On Affine Tropical F5 Algorithms

Tristan Vaccon, Thibaut Verron, Kazuhiro Yokoyama

Université de Limoges, JKU Linz, Rikkyo University

ISSAC 2018

Motivations for p-adic and tropical GB

Motivations for p-adic and tropical GB

Motivations for finite-precision GB over p-adics

- Some varieties in arithmetic geometry are defined over p-adics.
- Better understanding of the behaviour of the computation.

Motivations in tropical geometry

- Tropical GB can be used to decide if a point belong to some given tropical algebraic geometry.
- They are a more stable variant to classical GB (often **much more stable**, see *Vaccon 2015*).

Motivations for Affine Tropical F5 algorithms

Need for affine algorithms

Motivations for Affine Tropical F5 algorithms

Need for affine algorithms

- Last year, in *Vaccon-Yokoyama 2017*, we presented a Tropical F5 algorithm for homogeneous polynomials.

Motivations for Affine Tropical F5 algorithms

Need for affine algorithms

- Last year, in *Vaccon-Yokoyama 2017*, we presented a Tropical F5 algorithm for homogeneous polynomials.
- In a tropical setting: **no de-homogenization** possible.

Table of contents

Tropical term ordering

Definition (Affine tropical term ordering)

Let $\Gamma = \mathbb{R}^n$, and let $w \in \Gamma$ **be given**.

Tropical term ordering

Definition (Affine tropical term ordering)

Let $\Gamma = \mathbb{R}^n$, and let $w \in \Gamma$ **be given**. Let $<_{mon}$ be a monomial order on $K[X_1, \dots, X_n] = K[x]$. K is a field with a valuation.

Tropical term ordering

Definition (Affine tropical term ordering)

Let $\Gamma = \mathbb{R}^n$, and let $w \in \Gamma$ **be given**. Let $<_{mon}$ be a monomial order on $K[X_1, \ldots, X_n] = K[x]$. K is a field with a valuation.
Then we define a **tropical term order** on the terms ax^α of $K[x]$ by comparing:

Tropical term ordering

Definition (Affine tropical term ordering)

Let $\Gamma = \mathbb{R}^n$, and let $w \in \Gamma$ **be given**. Let $<_{mon}$ be a monomial order on $K[X_1, \ldots, X_n] = K[x]$. K is a field with a valuation.
Then we define a **tropical term order** on the terms ax^α of $K[x]$ by comparing:

- First $|x^\alpha|$.

Tropical term ordering

Definition (Affine tropical term ordering)

Let $\Gamma = \mathbb{R}^n$, and let $w \in \Gamma$ **be given**. Let $<_{mon}$ be a monomial order on $K[X_1, \dots, X_n] = K[x]$. K is a field with a valuation.
Then we define a **tropical term order** on the terms ax^α of $K[x]$ by comparing:

- First $|x^\alpha|$.
- If equality, then $val(a) + w \cdot \alpha$ (smallest is biggest).

Tropical term ordering

Definition (Affine tropical term ordering)

Let $\Gamma = \mathbb{R}^n$, and let $w \in \Gamma$ **be given**. Let $<_{mon}$ be a monomial order on $K[X_1, \ldots, X_n] = K[x]$. K is a field with a valuation.
Then we define a **tropical term order** on the terms ax^α of $K[x]$ by comparing:

- First $|x^\alpha|$.
- If equality, then $val(a) + w \cdot \alpha$ (smallest is biggest).
- If equality, using $<_{mon}$.

Tropical term ordering

Definition (Affine tropical term ordering)

Let $\Gamma = \mathbb{R}^n$, and let $w \in \Gamma$ **be given**. Let $<_{mon}$ be a monomial order on $K[X_1, \ldots, X_n] = K[x]$. K is a field with a valuation.
Then we define a **tropical term order** on the terms ax^α of $K[x]$ by comparing:

- First $|x^\alpha|$.
- If equality, then $val(a) + w \cdot \alpha$ (smallest is biggest).
- If equality, using $<_{mon}$.

Tropical term ordering

Definition (Affine tropical term ordering)

Let $\Gamma = \mathbb{R}^n$, and let $w \in \Gamma$ **be given**. Let $<_{mon}$ be a monomial order on $K[X_1, \ldots, X_n] = K[x]$. K is a field with a valuation.
Then we define a **tropical term order** on the terms ax^α of $K[x]$ by comparing:

- First $|x^\alpha|$.
- If equality, then $val(a) + w \cdot \alpha$ (smallest is biggest).
- If equality, using $<_{mon}$.

We can define $in(f)$ for $f \in K[x_1, \ldots, x_n]$ as its initial term.

