ZpL: a p-adic precision package

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The p-adic integers form a subring $\mathbb{Z}_p$ of $\mathbb{Q}_p$.

Summary on p-adics

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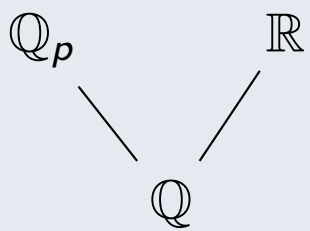
Remark



$$\begin{array}{ccc} \mathbb{Z}_p & \longrightarrow & \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \mathbb{Z}/p\mathbb{Z} & \to & \mathbb{Z}/p\mathbb{Z} \end{array}$$

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My personal (long-term) motivation

Computing (some) moduli spaces of p-adic Galois representations.

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We present the ZpL package for Sage. It features:

- Tracking of the optimal behaviour of p-adic precision.
- Portability: any existing p-adic code in Sage can use ZpL without any modification (outside of declaration of the fields).

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Finite-precision p-adics: "interval arithmetc"

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Example

The order of $\ldots 604, 3$ in $\mathbb{Q}_3$ is 3.

Precision formulae

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Proposition (division)

$$\frac{xp^a + O(p^b)}{yp^c + O(p^d)} = \frac{x}{y} p^{a-c} + O(p^{\min(d+a-2c,b-c)})$$

In particular,

$$\frac{1}{p^c y + O(p^d)} = y^{-1} p^{-c} + O(p^{d-2c})$$

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Drawbacks

- Precision: accumulation of loss in precision can happen (when used naively).
- Consequence: results can be far from optimal precision.

Optimality, intrinsicness

Step-by-step analysis is not optimal.

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The Main lemma of p-adic differential precision

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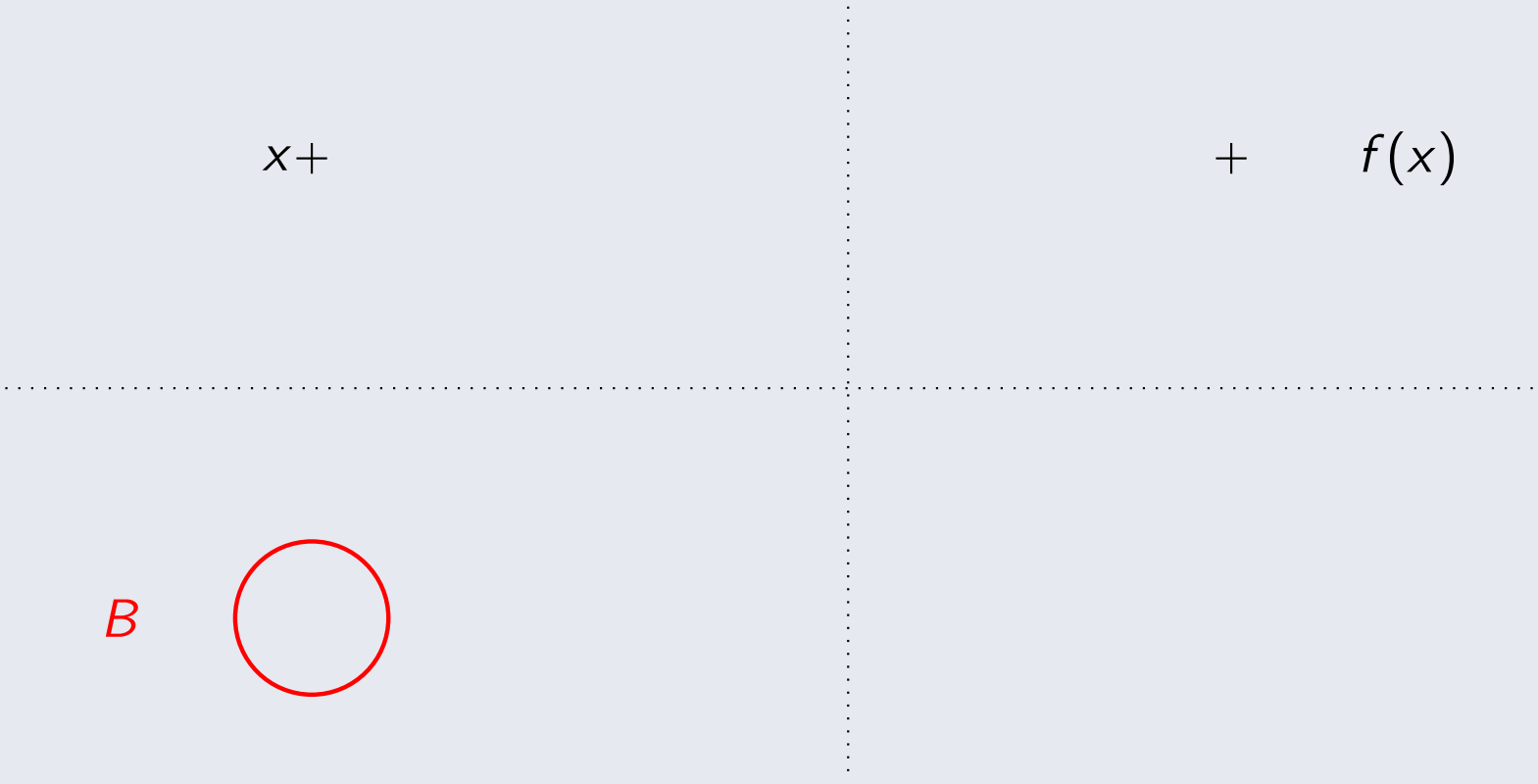
Interpretation



