A Polynomial-Division-Based Algorithm for Computing Linear Recurrence Relations

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Problem.

Input: A sequence $w =$

i	0	1	2	3	4	5	6	7	8	9	10
[redacted]	1	2	3	4	1	-18	-97	-376	-1299	-4238	-13397

.

Output: A **minimal generator**.

Solution.

- The BERLEKAMP–MASSEY algorithm (BM). [BERLEKAMP, 1968], [MASSEY, 1969]

  → Returns $w_{i+3} - 4\,w_{i+2} + w_{i+1} + 6\,w_i = 0$ for all $i + 3 \leq 10$.

- Many applications!
  - Sparse polynomial interpolation.
  - Sparse linear sytem solving.
  - Modular rational reconstruction – Padé approximants.
  - Polynomial approximation / extrapolation.

New application: change of ordering

Input:

- A Gröbner basis $\mathcal{G}_1$ for a monomial ordering $\prec_1$;
- another monomial ordering $\prec_2$.

Output: A Gröbner basis $\mathcal{G}_2$ for $\prec_2$.

Sparse-FGLM

[Faugère, Mou, 2011, 2017]

1. T_x, T_y: multiplication matrices for $\mathcal{G}_1$;
   r: random vector.
2. **Make** a multi-dimensional sequence $w_{i,j} = \langle r, T_x^i \cdot T_y^j \cdot \mathbf{1} \rangle$:
3. Call the Berlekamp–Massey–Sakata algorithm (BMS) on $\boldsymbol{w}$ and $\prec_2$.
4. Return $\mathcal{G}_2 = \langle x^3 + \cdots, y^3 + \cdots, x\, y + \cdots \rangle$.

[Sakata, 1988, 1990, 2009]

$\boldsymbol{w} =$

$i \backslash j$	0	1	2	3	4	5
0	1	2	3	4	1	−18
1	6	13	27	59	131	303
2	11	22	40	72	116	152
3	−104	−204	−404	−788	−1524	−2884
4	418	836	1656	3280	6448	12576

Univariate.

- Berlekamp–Massey. [Berlekamp, 1968] [Massey, 1969]
  → **fast** truncated **GCD**.

Multivariate.

- BMS. [Sakata, 1988, 1990, 2009]
  → **Polynomial additions** and **shifts**.
- Padé approximants. [Fitzpatrick, Flynn, 1992]
  → Truncated **Gröbner basis**.
- Scalar-FGLM. [B., Boyer, Faugère, 2015, 2017]
  → **Gaussian elim.** on a **multi-Hankel** matrix.
- Artinian Gorenstein border bases (AGbb). [Mourrain, 2017]
  → **Gram–Schmidt** process.

Univariate.

- Berlekamp–Massey. [Berlekamp, 1968] [Massey, 1969]

  → **fast** truncated **GCD**.

Goal.

→ Polynomial arithmetic **only**.

Multivariate.

- BMS. [Sakata, 1988, 1990, 2009]

  → **Polynomial additions** and **shifts**.

- Padé approximants. [Fitzpatrick, Flynn, 1992]

  → Truncated **Gröbner basis**.

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- Artinian Gorenstein border bases (AGbb). [Mourrain, 2017]

  → **Gram–Schmidt** process.

For $\boldsymbol{w} =$

i	0	1	2	3	4	5	6	7
	1	2	3	4	1	-18	-97	-376

,

$w_{i+3} - 4\,w_{i+2} + w_{i+1} + 6\,w_i = 0$ is a **valid relation** for all $i+3 \le 7$ **iff**

$$\begin{array}{c} \\ 1 \end{array} \begin{array}{cccccccc} 1 & x & x^2 & x^3 & x^4 & x^5 & x^6 & x^7 \\ (\,1 & 2 & 3 & 4 & 1 & -18 & -97 & -376\,) \end{array} \begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \end{array} \begin{pmatrix} 6 \\ 1 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = (\ 0\ ),$$

For $\boldsymbol{w} =$

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For $\boldsymbol{w} =$

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,

$w_{i+3} - 4\,w_{i+2} + w_{i+1} + 6\,w_i = 0$ is a **valid relation** for all $i + 3 \leq 7$ **iff**

$$\begin{array}{c} \\ 1 \end{array} \begin{pmatrix} \overset{1}{1} & \overset{x}{2} & \overset{x^2}{3} & \overset{x^3}{4} & \overset{x^4}{1} & \overset{x^5}{-18} & \overset{x^6}{-97} & \overset{x^7}{-376} \end{pmatrix} \begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ 1 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}.$$

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$$\begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & -18 \\ 4 & 1 & -18 & -97 \\ 1 & -18 & -97 & -376 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \end{array} \begin{pmatrix} 6 \\ 1 \\ -4 \\ 1 \end{pmatrix} = \begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For $\boldsymbol{w} =$

i	0	1	2	3	4	5	6	7
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,

$w_{i+3} - 4\,w_{i+2} + w_{i+1} + 6\,w_i = 0$ is a **valid relation** for all $i+3 \leq 7$ **iff**

