A symplectic Kovacic's algorithm in dimension 4

Thierry COMBOT

University of Burgundy, Dijon

Joint work with Camilo SANABRIA,
Universidad de los Andes, Bogota

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A matrix $M \in GL_{2n}(\mathbb{K})$ is symplectic $\Leftrightarrow M^t JM = J$ where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

A matrix $M \in GL_{2n}(\mathbb{K})$ is projectively symplectic $\Leftrightarrow M^t JM = \lambda J$ for some $\lambda \in \mathbb{K}^*$.

Set of symplectic/projective symplectic matrices: $SP_{2n}(\mathbb{K}),\ PSP_{2n}(\mathbb{K})$

The Lie algebras:

$$\mathfrak{sp}_{2n}(\mathbb{K}) = \{M \in M_{2n}(\mathbb{K}),\ M^t J + JM = 0\},$$

$$\mathfrak{psp}_{2n}(\mathbb{K}) = \{M \in M_{2n}(\mathbb{K}),\ \exists \lambda \in \mathbb{K},\ M^t J + JM = \lambda J\}.$$

Galois group of a linear differential operator $L \in \mathbb{K}(z)[\partial]$:
group of automorphisms of the field generated by the solutions of L fixing $\mathbb{K}(z)$

An operator L of order $2n$ is (projectively) symplectic

$$\Leftrightarrow$$

$\mathrm{Gal}(L)$ isomorphic to a subgroup of $SP_{2n}(\mathbb{K})$ (resp. $PSP_{2n}(\mathbb{K})$).

A more workable definition:

Proposition

An operator L of order $2n$ is (projectively) symplectic, if and only if there exists an invertible matrix $P \in M_{2n}(\mathbb{K}(z))$ such that

$$P^{-1}AP + P'P \in \mathfrak{sp}_{2n}(\mathbb{K}(z)), \quad \textit{resp.} \quad P^{-1}AP + P'P \in \mathfrak{psp}_{2n}(\mathbb{K}(z))$$

with A is the companion matrix of L.

Proposition

The operator L projectively symplectic $\Leftrightarrow$ $\exists$ *an invertible antisymmetric matrix* $W \in M_{2n}(\mathbb{K}(z))$ *such that*

$$A^t W + WA + W' + \lambda W = 0$$

for a $\lambda \in \mathbb{K}(z)$, *and L is symplectic for* $\lambda = 0$.

The gauge transformation matrix can be obtained by $W = P^t JP$.

IsSymplectic

Input: A linear differential operator L of order $2n$ with coefficients in $\mathbb{K}(z)$.

Output: A projective symplectic structure if it exists.

1. Write down the system $A^t W + WA + W' = 0$.
2. Compute a basis $B = \{W_1, \dots, W_m\}$ of the hyperexponential solutions.
3. For each exponential type of a solution in B, look for linear combinations over $\mathbb{K}$ of the W_i's with same exponential type such that $\det(a_1 W_{i_1} + \dots + a_p W_{i_p}) \neq 0$. If there are none, return []. Else return $a_1 W_{i_1} + \dots + a_p W_{i_p}$.

Definition

A Liouvillian solution of L is a solution of L built by successive integrations, exponentiations and algebraic extensions of $\mathbb{K}(z)$.

The purpose of original Kovacic algorithm is to compute Liouvillian solutions of an operator $L \in \mathbb{K}(z)[\partial]$ of order 2.

We want here to generalize it to operators of order 4, but using the additional constraint that L should be symplectic.

The vector space $\mathcal{L}$ of Liouvillian solutions is a subspace of $\mathbb{C}^4$.

The differential Galois group of L stabilize $\mathcal{L}$, and its reduction to $\mathcal{L}$ is a virtually solvable group.

Two cases appears:

- There exists a sub vector space stable by the Galois group: this can be tested by trying to factorize L.
- There is none except the trivial ones, and L is irreducible.