Tropical GB

Definition

For I an ideal in $K[x]$, we define $in(I)$ as the ideal generated by the $in(f)$ for $f \in I$.

Tropical GB

Definition

For I an ideal in $K[x]$, we define $in(I)$ as the ideal generated by the $in(f)$ for $f \in I$.

Definition

G is a **tropical Gröbner basis** if $in(G)$ generates $in(I)$ as a monoid.

Tropical GB

Definition

For I an ideal in $K[x]$, we define $in(I)$ as the ideal generated by the $in(f)$ for $f \in I$.

Definition

G is a **tropical Gröbner basis** if $in(G)$ generates $in(I)$ as a monoid.

Main differences with classical GB

- Reduction is best done using matrices and an adapted row-echelon form algorithm.
- Proofs are a little bit more involved as there is no easy way to use noetherianity for terms in place of monomials.

Table of contents

Table of contents

Order and signature

Definition (Signature)

We assume a monomial order $\leq_s$ on $K[x_1, \ldots, x_n]^s$ (with canonical basis $(e_1, \ldots, e_s)$) is given.

Order and signature

Definition (Signature)

We assume a monomial order $\leq_s$ on $K[x_1, \dots, x_n]^s$ (with canonical basis $(e_1, \dots, e_s)$) is given. We can then order the $x^\alpha e_i$'s and then can define LM's in $K[x_1, \dots, x_n]^s$.

Order and signature

Definition (Signature)

We assume a monomial order $\leq_s$ on $K[x_1, \dots, x_n]^s$ (with canonical basis $(e_1, \dots, e_s)$) is given. We can then order the $x^\alpha e_i$'s and then can define LM's in $K[x_1, \dots, x_n]^s$.
For $f \in I$, its **signature** is:

$$Sign(f) = min\{x^\alpha e_k \text{ s.t. } f = \sum a_i f_i \text{ and } LM(\sum a_i e_i) = x^\alpha e_k\}.$$

Filtration

Filtration by signature

The following is a **vector space filtration** of I.

Filtration

Filtration by signature

The following is a **vector space filtration** of I.

$$I = \bigcup_{\nearrow x^\alpha e_i} I_{\leq x^\alpha e_i},$$

with $I_{\leq x^\alpha e_i} = \{f \in I, \text{ s.t. } Sign(f) \leq_s x^\alpha e_i\}$.

Filtration

Filtration by signature

The following is a **vector space filtration** of I.

$$I = \bigcup_{\nearrow x^\alpha e_i} I_{\leq x^\alpha e_i},$$

with $I_{\leq x^\alpha e_i} = \{f \in I, \text{ s.t. } Sign(f) \leq_s x^\alpha e_i\}$.

Example

For grevlex and any w on $\mathbb{Q}_2[x, y, z]$, $f_1 = x$, $f_2 = y^2$:

- $xy^2 \in I_{\leq y^2 e_1}$,

Filtration

Filtration by signature

The following is a **vector space filtration** of I.

$$I = \bigcup_{\nearrow x^\alpha e_i} I_{\leq x^\alpha e_i},$$

with $I_{\leq x^\alpha e_i} = \{f \in I, \text{ s.t. } Sign(f) \leq_s x^\alpha e_i\}$.

Example

For grevlex and any w on $\mathbb{Q}_2[x, y, z]$, $f_1 = x$, $f_2 = y^2$:

- $xy^2 \in I_{\leq y^2 e_1}$,
- $y^3 \in I_{\leq y e_2}$.

The F5 elimination criterion, in the classical case

Proposition (Faugère 2002)

If $x^\alpha \in LM(I_{\leq i-1})$, then the filtration is constant at

$$I_{\leq x^\alpha e_i}.$$

The F5 elimination criterion, in the classical case

Proposition (Faugère 2002)

If $x^\alpha \in LM(I_{\leq i-1})$, then the filtration is constant at

$$I_{\leq x^\alpha e_i}.$$

Proposition (Faugère 2002)

If $(f_1, \dots, f_s)$ is a regular sequence, then

The F5 elimination criterion, in the classical case

Proposition (Faugère 2002)

If $x^\alpha \in LM(I_{\leq i-1})$, *then the filtration is constant at*

$$I_{\leq x^\alpha e_i}.$$

Proposition (Faugère 2002)

If $(f_1, \dots, f_s)$ *is a regular sequence, then the filtration is constant only at* $I_{\leq x^\alpha e_i}$.

Table of contents

$\mathfrak{S}$-GB

Definition (Arri-Perry 2011 for classical case)

$G \subset I = \langle f_1, \ldots, f_s \rangle$ is an $\mathfrak{S}$-GB of I if it is a Gröbner basis for every grade of the signature filtration of I.