$$\begin{array}{c} \\ x^4 \\ x^3 \\ x^2 \\ x \\ 1 \\ 1/x \\ 1/x^2 \\ 1/x^3 \end{array}
\begin{array}{c} \begin{array}{cccc} x^3 & x^2 & x & 1 \end{array} \\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & -18 \\ 4 & 1 & -18 & -97 \\ 1 & -18 & -97 & -376 \\ -18 & -97 & -376 & 0 \\ -97 & -376 & 0 & 0 \\ -376 & 0 & 0 & 0 \end{pmatrix} \end{array}
\begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \end{array}
\begin{pmatrix} 6 \\ 1 \\ -4 \\ 1 \end{pmatrix} =
\begin{array}{c} \\ x^4 \\ x^3 \\ x^2 \\ x \\ 1 \\ 1/x \\ 1/x^2 \\ 1/x^3 \end{array}
\begin{array}{c} x^3 \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ f_{x^2} \\ f_x \\ f_1 \end{pmatrix} \end{array} \Leftrightarrow$$

$$\begin{aligned} (x^7 + 2\,x^6 + 3\,x^5 + 4\,x^4 + x^3 - 18\,x^2 - 97\,x - 376)\,(x^3 - 4\,x^2 + x + 6) \bmod x^8 & \\ = f_{x^2}\,x^2 + f_x\,x + f_1, & \\ PC \bmod I = F. & \end{aligned}$$

Algorithm.

Input: A sequence $\boldsymbol{w} =$

i	0	1	$\cdots$	D
	w_0	w_1	$\cdots$	w_D

up to w_D.

Output: A minimal linear recurrence relation satisfied by $w_0, \ldots, w_D$.

1. $B := x^{D+1}$, $P := w_0\, x^D + w_1\, x^{D-1} + \cdots + w_D$.
2. Compute $PC = F \mod B$ s.t. $\deg F < \deg C$ and $\deg F$ maximal:

→ **Extended Euclidean algorithm** on B and P,

→ Stop when reaching the **first remainder** F and its **associated cofactor** C of P.

Example.

$\boldsymbol{w} =$

i	0	1	2	3	4	5	6	7
	1	2	3	4	1	-18	-97	-376

Remainders	Cofactors
$B = x^8$	0
$P = x^7 + 2\,x^6 + 3\,x^5 + 4\,x^4 + x^3 - 18\,x^2 - 97\,x - 376$	1
$-x^6 - 2\,x^5 - 7\,x^4 - 20\,x^3 - 61\,x^2 - 182\,x + 752$	$x - 2$
$-4\,x^5 - 16\,x^4 - 60\,x^3 - 200\,x^2 + 655\,x - 376$	$x^2 - 2\,x + 1$
$1299\,x^2 - 958\,x - 2256$	$x^3 - 4\,x^2 + x + 6$

For $\boldsymbol{w} =$

$i \backslash j$	0	1	2	3	4
0	1	2	3	4	1
1	6	13	27	59	
2	11	22			
3	−104				

and the DRL$(y \prec x)$ ordering, $w_{i+1,j+1} + w_{i,j+2} - 2\,w_{i+1,j} - 3\,w_{i,j+1} + 2\,w_{i,j} = 0$ is a **valid relation** for all $(i+1, j+1) \preceq (1,3)$ **iff**

$$\begin{matrix} & 1 & y & x & y^2 & x\,y & x^2 & y^3 & x\,y^2 & x^2\,y & x^3 & y^4 & x\,y^3 \\ 1 & (\,1 & 2 & 6 & 3 & 13 & 11 & 4 & 27 & 22 & -104 & 1 & 59\,) \end{matrix} \begin{matrix} 1 \\ y \\ x \\ y^2 \\ x\,y \\ x^2 \\ y^3 \\ x\,y^2 \\ x^2\,y \\ x^3 \\ y^4 \\ x\,y^3 \end{matrix} \begin{pmatrix} 2 \\ -3 \\ -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = (\,0\,),$$

For $\boldsymbol{w} =$

$_i{}^j$	0	1	2	3	4
0	1	2	3	4	1
1	6	13	27	59	
2	11	22			
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and the $\mathrm{DRL}(y \prec x)$ ordering, $w_{i+1,j+1} + w_{i,j+2} - 2\,w_{i+1,j} - 3\,w_{i,j+1} + 2\,w_{i,j} = 0$ is a **valid relation** for all $(i+1, j+1) \preceq (1,3)$ **iff**

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$$\begin{array}{c} \\ 1 \end{array} \begin{pmatrix} \overset{1}{1} & \overset{y}{2} & \overset{x}{6} & \overset{y^2}{3} & \overset{x\,y}{13} & \overset{x^2}{11} & \overset{y^3}{4} & \overset{x\,y^2}{27} & \overset{x^2\,y}{22} & \overset{x^3}{-104} & \overset{y^4}{1} & \overset{x\,y^3}{59} \end{pmatrix} \begin{array}{c} 1 \\ y \\ x \\ y^2 \\ x\,y \\ x^2 \\ y^3 \\ x\,y^2 \\ x^2\,y \\ x^3 \\ y^4 \\ x\,y^3 \end{array} \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ -3 \\ -2 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix},$$

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$$\begin{array}{c} \\ 1 \\ y \\ x \\ y^2 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & y & x & y^2 & x\,y \end{array} \\ \begin{pmatrix} 1 & 2 & 6 & 3 & 13 \\ 2 & 3 & 13 & 4 & 27 \\ 6 & 13 & 11 & 27 & 22 \\ 3 & 4 & 27 & 1 & 59 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ y \\ x \\ y^2 \\ x\,y \end{array}\begin{pmatrix} 2 \\ -3 \\ -2 \\ 1 \\ 1 \end{pmatrix} = \begin{array}{c} 1 \\ y \\ x \\ y^2 \end{array}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For $\boldsymbol{w} =$