Theorem

A proper algebraic subgroup of $SP_4(\mathbb{C})$ is up to conjugacy generated by elements of the form

$$\begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & \star & \star \end{pmatrix} \text{ or } \begin{pmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & 0 & \star \end{pmatrix} \text{ or }$$

$$\begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ 0 & 0 & \star & \star \\ 0 & 0 & \star & \star \end{pmatrix}, \begin{pmatrix} 0 & 0 & \star & \star \\ 0 & 0 & \star & \star \\ \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \end{pmatrix}.$$

The important point of the symplectic condition: the complicated finite groups of $SL_4(\mathbb{C})$ do not appear!

If L is reducible, then it admits a factorization in two operators of order 2 $\Rightarrow$ apply Kovacic algorithm on each factor

If L is irreducible, then it admits a LCLM factorization in a quadratic extension of $\mathbb{K}(z)$.

Proposition

The kernel of a Poisson structure W is an invariant vector space.

Proposition

Let L be an irreducible operator with symplectic structure W_1.

- *All projective Poisson structures are symplectic and their Pfaffian $\in \mathbb{K}(z)$.*
- *If $Gal(L) = \mathbb{Z}_2 \ltimes G_1,\ G_1 \subset SL_2(\mathbb{K})$, then L admits two projective symplectic structures in a quadratic extension of $\mathbb{K}(z)$.*
- *If L admits a projective symplectic structure $W_2 \neq W_1$, then $\exists \lambda \in \mathbb{C}$ such that $W_1 + \lambda W_2$ is a strict Poisson structure in a quadratic extension of $\mathbb{K}(z)$.*

Example:

$$L = Dz^4 + 2\,\frac{(z-1)\,Dz^3}{z\,(z-2)} - \frac{\left(16\,z^5 - 80\,z^4 + 128\,z^3 - 63\,z^2 - 2\,z + 4\right)Dz^2}{4z^2\,(z-2)^2} -$$

$$\frac{\left(32\,z^4 - 128\,z^3 + 144\,z^2 - 33\,z + 1\right)Dz}{4z^2\,(z-2)^2} + \frac{(z-1)\left(4\,z^5 - 20\,z^4 + 32\,z^3 - 21\,z^2 + 10\,z + 2\right)}{z^2\,(z-2)^2}$$

admits 3 projective symplectic structures W_1, W_2, W_3

$$\begin{pmatrix} 0 & \frac{4\,z^5-16\,z^4+20\,z^3-10\,z^2+5\,z-1}{\sqrt{z}} & \sqrt{z}\,(z-2)\,(3\,z-1) & z^{3/2}\,(z-2)\,(2-2z) \\ -\frac{4\,z^5-16\,z^4+20\,z^3-10\,z^2+5\,z-1}{\sqrt{z}} & 0 & \frac{(z-2)\left(8\,z^3-8\,z^2-1\right)}{4\sqrt{z}} & -1/2\,\sqrt{z}\,(z-2) \\ -\sqrt{z}\,(z-2)\,(3\,z-1) & -1/4\,\frac{(z-2)\left(8\,z^3-8\,z^2-1\right)}{\sqrt{z}} & 0 & z^{3/2}\,(z-2) \\ -z^{3/2}\,(z-2)\,(-2\,z+2) & 1/2\,\sqrt{z}\,(z-2) & -z^{3/2}\,(z-2) & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{4z^5-24z^4+52z^3-50z^2+19z+1}{\sqrt{z-2}} & z\sqrt{z-2}\,(3z-5) & z\,(z-2)^{\frac{3}{2}}\,(2-2z) \\ -\frac{4z^5-24z^4+52z^3-50z^2+19z+1}{\sqrt{z-2}} & 0 & \frac{z\left(8z^3-40z^2+64z-33\right)}{4\sqrt{z-2}} & -1/2\,z\sqrt{z-2} \\ -z\sqrt{z-2}\,(3z-5) & -1/4\,\frac{z\left(8z^3-40z^2+64z-33\right)}{\sqrt{z-2}} & 0 & z\,(z-2)^{3/2} \\ -z\,(z-2)^{3/2}\,(-2z+2) & 1/2\,z\sqrt{z-2} & -z\,(z-2)^{3/2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\frac{\sqrt{z(z-2)}\left(16z^3-48z^2+32z+1\right)}{4z(z-2)} & \frac{\sqrt{z(z-2)}(z-1)}{z(z-2)} & \sqrt{z\,(z-2)} \\ 1/4\,\frac{\sqrt{z(z-2)}\left(16z^3-48z^2+32z+1\right)}{z(z-2)} & 0 & -\sqrt{z\,(z-2)} & 0 \\ -\frac{\sqrt{z(z-2)}(z-1)}{z(z-2)} & \sqrt{z\,(z-2)} & 0 & 0 \\ -\sqrt{z\,(z-2)} & 0 & 0 & 0 \end{pmatrix}$$