$\mathfrak{S}$-GB

Definition (Arri-Perry 2011 for classical case)

$G \subset I = \langle f_1, \dots, f_s \rangle$ is an $\mathfrak{S}$-GB of I if it is a Gröbner basis for every grade of the signature filtration of I. In other words,

$\mathfrak{S}$-GB

Definition (Arri-Perry 2011 for classical case)

$G \subset I = \langle f_1, \dots, f_s \rangle$ is an $\mathfrak{S}$-GB of I if it is a Gröbner basis for every grade of the signature filtration of I. In other words, $\forall i, x^\alpha$,

$$\{LM(x^\beta g), \text{s.t. } g \in G,\ x^\beta g \in I_{\leq x^\alpha e_i}\} = LM(I_{\leq x^\alpha e_i}).$$

$\mathfrak{S}$-GB

Definition (Arri-Perry 2011 for classical case)

$G \subset I = \langle f_1, \dots, f_s \rangle$ is an $\mathfrak{S}$-GB of I if it is a Gröbner basis for every grade of the signature filtration of I. In other words, $\forall i, x^\alpha$,

$$\{LM(x^\beta g), \text{s.t. } g \in G,\ x^\beta g \in I_{\leq x^\alpha e_i}\} = LM(I_{\leq x^\alpha e_i}).$$

Remark

The F5 algorithm computes an $\mathfrak{S}$-GB.

Buchberger's criterion and algorithm

Buchberger's algorithm

We start from $G := F \subset K[X]$, S the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop h from S. $\overline{h}$ its reduction by G.
3. If $\overline{h} \neq 0$, add $\overline{h}$ to G and update S.

Buchberger's criterion and algorithm

Buchberger's algorithm

We start from $G := F \subset K[X]$, S the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop h from S. $\overline{h}$ its reduction by G.
3. If $\overline{h} \neq 0$, add $\overline{h}$ to G and update S.

Basic Tropical F5 algorithm

First idea of F5 (Faugère 2002 for classical case)

We start from $G := F = (f_1, \dots, f_s) \subset K[X]$, **homogeneous**. S is the set of the S-polynomials of F.

Basic Tropical F5 algorithm

First idea of F5 (Faugère 2002 for classical case)

We start from $G := F = (f_1, \dots, f_s) \subset K[X]$, **homogeneous**. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)

Basic Tropical F5 algorithm

First idea of F5 (Faugère 2002 for classical case)

We start from $G := F = (f_1, \ldots, f_s) \subset K[X]$, **homogeneous**. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest degree in S.

Basic Tropical F5 algorithm

First idea of F5 (Faugère 2002 for classical case)

We start from $G := F = (f_1, \ldots, f_s) \subset K[X]$, **homogeneous**. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest degree in S. Apply F5 criterion to remove the ones that are useless.

Basic Tropical F5 algorithm

First idea of F5 (Faugère 2002 for classical case)

We start from $G := F = (f_1, \dots, f_s) \subset K[X]$, **homogeneous**. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest degree in S. Apply F5 criterion to remove the ones that are useless.
3. Select **reducers (each with smallest signature possible)** in G.

Basic Tropical F5 algorithm

First idea of F5 (Faugère 2002 for classical case)

We start from $G := F = (f_1, \dots, f_s) \subset K[X]$, **homogeneous**. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest degree in S. Apply F5 criterion to remove the ones that are useless.
3. Select **reducers (each with smallest signature possible)** in G. Make a Macaulay matrix M from them.

Basic Tropical F5 algorithm

First idea of F5 (Faugère 2002 for classical case)

We start from $G := F = (f_1, \dots, f_s) \subset K[X]$, **homogeneous**. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest degree in S. Apply F5 criterion to remove the ones that are useless.
3. Select **reducers (each with smallest signature possible)** in G. Make a Macaulay matrix M from them. Rows are ordered by signature.

Basic Tropical F5 algorithm

First idea of F5 (Faugère 2002 for classical case)

We start from $G := F = (f_1, \dots, f_s) \subset K[X]$, **homogeneous**. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest degree in S. Apply F5 criterion to remove the ones that are useless.
3. Select **reducers (each with smallest signature possible)** in G. Make a Macaulay matrix M from them. Rows are ordered by signature. $\widetilde{M}$ the **tropical row-reduction** (no choice of pivot) of M.