$i \backslash j$	0	1	2	3	4
0	1	2	3	4	1
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2	11	22			
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$$
\begin{array}{r} \\ x^2y^2 \\ x^2y \\ xy^2 \\ x^2 \\ xy \\ y^2 \\ x^2/y \\ x \\ y \\ y^2/x \\ x^2/y^2 \\ x/y \end{array}
\begin{array}{c} \begin{array}{ccccc} xy^2 & xy & y^2 & x & y \end{array} \\ \begin{pmatrix} 1 & 2 & 6 & 3 & 13\\ 2 & 3 & 13 & 4 & 27\\ 6 & 13 & 11 & 27 & 22\\ 3 & 4 & 27 & 1 & 59\\ 13 & 27 & 22 & 59 & 0\\ 11 & 22 & -104 & 0 & 0\\ 4 & 1 & 59 & 0 & 0\\ 27 & 59 & 0 & 0 & 0\\ 22 & 0 & 0 & 0 & 0\\ -104 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 59 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}
\begin{array}{r} 1 \\ y \\ x \\ y^2 \\ xy \end{array}
\begin{pmatrix} 2\\ -3\\ -2\\ 1\\ 1 \end{pmatrix}
=
\begin{array}{r} \\ x^2y^2 \\ x^2y \\ xy^2 \\ x^2 \\ xy \\ y^2 \\ x^2/y \\ x \\ y \\ y^2/x \\ x^2/y^2 \\ x/y \end{array}
\begin{array}{c} xy^2 \\ \begin{pmatrix} 0\\ 0\\ 0\\ 0\\ f_{x^2y^3}\\ f_{xy^4}\\ f_{x^3y}\\ f_{x^2y^2}\\ f_{xy^3}\\ f_{y^4}\\ f_{x^3}\\ f_{x^2y} \end{pmatrix} \end{array}
\Leftrightarrow
$$

$$
\begin{aligned}
(x^3\,y^4 + 2\,x^3\,y^3 + 6\,x^2\,y^4 + \cdots + x^3 + 59\,x^2\,y)\,(x\,y + y^2 - 2\,x - 3\,y + 2) \bmod (x^4, y^5)& \\
= f_{x^2y^3}\,x^2\,y^3 + \cdots + f_{x^2y}\,x^2\,y,& \\
PC \bmod I = F.&
\end{aligned}
$$

Main tools.

- **Mirror** of the **truncated generating series**.

$$\rightarrow \quad P = \sum_{(i,j)\preceq(1,3)} w_{i,j}\,\frac{x^3\,y^4}{x^i\,y^j}.$$

- Computation modulo a **monomial ideal**.

$$\rightarrow \quad \bmod\,(x^4, y^5).$$

Scalar-FGLM

→ **Gaussian elim.** on a **multi-Hankel** matrix.

→ Polynomial viewpoint **possible**.

→ **Unified** algorithm for BMS and Scalar-FGLM.

Definition.

Assuming:

- $\prec$ is a monomial degree ordering on $x_1, \ldots, x_n$;
- $a = x_1^{a_1} \cdots x_n^{a_n}$.
- w is an n-dimensional sequence known **from** $w_{0,\ldots,0}$ **up to** $w_{a_1,\ldots,a_n}$.

Then:

1. $M = x_1^{D_1} \cdots x_n^{D_n} := \mathsf{LCM}(1, \ldots, a)$.
2. $P = \sum_{m \preceq a} w_{m_1,\ldots,m_n} \frac{M}{m}$ is the **mirror** of the **truncated generating series**.
3. For a polynomial $C_g = g + \sum_{m \prec g} \gamma_m m$, the pair $R_g = [F_g, C_g]$ always satisfies

$$P C_g \mod (x_1^{D_1+1}, \ldots, x_n^{D_n+1}) = F_g.$$

Theorem (simplified to the BMS case).

Then:

1. $M = x_1^{D_1} \cdots x_n^{D_n} := \mathsf{LCM}(1, \ldots, a)$;
2. $P = \sum_{m \preceq a} w_{m_1, \ldots, m_n} \frac{M}{m}$ is the **mirror** of the **truncated generating series**;
3. A pair $R_g = [F_g, C_g]$ corresponds to a **valid relation** C_g **iff**

$$F_g = P C_g \mod (x_1^{D_1+1}, \ldots, x_n^{D_n+1}) \text{ and } \mathsf{LM}(F_g) \prec \frac{M}{\max_{\sigma g \prec a}(\sigma)}.$$

Scalar-FGLM

→ More general theorem.

Theorem (simplified to the BMS case).

Then:

1. $M = x_1^{D_1} \cdots x_n^{D_n} := \mathsf{LCM}(1, \ldots, a)$;
2. $P = \sum_{m \preceq a} w_{m_1,\ldots,m_n} \frac{M}{m}$ is the **mirror** of the **truncated generating series**;
3. A pair $R_g = [F_g, C_g]$ corresponds to a **valid relation** C_g **iff**

$$F_g = P C_g \mod (x_1^{D_1+1}, \ldots, x_n^{D_n+1}) \text{ and } \mathsf{LM}(F_g) \prec \frac{M}{\max_{\sigma g \prec a}(\sigma)}.$$

Example.

$\boldsymbol{w} =$

$i \backslash j$	0	1	2	3	4
0	1	2	3	4	1
1	6	13	27	59	
2	11	22			
3	-104				

, $P = x^3 y^4 + \cdots + 59 x^2 y$, $C_{xy} = x y + y^2 - 2x - 3y + 2$:

$F_{xy} = P C_{xy} \mod (x^4, y^5) = -40 x^2 y^3 + \cdots$, $\max_{\sigma xy \prec xy^3} = y^2$ and $x^2 y^3 \prec \frac{x^3 y^4}{y^2}$.