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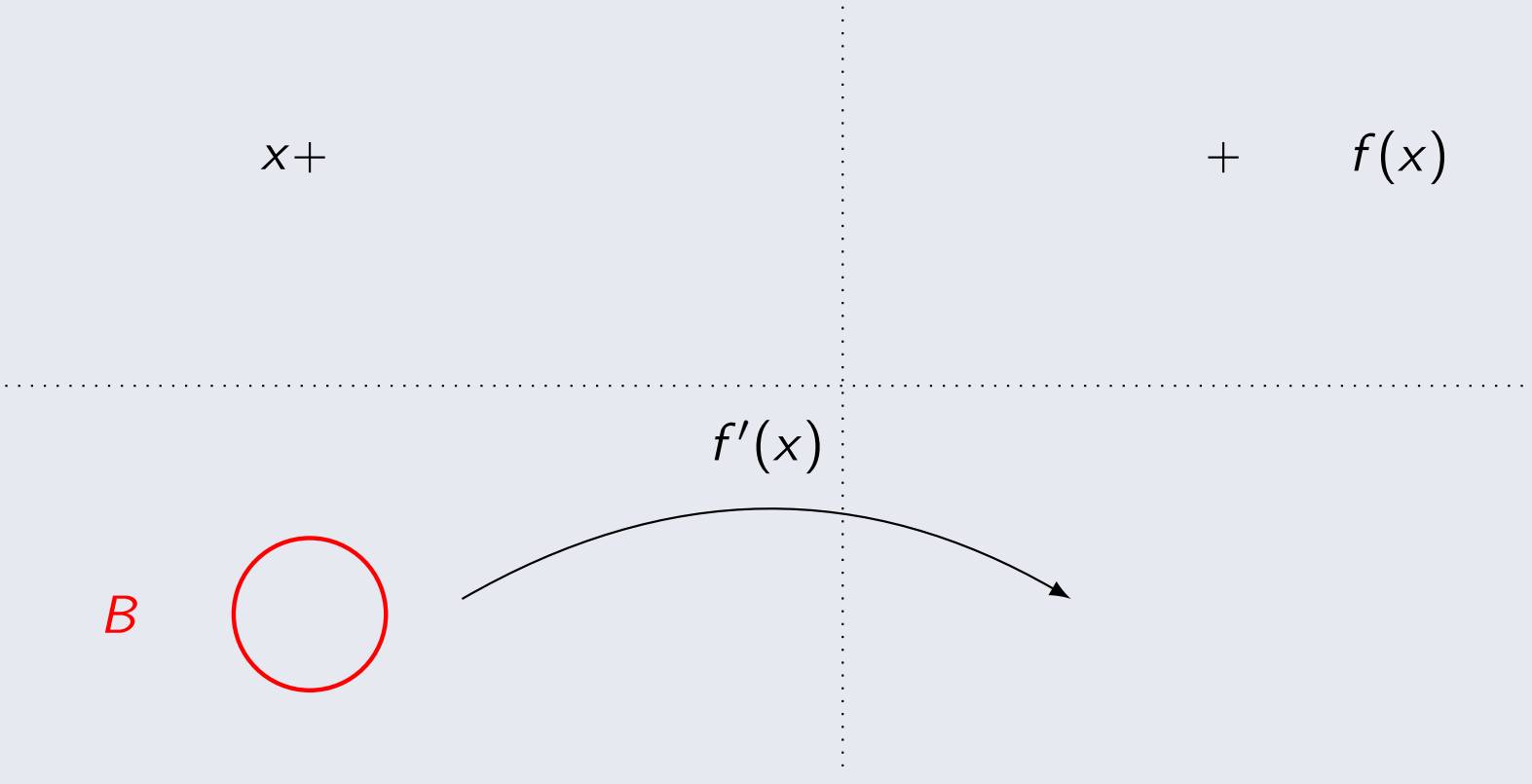