$$\det(\lambda_1 W_1 + \lambda_2 W_2 + \lambda_3 W_3) = z^2(z-2)^2(\lambda_1^2 + \lambda_2^2 - \lambda_3^2)^2$$

$\Rightarrow$ L admits several LCLM factorizations in quadratic extensions.

$$L = LCLM\left(Dz^2 - \frac{Dz}{2(z-2)} - 2z - \frac{1}{\sqrt{z}(z-2)} + 2,\right.$$
$$\left. Dz^2 - \frac{Dz}{2(z-2)} - 2z + \frac{1}{\sqrt{z}(z-2)} + 2\right)$$

The Kovacic algorithm can be applied on the order 2 factors with base field $\mathbb{K}(z, \sqrt{z})$.

SymplecticKovacic

Input: An order 4 symplectic differential operator $L \in \mathbb{K}(z)[\partial]$.
Output: A basis of the vector space of Liouvillian solutions of L.

1. Factorize L. If factors are order 1, return solutions of L.
2. A single order 2 factor $\tilde{L}$, apply Kovacic algorithm to $\tilde{L}$.
   1. If $\tilde{L}$ is solvable, then return solutions of L.
   2. Else, compute hyperexponential solutions of L.
   3. If one, then $L = ML_0$, L_0 order 1. Compute hyperexponential solutions of M.
   4. If one, then $M = NM_0$, M_0 order 1. Return solutions of $M_0 L_0$.
   5. Else return a solution of L_0. Else return [].
3. Two order 2 factors L_1, L_2, apply Kovacic's algorithm to them
   1. If L_1 is not solvable, return Liouvillian solutions of L_2.
   2. If L_1, L_2 are solvable return the solutions.
   3. Compute an LCLM factor of L. If two factors, apply Kovacic algorithm and return Liouvillian solutions. Else return [].

1. Compute projective Poisson structures. Less than 2, return [].
2. Else denote W_1, W_2 symplectic structures with W_1 in $\mathbb{K}(z)$, and W_2 in $\mathbb{K}(z, \sqrt{w(z)})$.
3. Solve $\det(W_1 + \lambda W_2) = 0$, compute the conjugate kernels V_1, V_2.
4. Compute differential system associated to L restricted to V_1. Apply cyclic vector to obtain $\tilde{L} \in \mathbb{K}(z, \sqrt{w(z)})[\partial]$.
5. If $Sym^2(\tilde{L})$ has solutions in $\mathbb{K}(z, \sqrt{w(z)})$, return Liouvillian solutions of the form
$$e^{\int \sqrt{\alpha(z)+\sqrt{w(z)}\beta(z)dz}}.$$
6. If for $i \in \{6, 8, 12\}$, $Sym^i(\tilde{L})$ has solutions in $\mathbb{K}(z, \sqrt{w(z)})$, return solutions of the form
$$e^{\int \alpha(z)+\sqrt{w(z)}\beta(z)dz} F(p(z) + \sqrt{w(z)}r(z))$$
where F is a solution of a standard equation. Else return [].