Basic Tropical F5 algorithm

First idea of F5 (Faugère 2002 for classical case)

We start from $G := F = (f_1, \ldots, f_s) \subset K[X]$, **homogeneous**. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest degree in S. Apply F5 criterion to remove the ones that are useless.
3. Select **reducers (each with smallest signature possible)** in G. Make a Macaulay matrix M from them. Rows are ordered by signature. $\widetilde{M}$ the **tropical row-reduction** (no choice of pivot) of M.
4. Add to G the rows of $\widetilde{M}$ providing new leading monomials for a given signature.

Basic Tropical F5 algorithm

First idea of F5 (Faugère 2002 for classical case)

We start from $G := F = (f_1, \ldots, f_s) \subset K[X]$, **homogeneous**. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest degree in S. Apply F5 criterion to remove the ones that are useless.
3. Select **reducers (each with smallest signature possible)** in G. Make a Macaulay matrix M from them. Rows are ordered by signature. $\widetilde{M}$ the **tropical row-reduction** (no choice of pivot) of M.
4. Add to G the rows of $\widetilde{M}$ providing new leading monomials for a given signature. Update S.

Macaulay matrix processing in F5

The Symbolic preprocessing, in degree d; $z \in LM(f_1)$; over $\mathbb{Q}_3$

	x^3	x^2y	xy^2	y^3	x^2z	xyz	$\dots$
ye_2	1		3				$\dots$
xe_3	1			2		5	$\dots$

Macaulay matrix processing in F5

The Symbolic preprocessing, in degree d; $z \in LM(f_1)$; over $\mathbb{Q}_3$

	x^3	x^2y	xy^2	y^3	x^2z	xyz	$\dots$
x^2e_1					1	2	$\dots$
$z\ e_2$					1		$\dots$
ye_2	1		3				$\dots$
xe_3	1			2		5	$\dots$

Macaulay matrix processing in F5

The Symbolic preprocessing, in degree d; $z \in LM(f_1)$; over $\mathbb{Q}_3$

	x^3	x^2y	xy^2	y^3	x^2z	xyz	$\dots$
x^2e_1					1	2	$\dots$
ze_2					1		$\dots$
ye_2	1		3				$\dots$
xe_3	1			2		5	$\dots$

Macaulay matrix processing in F5

The Symbolic preprocessing, in degree d; $z \in LM(f_1)$; over $\mathbb{Q}_3$

	x^3	x^2y	xy^2	y^3	x^2z	xyz	...
~~x^2e_1~~					~~1~~	~~2~~	~~...~~
~~$z\,e_2$~~					~~1~~		~~...~~
ye_2	1		3				...
xe_3	1			2		5	...

Macaulay matrix processing in F5

The Symbolic preprocessing, in degree d; $z \in LM(f_1)$; over $\mathbb{Q}_3$

	x^3	x^2y	xy^2	y^3	x^2z	xyz	…
~~x^2e_1~~					1	2	…
~~$z\,e_2$~~					1		…
ye_2	1		3				…
xe_3	1			2		5	…

Macaulay matrix processing in F5

The Symbolic preprocessing, in degree d; $z \in LM(f_1)$; over $\mathbb{Q}_3$

	x^3	x^2y	xy^2	y^3	x^2z	xyz	$\dots$
ye_2	1		3				$\dots$
xe_3	1			2		5	$\dots$
ye_3			1		4	1	$\dots$

Macaulay matrix processing in F5

The Symbolic preprocessing, in degree d; $z \in LM(f_1)$; over $\mathbb{Q}_3$

	x^3	x^2y	xy^2	y^3	x^2z	xyz	$\dots$
xe_1						1	$\dots$
ye_2	1		3				$\dots$
xe_3	1			2		5	$\dots$
ye_3			1		4	1	$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	x^3	x^2y	xy^2	y^3	x^2z	xyz	$\dots$
xe_1						1	$\dots$
ye_2	1		3				$\dots$
xe_3	1			2		5	$\dots$
ye_3			1		4	1	$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	x^3	x^2y	xy^2	y^3	x^2z	xyz	...
xe_1						1	...
ye_2	1		3				...
xe_3	1			2		5	...
ye_3					4	1	...

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	x^3	x^2y	xy^2	y^3	x^2z	xyz	...
xe_1						1	...
ye_2	1		3				...
xe_3	1			2		5	...
ye_3					4	1	...