$\rightarrow$ **valid relation**

$R_1 = [F_1, 1] = [P, 1]$ $= [x^3\, y^4 + \cdots, 1]$				

$R_y = [F_y, y-2]$ $= [-x^3\, y^3 + \cdots, y + \cdots]$				
$y \uparrow$				
$R_1 = [F_1, 1] = [P, 1]$ $= [x^3\, y^4 + \cdots, 1]$				

$R_y = [F_y, y-2]$ $= [-x^3\, y^3 + \cdots, y + \cdots]$				
$y \uparrow$				
$R_1 = [F_1, 1] = [P, 1]$ $= [x^3\, y^4 + \cdots, 1]$	$\overset{x}{\rightarrow}$	$R_x = [F_x, x+y-8]$ $= [-24\, x^2\, y^4 + \cdots, x + \cdots]$		

$R_{y^2} = [F_{y^2}, 24\, y^2 + \cdots]$ $= [\cdot\, x^3\, y^2 + \cdots, y^2 + \cdots]$				
$y \uparrow$				
$R_y = [F_y, y - 2]$ $= [-x^3\, y^3 + \cdots, y + \cdots]$				
$y \uparrow$				
$R_1 = [F_1, 1] = [P, 1]$ $= [x^3\, y^4 + \cdots, 1]$	$\xrightarrow{x}$	$R_x = [F_x, x + y - 8]$ $= [-24\, x^2\, y^4 + \cdots, x + \cdots]$		

$R_{y^2} = [F_{y^2}, 24\, y^2 + \cdots]$ $= [\cdot\, x^3\, y^2 + \cdots, y^2 + \cdots]$				
$y \uparrow$				
$R_y = [F_y, y - 2]$ $= [-x^3\, y^3 + \cdots, y + \cdots]$	$\xrightarrow{x}$	$R_{xy} = [F_{xy}, x\, y + \cdots]$ $= [\cdot\, x^2\, y^3 + \cdots, x\, y + \cdots]$		
$y \uparrow$				
$R_1 = [F_1, 1] = [P, 1]$ $= [x^3\, y^4 + \cdots, 1]$	$\xrightarrow{x}$	$R_x = [F_x, x + y - 8]$ $= [-24\, x^2\, y^4 + \cdots, x + \cdots]$		

LM $(F_{xy}) = x^2\, y^3$ **small** $\rightarrow$ **valid relation**.

$R_{y^2} = [F_{y^2}, 24\, y^2 + \cdots]$ $= [\cdot\, x^3\, y^2 + \cdots, y^2 + \cdots]$				
$y \uparrow$				
$R_y = [F_y, y - 2]$ $= [-x^3\, y^3 + \cdots, y + \cdots]$	$\xrightarrow{x}$	$R_{xy} = [F_{xy}, x\, y + \cdots]$ $= [\cdot\, x^2\, y^3 + \cdots, x\, y + \cdots]$		
$y \uparrow$				
$R_1 = [F_1, 1] = [P, 1]$ $= [x^3\, y^4 + \cdots, 1]$	$\xrightarrow{x}$	$R_x = [F_x, x + y - 8]$ $= [-24\, x^2\, y^4 + \cdots, x + \cdots]$	$\xrightarrow{x}$	$R_{x^2} = [F_{x^2}, x^2 + \cdots]$ $= [\cdot\, x^2\, y^3 + \cdots, x^2 + \cdots]$

LM $(F_{xy}) = x^2\, y^3$ **small** $\rightarrow$ **valid relation**.

$R_{y^3} = [F_{y^3}, y^3 + \cdots]$ $= [\cdot\, x\, y^4 + \cdots, y^3 + \cdots]$				
$y \uparrow$				
$R_{y^2} = [F_{y^2}, 24\, y^2 + \cdots]$ $= [\cdot\, x^3\, y^2 + \cdots, y^2 + \cdots]$				
$y \uparrow$				
$R_y = [F_y, y - 2]$ $= [-x^3\, y^3 + \cdots, y + \cdots]$	$\xrightarrow{x}$	$R_{xy} = [F_{xy}, x\, y + \cdots]$ $= [\cdot\, x^2\, y^3 + \cdots, x\, y + \cdots]$		
$y \uparrow$				
$R_1 = [F_1, 1] = [P, 1]$ $= [x^3\, y^4 + \cdots, 1]$	$\xrightarrow{x}$	$R_x = [F_x, x + y - 8]$ $= [-24\, x^2\, y^4 + \cdots, x + \cdots]$	$\xrightarrow{x}$	$R_{x^2} = [F_{x^2}, x^2 + \cdots]$ $= [\cdot\, x^2\, y^3 + \cdots, x^2 + \cdots]$

LM $(F_{xy}) = x^2\, y^3$ **small** $\rightarrow$ **valid relation**.

Goal.

→ Find **small** F in $\left\langle x_1^{D_1+1}, \ldots, x_n^{D_n+1}, P = \sum_{m \prec a} w_{m_1,\ldots,m_n} \frac{M}{m} \right\rangle$.

Normal forms of polynomials.

$\mathrm{NormalForm}\ (F_g, [F_{t_1}, \ldots, F_{t_r}])$ returns **quotients** $Q_{t_1}, \ldots, Q_{t_r}$ and **remainder** F_h s.t.

$$F_g = Q_{t_1} F_{t_1} + \cdots + Q_{t_r} F_{t_r} + F_h.$$

Extension to pairs.