A D_8 example:

$$Dz^4+\frac{\left(15\,z^2+32\,z-15\right)\,Dz^3}{(3\,z+5)\,(z-1)\,z}+\frac{\left(4860\,z^4+19341\,z^3+17209\,z^2-27465\,z+3975\right)\,Dz^2}{128z^2\,(3\,z+5)^2\,(z-1)^2}+$$

$$\frac{9\,\left(20\,z^2+153\,z+35\right)\,Dz}{256z^3\,(3\,z+5)\,(z-1)}-\frac{9}{65536}\,\frac{240\,z^4+50169\,z^3-153939\,z^2-84805\,z-44625}{z^4\,(3\,z+5)^2\,(z-1)^2}$$

LCLM with its conjugate of

$$Dz^2+\frac{3(20z+37\sqrt{z}+21)}{256z^2(\sqrt{z}+1)^2}.$$

Solutions:

$$\sqrt{z}(1+\sqrt{z})^{\frac{1}{4}}e^{\frac{1}{16}\int\frac{1}{z\sqrt{1+\sqrt{z}}}dz},\sqrt{z}(1+\sqrt{z})^{\frac{1}{4}}e^{-\frac{1}{16}\int\frac{1}{z\sqrt{1+\sqrt{z}}}dz},$$

$$\sqrt{z}(1-\sqrt{z})^{\frac{1}{4}}e^{\frac{1}{16}\int\frac{1}{z\sqrt{1-\sqrt{z}}}dz},\sqrt{z}(1-\sqrt{z})^{\frac{1}{4}}e^{-\frac{1}{16}\int\frac{1}{z\sqrt{1-\sqrt{z}}}dz}$$

An A_5 example.LCLM with its conjugate of

$$Dz^2 + \frac{1}{2z}Dz + \frac{739z^{3/2} + 864z^2 + 611\sqrt{z} - 314z + 800}{14400z^2(z-1)^2}.$$

Solutions:

$$\sqrt[12]{\frac{z^2P(z)(\sqrt{z}-1)^2}{(5589\sqrt{z}-800)^3}}\mathcal{L}\left(-\frac{1}{6}, 5, \sqrt{99\frac{(27945z - 19967\sqrt{z} + 1600)^2}{(5589\sqrt{z}-800)^3(1-\sqrt{z})}}\right)$$

with $P = 251894530944z^2 - 360031369239z^{3/2} + 134021894211z - 17568425600\sqrt{z} + 765440000$, and their conjugates $\sqrt{z} \mapsto -\sqrt{z}$.

$\mathcal{L}$ is a solution of the Legendre differential equation.

An A_4 example. LCLM with its conjugate of

$$Dz^2 + \frac{108z^2 + 648z^{3/2} + 1505z + 1498\sqrt{z} + 560}{576(\sqrt{z}+1)^2 z^2 (2+\sqrt{z})^2}.$$

Solutions:

$$\frac{(189z^2 + 810z^{\frac{3}{2}} + 1118z + 526\sqrt{z} + 20)z^{\frac{5}{12}}}{(P(z)Q(z)^{14}(2+\sqrt{z})^6(\sqrt{z}+1)^6)^{\frac{1}{24}}} {}_2F_1\left(\frac{13}{24}, \frac{25}{24}, \frac{5}{4}, \frac{P(z)Q(z)^{-2}}{45(z+3\sqrt{z}+2)^2}\right)$$

with

$$P = 67191201z^6 + 863886870z^{11/2} + 4900709061z^5 + 16136882532z^{9/2} + 34114858452z^4 + 48314544768z^{7/2} + 46335734636z^3 + 29648385408z^{5/2} + 12093966336z^2 + 2856633184z^{3/2} + 318081360z + 10315200\sqrt{z} + 104000$$

$$Q = 945z^2 + 3240z^{3/2} + 3354z + 1052\sqrt{z} + 20$$