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^2y	xy^2	y^3	x^2z	x^3	$\dots$
xe_1	1						$\dots$
ye_2			3			1	$\dots$
xe_3	5			2		1	$\dots$
ye_3	1		1		4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^2y	xy^2	y^3	x^2z	x^3	$\dots$
xe_1	1						$\dots$
ye_2			3			1	$\dots$
xe_3	5			2		1	$\dots$
ye_3	1		1		4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^2y	xy^2	y^3	x^2z	x^3	$\dots$
xe_1	1						$\dots$
ye_2			3			1	$\dots$
xe_3	0			2		1	$\dots$
ye_3	1		1		4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^2y	xy^2	y^3	x^2z	x^3	$\dots$
xe_1	1						$\dots$
ye_2			3			1	$\dots$
xe_3	0			2		1	$\dots$
ye_3	0		1		4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^2y	xy^2	y^3	x^2z	x^3	$\dots$
xe_1	1						$\dots$
ye_2			3			1	$\dots$
xe_3	0			2		1	$\dots$
ye_3	0		1		4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^2y	xy^2	y^3	x^2z	x^3	$\dots$
xe_1	1						$\dots$
ye_2			3			1	$\dots$
xe_3	0			2		1	$\dots$
ye_3	0		1		4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^2y	xy^2	y^3	x^2z	x^3	...
xe_1	1						...
ye_2			3			1	...
xe_3	0			2		1	...
ye_3	0		1		4		...

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^2y	xy^2	y^3	x^2z	x^3	$\dots$
xe_1	1						$\dots$
ye_2			3			1	$\dots$
xe_3	0			2		1	$\dots$
ye_3	0		1		4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^3	xy^2	y^3	x^2z	x^2y	$\dots$
xe_1	1						$\dots$
ye_2		1	3				$\dots$
xe_3	0	1		2			$\dots$
ye_3	0		1		4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^3	xy^2	y^3	x^2z	x^2y	$\dots$
xe_1	1						$\dots$
ye_2		1	3				$\dots$
xe_3	0	0	-3	2			$\dots$
ye_3	0		1		4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^3	xy^2	y^3	x^2z	x^2y	$\dots$
xe_1	1						$\dots$
ye_2		1	3				$\dots$
xe_3	0	0	-3	2			$\dots$
ye_3	0		1		4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^3	xy^2	y^3	x^2z	x^2y	$\dots$
xe_1	1						$\dots$
ye_2		1	3				$\dots$
xe_3	0	0	-3	2			$\dots$
ye_3	0		1		4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^3	xy^2	y^3	x^2z	x^2y	$\dots$
xe_1	1						$\dots$
ye_2		1	3				$\dots$
xe_3	0	0	-3	2			$\dots$
ye_3	0		1		4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^3	xy^2	y^3	x^2z	x^2y	$\dots$
xe_1	1						$\dots$
ye_2		1		3			$\dots$
xe_3	0	0	2	-3			$\dots$
ye_3	0			1	4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^3	xy^2	y^3	x^2z	x^2y	$\dots$
xe_1	1						$\dots$
ye_2		1		3			$\dots$
xe_3	0	0	2	-3			$\dots$
ye_3	0			1	4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^3	xy^2	y^3	x^2z	x^2y	$\dots$
xe_1	1						$\dots$
ye_2		1		3			$\dots$
xe_3	0	0	2	-3			$\dots$
ye_3	0			1	4		$\dots$

Macaulay matrix processing in F5

The Tropical Reduction, in degree d; $x \in LM(f_1)$; over $\mathbb{Q}_3$

	xyz	x^3	xy^2	y^3	x^2z	x^2y	$\dots$
xe_1	1						$\dots$
ye_2		1		3			$\dots$
xe_3	0	0	2	-3			$\dots$
ye_3	0			1	4		$\dots$

About correctness and termination

Correctness and termination

Can be proved using an **adapted Buchberger termination criterion** for $\mathfrak{S}$-GB. Induction on signature is crucial.

About correctness and termination

Correctness and termination

Can be proved using an **adapted Buchberger termination criterion** for $\mathfrak{S}$-GB. Induction on signature is crucial.

Table of contents

Table of contents

Non-homogeneousness

What is lacking?

- In the homogeneous case, we used heavily that we can work **degree by degree**, with one finite-dimensional vector space by degree.
- The degree of a polynomial and the degree of its signature are related.

First solution

Working iteratively in the f_i's (Position Over Term order for the signatures).

- Can still have the full F5 Elimination Criterion.
- More time spent in building matrices (one for every necessary f_i and degree). In practice, often gives approx. same timings.

Table of contents

Definitions

Signatures and sugar-degree

We define an order on the monomials of $K[x_1, \ldots, x_n]^s$ by taking into account for $x^\alpha e_i$

1. Position, then
2. **sugar-degree**: $|x^\alpha| + |f_i|$, then
3. classical monomial order.

We can define signatures accordingly (minimal leading monomial...).

Main motivation for sugar-degree

We can work with one finite-dimensional vector space by sugar-degree, and hence with only one matrix by sugar-degree.