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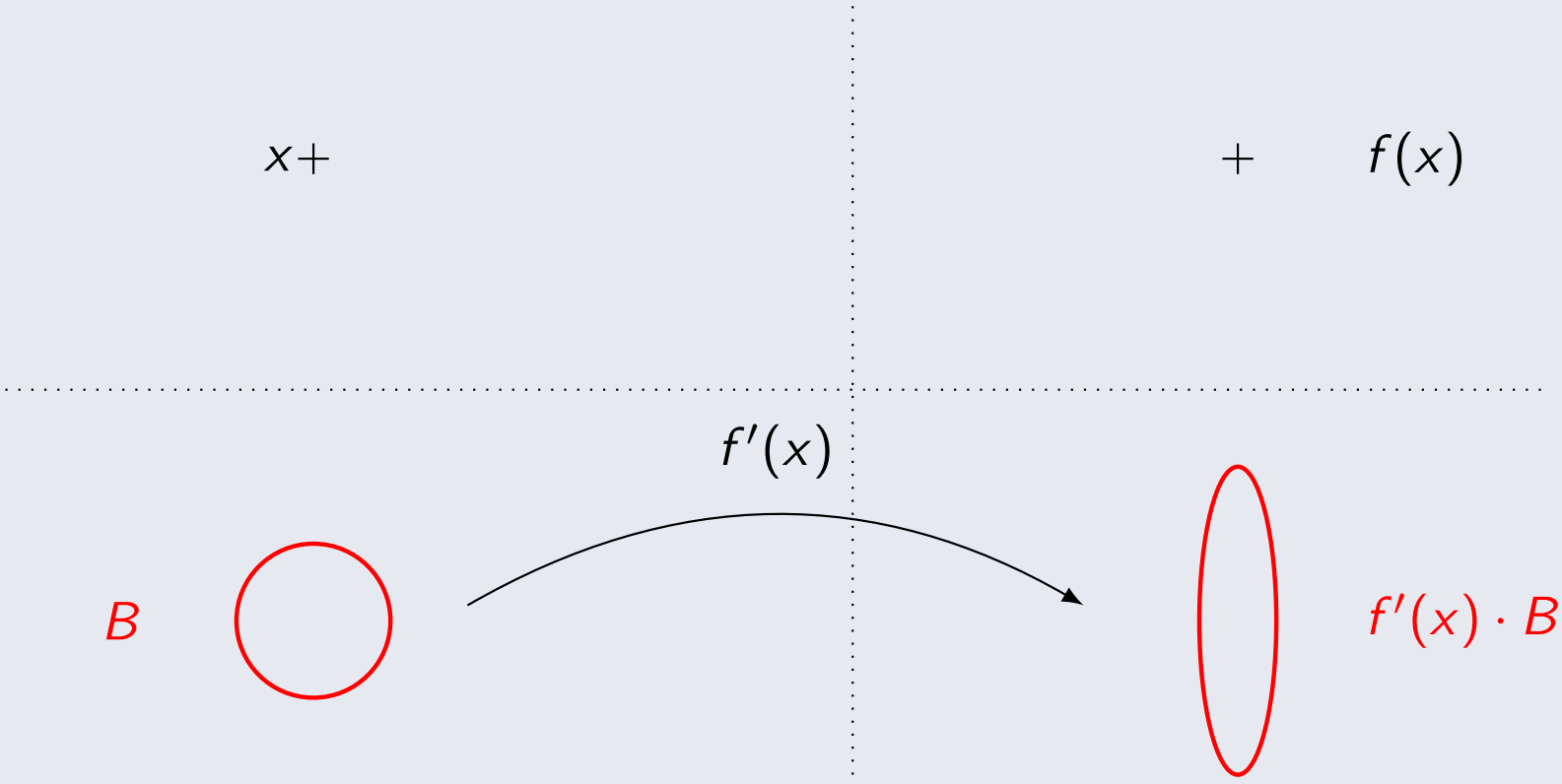