$\mathrm{NormalForm}\ (R_g, [R_{t_1}, \ldots, R_{t_r}])$ returns **quotients** $Q_{t_1}, \ldots, Q_{t_r}$ and **remainder** R_h s.t.

$$\begin{aligned} F_g &= Q_{t_1} F_{t_1} + \cdots + Q_{t_r} F_{t_r} + F_h. \\ C_g &= Q_{t_1} C_{t_1} + \cdots + Q_{t_r} C_{t_r} + C_h. \end{aligned}$$

Example.

$\mathrm{NormalForm}\,(x^3 y^5, [x^3 y^4 + 2 x^3 y^3, x^4, y^5])$ yields

$$x^3 y^5 = (y-2)(x^3 y^4 + 2 x^3 y^3) + Q_{x^4}\, x^4 + Q_{y^5}\, y^5 + (4 x^3 y^3 + \cdots).$$

Goal.

$\rightarrow$ Find **small** F in $\left\langle x_1^{D_1+1}, \ldots, x_n^{D_n+1}, P = \sum_{m \prec a} w_{m_1,\ldots,m_n} \frac{M}{m} \right\rangle$.

Normal forms of polynomials.

$\mathrm{NormalForm}\,(F_g, [F_{t_1}, \ldots, F_{t_r}])$ returns **quotients** $Q_{t_1}, \ldots, Q_{t_r}$ and **remainder** F_h s.t.

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$\mathrm{NormalForm}\,(R_g, [R_{t_1}, \ldots, R_{t_r}])$ returns **quotients** $Q_{t_1}, \ldots, Q_{t_r}$ and **remainder** R_h s.t.

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Example.

$\mathrm{NormalForm}\,([x^3\, y^5, 0], [[x^3\, y^4 + 2\, x^3\, y^3, 1], [x^4, 0], [y^5, 0]])$ yields

$$[x^3\, y^5, 0] = (y-2)\, [x^3\, y^4 + 2\, x^3\, y^3, 1] + Q_{x^4}\, [x^4, 0] + Q_{y^5}\, [y^5, 0] + [4\, x^3\, y^3 + \cdots, -y + 2].$$

Polynomial Scalar-FGLM (simplified).

1. **Start** with $R_{B_1} = [x_1^{D_1+1}, 0], \ldots,$ $R_{B_n} = [x_n^{D_n+1}, 0]$,
   $$R_1 = [P, 1] = \left[\sum_{m \preceq a} w_{m_1,\ldots,m_n} \frac{M}{m}, 1\right],$$
   $R' = [R_{B_1}, \ldots, R_{B_n}]$ and $\mathcal{G} = \varnothing$.
2. **If** $P = 0$ **then return** 1,
   else add R_1 to R'.
3. **For** $g \preceq a$
   a. **If** $\exists h \mid g$ s.t. $C_h \in \mathcal{G}$ **then next**.
   b. **Make** a new pair R_g using **NormalForm** s.t. $\mathrm{LM}(C_g) = g$.
   c. **If** $\mathrm{LM}(F_g) \prec \frac{M}{\max_{\sigma \mid g \prec a}(\sigma)}$ **then** add C_g to $\mathcal{G}$, **else** add R_g to R'.

Theorem (simplified).

Assuming:

- $\prec$ is a monomial degree ordering on $x_1, \ldots, x_n$;
- $a = x_1^{a_1} \cdots x_n^{a_n}$;
- $\boldsymbol{w}$ is an n-dimensional sequence known **from** $w_{0,\ldots,0}$ **up to** $w_{a_1,\ldots,a_n}$;
- $\mathcal{G}$ the Gröbner basis and S the staircase satisfy $\max(S) \cdot \max(S \cup \mathrm{LM}(\mathcal{G})) \preceq a$.

Then:

Polynomial Scalar-FGLM **terminates** and **returns** $\mathcal{G}$ with **complexity** $O(\#S(\#S + \#\mathcal{G})\#T)$, where $T = \{m, m \preceq a\}$.

For $\boldsymbol{w} =$

$i \backslash j$	0	1	2	3	4	5
0	1	2	3	4	1	−18
1	6	13	27	59	131	
2	11	22	40	72		
3	−104	−204	−404			
4	418	836				
5	−1411					

, DRL$(y \prec x)$, $a = x^5$, $M = x^5\,y^5$.

1. **Start** with $R_{B_1} = [x^6, 0]$, $R_{B_2} = [y^6, 0]$ and $R_1 = [P, 1] = [x^5\,y^5 + 2\,x^5\,y^4 + \cdots, 1]$.
2. **Make** $R_y := \mathrm{NormalForm}\,(x^6\,R_{B_2}, [R_1, R_{B_1}, R_{B_2}]) = [-x^5\,y^4 + \cdots, y - 2]$.
3. **Make** $R_x := \mathrm{NormalForm}\,(y^6\,R_{B_1}, [R_1, R_{B_1}, R_{B_2}, R_y]) = [-24\,x^4\,y^5 + \cdots, x + y - 8]$.
4. **Make** $R_{y^2} := \mathrm{NormalForm}\,(R_1, [R_y, R_{B_1}, R_{B_2}, R_x]) = \left[-\frac{47}{24}\,x^5\,y^3 + \cdots, y^2 + \frac{7}{24}\,x - \frac{41}{24}\,y - \frac{4}{3}\right]$.
5. **Make** $R_{x\,y} := \mathrm{NormalForm}(y\,R_x, [R_{B_1}, R_{B_2}, R_y, R_x, R_{y^2}]) = [-206\,x^5\,y + \cdots, x\,y + y^2 - 2\,x - 3\,y + 2]$.
   - LM $(F_{x\,y}) = x^5\,y \prec x^2\,y^5 = \frac{x^5\,y^5}{x^3}$ so $x\,y + y^2 - 2\,x - 3\,y + 2 \in \mathcal{G}$!