Affine issue with signatures and sugar-degree

We want to proceed by **increasing sugar-degree** with one matrix by sugar-degree. Yet:

$$
\begin{aligned}
f_1 &= xy - 1 \\
f_2 &= y^2 - 1 \\
f_3 &= z^2 - x \\
f_4 &= z^2 - y
\end{aligned}
$$

$$
\begin{aligned}
x - y &= f_4 - f_3 \\
&= xf_2 - yf_1.
\end{aligned}
$$

Affine issue with signatures and sugar-degree

We want to proceed by **increasing sugar-degree** with one matrix by sugar-degree. Yet:

$$f_1 = xy - 1$$
$$f_2 = y^2 - 1$$
$$f_3 = z^2 - x$$
$$f_4 = z^2 - y$$

$$x - y = f_4 - f_3 = xf_2 - yf_1.$$

$$\text{Sugar-degree}(e_4 - e_3) = |f_4| = 2$$

Affine issue with signatures and sugar-degree

We want to proceed by **increasing sugar-degree** with one matrix by sugar-degree. Yet:

$$\begin{aligned} f_1 &= xy - 1 \\ f_2 &= y^2 - 1 \\ f_3 &= z^2 - x \\ f_4 &= z^2 - y \end{aligned}$$

$$\begin{aligned} x - y &= f_4 - f_3 \\ &= xf_2 - yf_1. \end{aligned}$$

$$\text{Sugar-degree}(e_4 - e_3) = |f_4| = 2$$
$$\text{Sugar-degree}(xe_2 - ye_1) = |xf_2| = 3$$

Affine issue with signatures and sugar-degree

We want to proceed by **increasing sugar-degree** with one matrix by sugar-degree. Yet:

$$f_1 = xy - 1$$
$$f_2 = y^2 - 1$$
$$f_3 = z^2 - x$$
$$f_4 = z^2 - y$$

$$\begin{aligned} x - y &= f_4 - f_3 \\ &= xf_2 - yf_1. \end{aligned}$$

$$\text{Sugar-degree}(e_4 - e_3) = |f_4| = 2$$
$$\text{Sugar-degree}(xe_2 - ye_1) = |xf_2| = 3$$

We can reach $x - y$ at sugar-degree 2 but its signature is xe_2 of sugar-degree 3.

Signature and sugar-degree

Definition

$$I_d = \mathrm{Vect}\left(\{x^\alpha e_i, \text{ s.t. } |x^\alpha f_i| \leq d\}\right).$$

For $f \in I_d$, we can define its d-signature $S_d(f)$ as signature when restricting to I_d.

Remark

For $f \in I_d$, we can have $S_d(f) \neq S_{d+1}(f)$.

Affine Tropical F5 algorithm

F5 Algorithm

We start from $G := F = (f_1, \ldots, f_s) \subset K[X]$. S is the set of the S-polynomials of F.

Affine Tropical F5 algorithm

F5 Algorithm

We start from $G := F = (f_1, \dots, f_s) \subset K[X]$. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)

Affine Tropical F5 algorithm

F5 Algorithm

We start from $G := F = (f_1, \ldots, f_s) \subset K[X]$. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest **sugar-degree** d in S.

Affine Tropical F5 algorithm

F5 Algorithm

We start from $G := F = (f_1, \dots, f_s) \subset K[X]$. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest **sugar-degree** d in S.
   Apply F5 criterion to remove the ones that are useless.

Affine Tropical F5 algorithm

F5 Algorithm

We start from $G := F = (f_1, \ldots, f_s) \subset K[X]$. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest **sugar-degree** d in S. Apply F5 criterion to remove the ones that are useless.
3. Select **reducers (each with smallest** d**-signature possible)** in G.

Affine Tropical F5 algorithm

F5 Algorithm

We start from $G := F = (f_1, \dots, f_s) \subset K[X]$. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest **sugar-degree** d in S. Apply F5 criterion to remove the ones that are useless.
3. Select **reducers (each with smallest** d**-signature possible)** in G. Make a Macaulay matrix M from them.

Affine Tropical F5 algorithm

F5 Algorithm

We start from $G := F = (f_1, \dots, f_s) \subset K[X]$. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest **sugar-degree** d in S. Apply F5 criterion to remove the ones that are useless.
3. Select **reducers (each with smallest** d**-signature possible)** in G. Make a Macaulay matrix M from them. Rows are ordered by d-signature.