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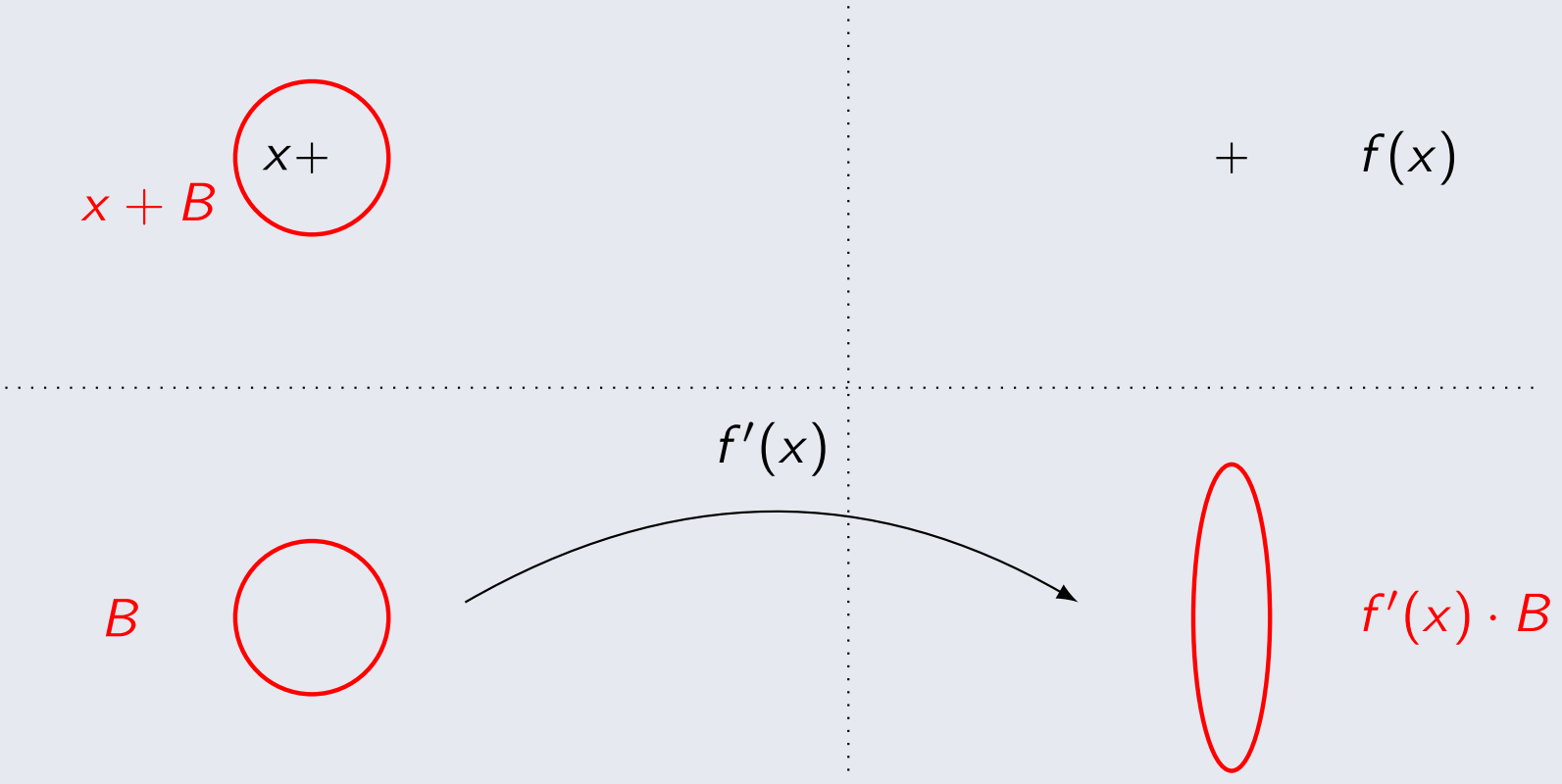