For $\boldsymbol{w} =$

$i \backslash j$	0	1	2	3	4	5
0	1	2	3	4	1	−18
1	6	13	27	59	131	
2	11	22	40	72		
3	−104	−204	−404			
4	418	836				
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1. **Start** with $R_{B_1} = [x^6, 0]$, $R_{B_2} = [y^6, 0]$ and $R_1 = [P, 1] = [x^5\, y^5 + 2\, x^5\, y^4 + \cdots, 1]$.
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6. **Make** $R_{x^2} := \mathrm{NormalForm}\,(R_1, [R_x, R_{B_1}, R_{B_2}, R_y, R_{y^2}]) = \left[\frac{114075}{47}\, x^3\, y^5 + \cdots, x^2 + x\, y + \cdots\right]$.
7. **Make** $R_{y^3} := \mathrm{NormalForm}(R_y, [R_{y^2}, R_{B_1}, R_{B_2}]) = \left[97\, x^5\, y^2 + \cdots, y^3 + \frac{7}{24}\, x\, y - \frac{89}{24}\, y^2 - \frac{7}{12}\, x + \frac{1}{8}\, y + \frac{79}{12}\right]$.
   - LM $(F_{y^3}) = x^5\, y^2 \prec x^3\, y^5 = \frac{x^5\, y^5}{x^2}$ so $y^3 + \frac{7}{24}\, x\, y - \frac{89}{24}\, y^2 - \frac{7}{12}\, x + \frac{1}{8}\, y + \frac{79}{12} \in \mathcal{G}$!
   - Further reduction to $y^3 - 4\, y^2 + y + 6 \in \mathcal{G}$.

For $\boldsymbol{w} =$

j / i	0	1	2	3	4	5
0	1	2	3	4	1	−18
1	6	13	27	59	131	
2	11	22	40	72		
3	−104	−204	−404			
4	418	836				
5	−1411					

, DRL$(y \prec x)$, $a = x^5$, $M = x^5 y^5$.

