & 0 & B_i \end{pmatrix}$$

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   - $dim(\mathcal{E}(A_i)) = m_i$.
   - $$r_i = \frac{size(A_i)}{\dim(\mathcal{E}(A_i))}$$

.

Step 3 & 4

$\hookrightarrow$ We can construct this factorization in $\mathcal{C}((x))$:

$$\left(\begin{array}{c|c|c|c} B_i & D_1 & \cdots & D_{m_i-1} \\ \hline & \ddots & \ddots & \vdots \\ \hline & & \ddots & D_1 \\ \hline & & & B_i \end{array}\right) \quad \textit{with } \; size(B_i) = r_i.$$

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How to compute Q ?

- Perform the ramification in $[B_i]$: $t = x^{1/r_i}$.
- No further ramification is needed.
- Proceed as in [Barkatou1997] using only recursive call of
  - Moser-reduction,
  - splitting lemma,
  - shifting the eigenvalues of B_0.

How to deal with series and precision ?

Consider the system

$$Y' = AY$$

where A is an n-dimensional matrix in $\mathcal{C}((x))$ with Poincaré rank q.

Problem 1 : How many initial terms are sufficient to consider in A in order to compute the exponential parts ?

Answer : nq+1. (Babbitt-Varadarajan[1983], Lutz-Schafke[1985])

Given $T \in \mathcal{E}(A)$, we know that its characteristic polynomial χ_T has coefficients in $\mathcal{C}$.
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Answer : $\mathrm{val}(T) \times n + 1$.

Problem 3 Now we need to compute a basis of $\ker(T - \lambda_i I_n)^{h_i}$ where $(\lambda - \lambda_i)^{h_i}$ is a factor of χ_T.

Problem 3 : Given $T \in \mathrm{Mat}_n(C((x)))$ we want to compute the first k terms of a basis $\mathcal{B}$ of the $\ker(T)$: $v \in \mathcal{B}$ such that $Tv = O(x^{\mathrm{val}(T)+k+1})$.

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How many terms to consider in T ?

Answer : The first k terms.

Key of the proof : Use Approximate Smith Normal Form.

$\longrightarrow$ Vaccon (2015)

Problem 4 : Computing $P[A] := P^{-1}AP - P^{-1}P'$ for sufficient precision.

From **Problem 1** we should compute $P[A]$ up to precision $val(P[A]) \times n + 1$. It is bounded by

$$N := (-\mathrm{val}(P^{-1}) - \mathrm{val}(A)) \times n + 1$$

How many initial terms do we need to consider in :

1. A, P, P^{-1} to compute $P[A]$ up to order N ? $\surd$
2. P to compute $val(P^{-1})$ and P^{-1} up to an order k ?

Problem 4.2 We can compute P for any precision k with $\mathrm{val}(P^{<k>}) = 0$. We will denoted by $P^{<k>}$.

Question : How to choose k such that $\mathrm{val}(P^{-1}) = \mathrm{val}({P^{<k>}}^{-1})$?

Answer : By Abramov-Barkatou[2018], one should choose k such that

- $\det(P^{<k>}) \neq 0$
- $k + \mathrm{val}({P^{<k>}}^{-1}) \geq 0$

Then

- $\mathrm{val}(P^{-1}) = \mathrm{val}({P^{<k>}}^{-1})$.
- The Laurent series expansions of P^{-1} and ${P^{<k>}}^{-1}$ coincide up to order $k + 2\,\mathrm{val}({P^{<k>}}^{-1})$.

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- Manage truncated series and precisions.

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4	1.263	0.246
6	4.041	0.256
9	32.097	0.827

Table – Time records (s) on some examples

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