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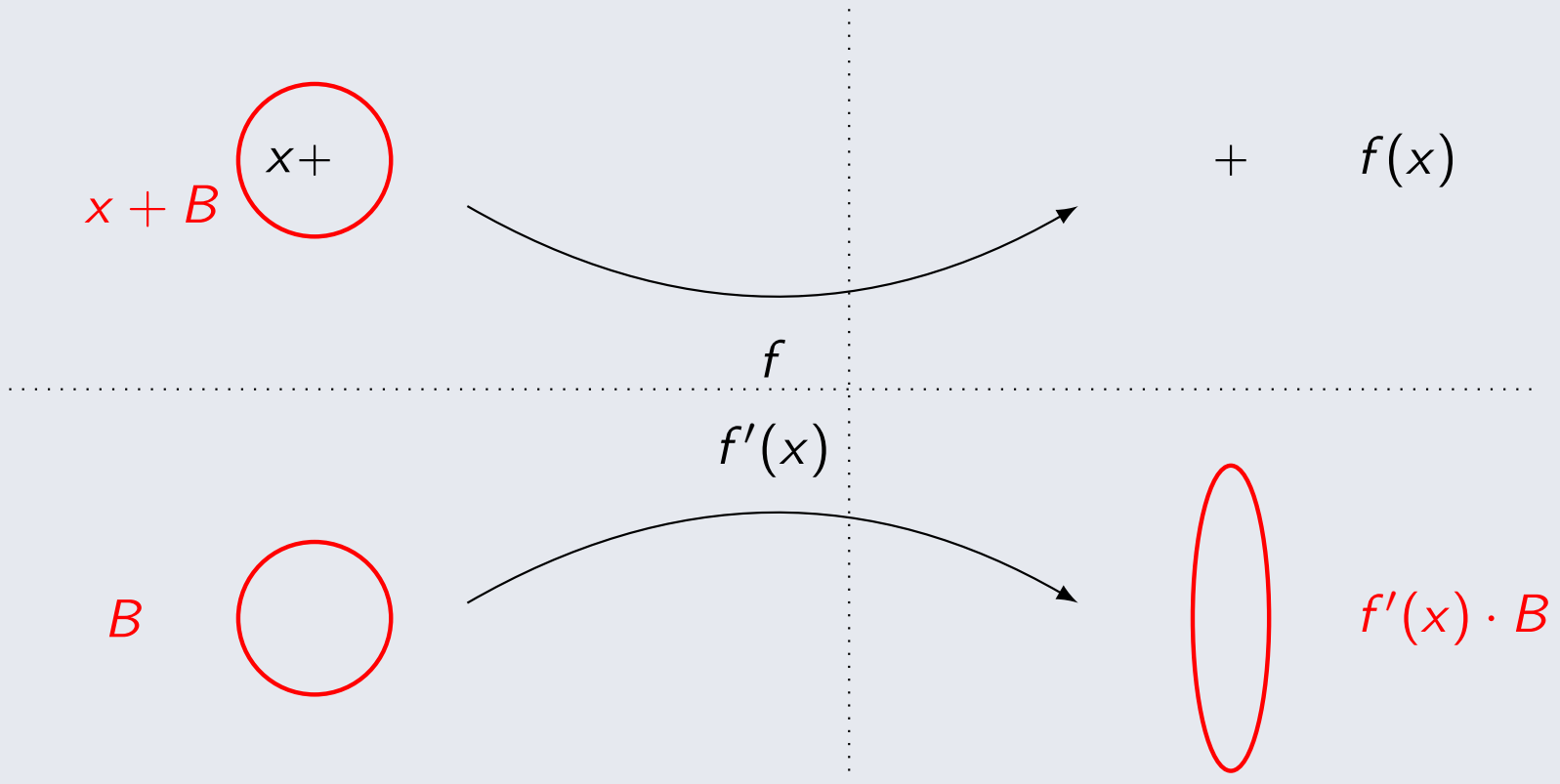