1. **Start** with $R_{B_1} = [x^6, 0]$, $R_{B_2} = [y^6, 0]$ and $R_1 = [P, 1] = [x^5 y^5 + 2 x^5 y^4 + \cdots, 1]$.
2. **Make** $R_y := \mathrm{NormalForm}\,(x^6 R_{B_2}, [R_1, R_{B_1}, R_{B_2}]) = [-x^5 y^4 + \cdots, y - 2]$.
3. **Make** $R_x := \mathrm{NormalForm}\,(y^6 R_{B_1}, [R_1, R_{B_1}, R_{B_2}, R_y]) = [-24 x^4 y^5 + \cdots, x + y - 8]$.
4. **Make** $R_{y^2} := \mathrm{NormalForm}\,(R_1, [R_y, R_{B_1}, R_{B_2}, R_x]) = \left[-\frac{47}{24} x^5 y^3 + \cdots, y^2 + \frac{7}{24} x - \frac{41}{24} y - \frac{4}{3}\right]$.
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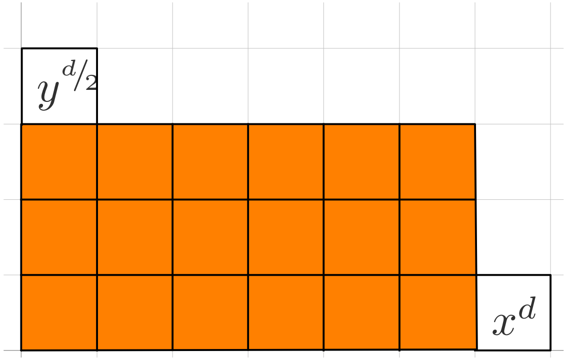
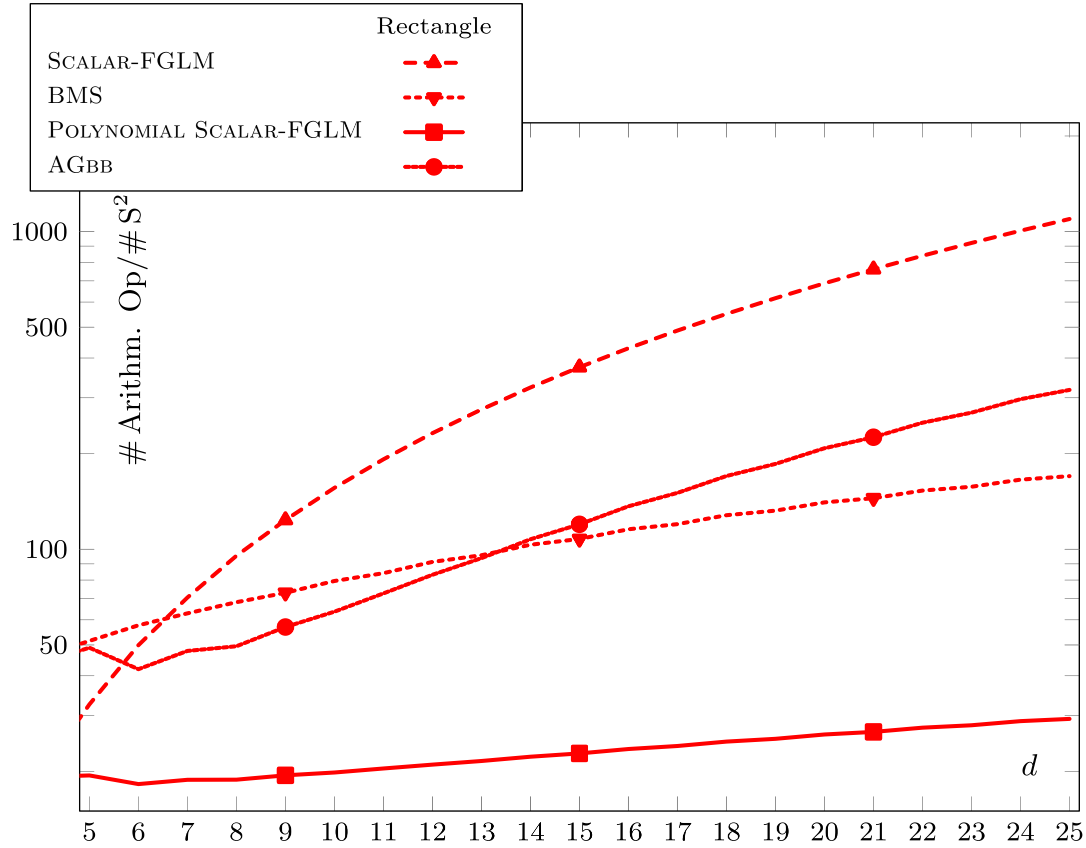
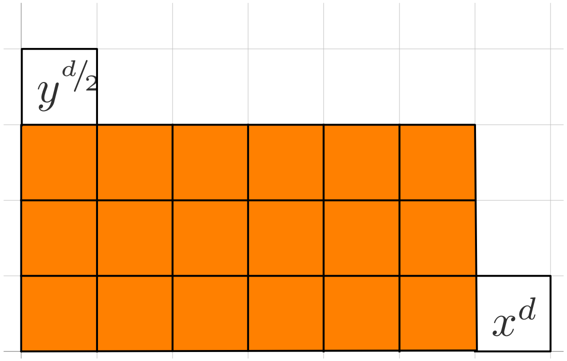
8. **Make** $R_{x^3} := \mathrm{NormalForm}\,(R_x, [R_{x^2}, R_{B_1}, R_{B_2}, R_{y^2}]) = \left[\frac{327009}{47} x^5 y^2 + \cdots, x^3 + x^2\, y + \cdots\right]$.
   - LM $(F_{x^3}) = x^5 y^2 \prec x^3 y^5 = \frac{x^5 y^5}{x^2}$ so $x^3 + x^2 y + \cdots \in \mathcal{G}$!
   - Further reduction to $x^3 + 5\, x^2 - 4\, y^2 + 7\, x + 19\, y - 19 \in \mathcal{G}$.