Affine Tropical F5 algorithm

F5 Algorithm

We start from $G := F = (f_1, \ldots, f_s) \subset K[X]$. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest **sugar-degree** d in S. Apply F5 criterion to remove the ones that are useless.
3. Select **reducers (each with smallest** d**-signature possible)** in G. Make a Macaulay matrix M from them. Rows are ordered by d-signature. $\widetilde{M}$ the **tropical row-reduction** (no choice of pivot) of M.

Affine Tropical F5 algorithm

F5 Algorithm

We start from $G := F = (f_1, \dots, f_s) \subset K[X]$. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest **sugar-degree** d in S. Apply F5 criterion to remove the ones that are useless.
3. Select **reducers (each with smallest** d**-signature possible)** in G. Make a Macaulay matrix M from them. Rows are ordered by d-signature. $\widetilde{M}$ the **tropical row-reduction** (no choice of pivot) of M.
4. Add to G the rows of $\widetilde{M}$ providing new leading monomials (for a given d-signature).

Affine Tropical F5 algorithm

F5 Algorithm

We start from $G := F = (f_1, \ldots, f_s) \subset K[X]$. S is the set of the S-polynomials of F.

1. While $S \neq \emptyset$, (**Buchberger's criterion**)
2. Pop all the $(x^\alpha f_1, x^\beta f_2)$ of the smallest **sugar-degree** d in S. Apply F5 criterion to remove the ones that are useless.
3. Select **reducers (each with smallest** d**-signature possible)** in G. Make a Macaulay matrix M from them. Rows are ordered by d-signature. $\widetilde{M}$ the **tropical row-reduction** (no choice of pivot) of M.
4. Add to G the rows of $\widetilde{M}$ providing new leading monomials (for a given d-signature). Update S.

An example : in sugar-degree 2

$$f_1 = xy - 1$$
$$f_2 = y^2 - 1$$
$$f_3 = z^2 - x$$
$$f_4 = z^2 - y$$

	z^2	x	y
e_3	1	−1	
e_4	1		−1

An example : in sugar-degree 2

$$f_1 = xy - 1$$
$$f_2 = y^2 - 1$$
$$f_3 = z^2 - x$$
$$f_4 = z^2 - y$$
$$f_5 = x - y$$
$$S_2(x-y) = e_4$$

	z^2	x	y
e_3	1	-1	
e_4	0	1	-1

An example : in sugar-degree 2

$$f_1 = xy - 1$$
$$f_2 = y^2 - 1$$
$$f_3 = z^2 - x$$
$$f_4 = z^2 - y$$
$$f_5 = x - y$$
$$S_2(x - y) = e_4$$

	z^2	x	y
e_3	1	−1	
e_4	0	1	−1

An example : in sugar-degree 3

$$f_1 = xy - 1$$
$$f_2 = y^2 - 1$$
$$f_3 = z^2 - x$$
$$f_4 = z^2 - y$$
$$f_5 = x - y$$
$$S_2(x-y) = e_4$$

	xy^2	xy	y^2	x	y	1
e_1		1				−1
ye_1	1				−1	
e_2			1			−1
xe_2	1			−1		
ye_4		1	−1			

An example : in sugar-degree 3

$$f_1 = xy - 1$$
$$f_2 = y^2 - 1$$
$$f_3 = z^2 - x$$
$$f_4 = z^2 - y$$
$$f_5 = x - y$$
$$S_2(x-y) = e_4$$

	xy^2	xy	y^2	x	y	1
e_1		1				-1
ye_1	1				-1	
e_2			1			-1
xe_2	0			-1	1	
ye_4		0	0			0

An example : in sugar-degree 3

$$f_1 = xy - 1$$
$$f_2 = y^2 - 1$$
$$f_3 = z^2 - x$$
$$f_4 = z^2 - y$$
$$f_5 = x - y$$
$$S_2(x - y) = e_4$$
$$f_6 = x - y$$
$$S_3(x - y) = xe_2$$

	xy^2	xy	y^2	x	y	1
e_1		1				-1
ye_1	1				-1	
e_2			1			-1
xe_2	0			-1	1	
ye_4		0	0			0

Conclusion

Affine case

- Can compute (tropical) $\mathfrak{S}_d$-GB iteratively (in d).

Conclusion

Affine case

- Can compute (tropical) $\mathfrak{S}_d$-GB iteratively (in d).
- The Buchberger criterion is valid for $\mathfrak{S}_d$-GB and $\mathfrak{S}$-GB: no issue with correctness and termination.

Conclusion

Affine case

- Can compute (tropical) $\mathfrak{S}_d$-GB iteratively (in d).
- The Buchberger criterion is valid for $\mathfrak{S}_d$-GB and $\mathfrak{S}$-GB: no issue with correctness and termination.
- In practice: loss in precision for tropical Gröbner bases using a good choice of parameter is **very satisfying**.