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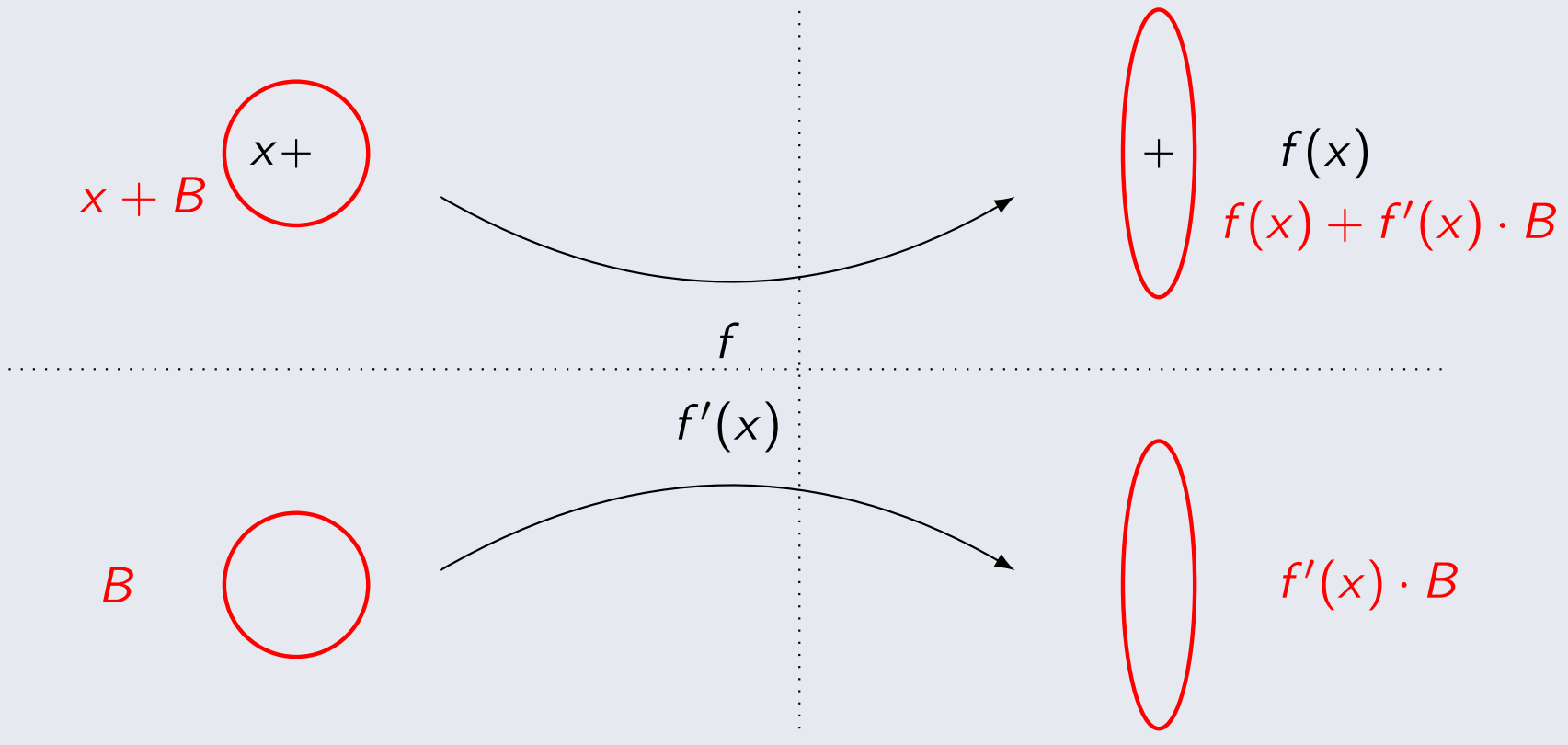

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This framework can be extended to **(complete) ultrametric K-vector spaces** (*e.g.* $\mathbb{F}_p((X))^n$, $\mathbb{Q}((X))^m$, ...).

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Small demo

Go to Sage session.

Specific motivations

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Main idea

Carry on a lattice of precision. It represents the precision on all the quantities computed.

Representing lattices

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Note that lattices are exact, since we may use row operations to reduce each column modulo the power of p on the diagonal (**HNF**).

Operations and variables

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  - In one model (`ZpLF`) we allow such submodules (at the cost of working with inexact objects).
  - In the main model (`ZpLC`) we add a new row $(0, \ldots, 0, p^C)$ for some cap C.

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The corresponding lattice (for ZpLC)

$$\begin{pmatrix} 729 & 0 & 729 & 729 & 1458 & 0 \\ 0 & 81 & 81 & 3486784320 & 0 & 162 \\ 0 & 0 & 10460353203 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3486784401 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3486784401 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3486784401 \end{pmatrix}$$

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Diffused digits

Can quantify the amount of precision lost by checking precision on individual variables. More precisely, how far interval arithmetic is from lattice precision: the **number of diffused digits**.

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To sum up

On ZpL

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- Two variants available in Sage.

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- Two variants available in Sage.
- Obtain the theoretical optimal precision, except in very small precision (where the precision is too small for the Lemma to apply).
- There is obviously an overhead cost (rough estimation available in the article).
- Well suited for exploration and understanding the behaviour of the precision.

References

Initial article

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Linear Algebra

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Thank you for your attention

Thanks