Conclusion

Affine case

- Can compute (tropical) $\mathfrak{S}_d$-GB iteratively (in d).
- The Buchberger criterion is valid for $\mathfrak{S}_d$-GB and $\mathfrak{S}$-GB: no issue with correctness and termination.
- In practice: loss in precision for tropical Gröbner bases using a good choice of parameter is **very satisfying**.
- Implementation available at: `https://gist.github.com/TristanVaccon`.

Conclusion

Affine case

- Can compute (tropical) $\mathfrak{S}_d$-GB iteratively (in d).
- The Buchberger criterion is valid for $\mathfrak{S}_d$-GB and $\mathfrak{S}$-GB: no issue with correctness and termination.
- In practice: loss in precision for tropical Gröbner bases using a good choice of parameter is **very satisfying**.
- Implementation available at: `https://gist.github.com/TristanVaccon`.

Conclusion

Affine case

- Can compute (tropical) $\mathfrak{S}_d$-GB iteratively (in d).
- The Buchberger criterion is valid for $\mathfrak{S}_d$-GB and $\mathfrak{S}$-GB: no issue with correctness and termination.
- In practice: loss in precision for tropical Gröbner bases using a good choice of parameter is **very satisfying**.
- Implementation available at: `https://gist.github.com/TristanVaccon`.

Future works

- Can FGLM be adapted to the tropical case?

Conclusion

Affine case

- Can compute (tropical) $\mathfrak{S}_d$-GB iteratively (in d).
- The Buchberger criterion is valid for $\mathfrak{S}_d$-GB and $\mathfrak{S}$-GB: no issue with correctness and termination.
- In practice: loss in precision for tropical Gröbner bases using a good choice of parameter is **very satisfying**.
- Implementation available at: `https://gist.github.com/TristanVaccon`.

Future works

- Can FGLM be adapted to the tropical case?
- Comparison with other methods (*e.g.*

Conclusion

Affine case

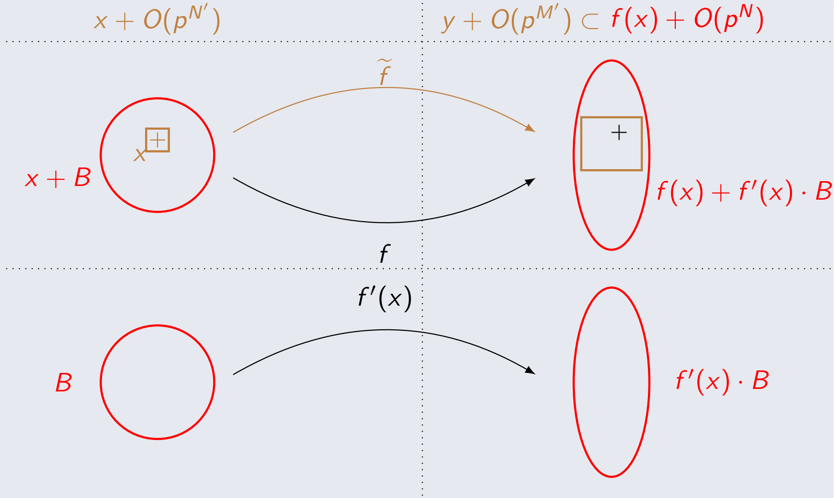
- Can compute (tropical) $\mathfrak{S}_d$-GB iteratively (in d).
- The Buchberger criterion is valid for $\mathfrak{S}_d$-GB and $\mathfrak{S}$-GB: no issue with correctness and termination.
- In practice: loss in precision for tropical Gröbner bases using a good choice of parameter is **very satisfying**.
- Implementation available at: `https://gist.github.com/TristanVaccon`.

Future works

- Can FGLM be adapted to the tropical case?
- Comparison with other methods (*e.g. Markwig-Ren 2016-2017*).

Thank you for your attention

Thank you



References

Classical Gröbner bases

- *A new efficient algorithm for computing Gröbner bases (F4)*, J.C. Faugère, 1999.
- *A new efficient algorithm for computing Gröbner bases without reduction to zero (F5)*, J.C. Faugère, 2002.
- *The F5 Criterion revised*, A.Arri and J.Perry, 2011.

Tropical Gröbner bases

- *Gröbner bases over fields with valuations*, A.Chan and D.Maclagan, 2013.
- *Matrix-F5 Algorithms and Tropical Gröbner Bases Computation*, **V.**, 2015.
- *A Tropical F5 Algorithm*, **V.** and Yokoyama K., 2017.