Rectangle

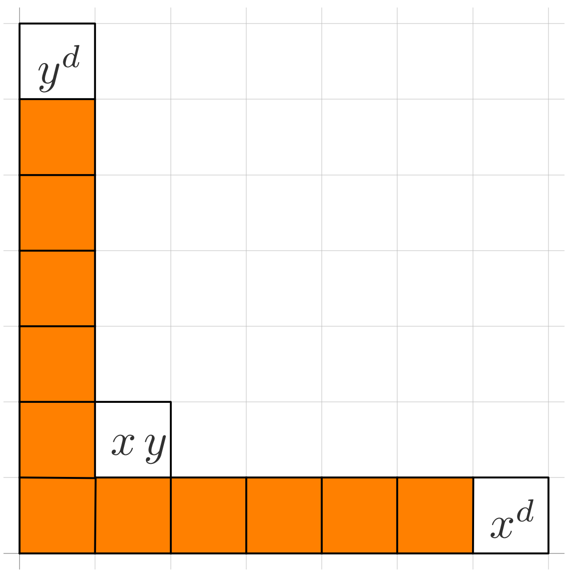

$y^{d/2}$
x^d

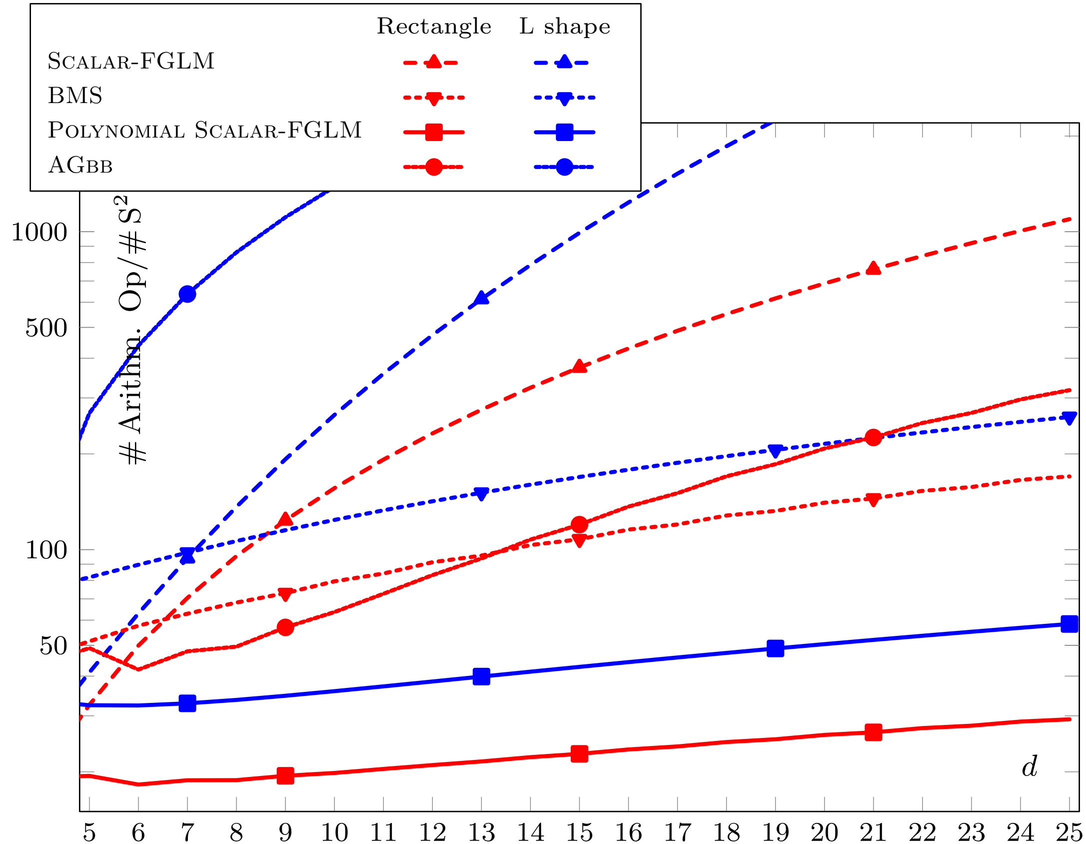

Rectangle
SCALAR-FGLM
BMS
POLYNOMIAL SCALAR-FGLM
AGBB
Arithm. Op/# S^2
1000
500
100
50
d
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25

Rectangle



L-shape





Conclusions.

- **Unified** algorithm extending **BMS** and **Scalar-FGLM**.
- Algorithm based on **polynomial arithmetic**:
  - Divisions.
  - Normal Forms.
- Reencoding as $PC_g \bmod I = F_g$ with $\mathrm{LM}(F_g)$ **small**.

Perspectives.

- **Adaptive** variant **reducing** the number of **queries**.
- Apply the same idea for **P-relations**:
  - Guessing sequences relations with **polynomial coefficients**.
  - **Quasi-commutative** polynomial arithmetic.

Conclusions.

- **Unified** algorithm extending BMS and SCALAR-FGLM.
- Algorithm based on **polynomial arithmetic**:
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Perspectives.

- **Adaptive** variant **reducing** the number of **queries**.
- Apply the same idea for **P-relations**:
  - Guessing sequences relations with **polynomial coefficients**.
  - **Quasi-commutative** polynomial arithmetic.

Thank you!

Informal version of the Algorithm.

Input:

- A monomial degree ordering $\prec$ on $x_1, \ldots, x_n$;
- All the monomials from 1 to $a = x_1^{a_1} \cdots x_n^{a_n}$ for $\prec$.
- An n-dimensional sequence $\boldsymbol{w}$ **from** $w_{0,\ldots,0}$ **up to** $w_{a_1,\ldots,a_n}$.

Output: A Gröbner basis for $\prec$ of linear recurrence relations satisfied by $w_{0,0}, \ldots, w_{a_1,\ldots,a_n}$.

1. $M = x_1^{D_1} \cdots x_n^{D_n} := \mathsf{LCM}(1, \ldots, a)$.
2. $B_1 := x_1^{D_1+1}, \ldots, B_n := x_n^{D_n+1}$, $P = \sum_{m \prec a} w_{m_1,\ldots,m_n} \frac{M}{m}$.
3. $R = [[P, 1]]$, $R' = \emptyset$, $\mathcal{G} = \emptyset$, $S = \emptyset$.
4. **While** $R \neq \emptyset$
   a. Take $R_m = [F_m, C_m]$ the first element of R and reduce it with the relations in R'.
   b. **If** $\mathsf{LM}(F_m)$ is as in the Theorem **then** add C_m to $\mathcal{G}$.
      Else no relations start with m and $\frac{M}{\mathsf{LM}(F_m)}$ so add them to S, add R_m to R'.
   c. **For all** $h \in \mathrm{Border}(S)$, **if** no R_h exists **then** make one with $\mathrm{NormalForm}$.
5. Return $\mathcal{G}$.

Theorem (simplified).

Assuming:

- $\prec$ is a monomial degree ordering on $x_1, \ldots, x_n$;
- $a = x_1^{a_1} \cdots x_n^{a_n}$;
- $\boldsymbol{w}$ is an n-dimensional sequence known **from** $w_{0,\ldots,0}$ **up to** $w_{a_1,\ldots,a_n}$;
- $\mathcal{G}$ the Gröbner basis and S the staircase satisfy $\max(S) \cdot \max(S \cup \mathrm{LM}(\mathcal{G})) \preceq a$.

Then:

1. Polynomial Scalar-FGLM **terminates** and **returns** $\mathcal{G}$;
2. $T = \{m, m \preceq a\}$;
3. The **complexity** of Polynomial Scalar-FGLM is $O(\#S(\#S + \#\mathcal{G})\#T)$.

Proof Sketch.

For all monomial $m \in S \cup \mathrm{LM}(\mathcal{G})$, we make one pair R_m with support $\#T$.

Furthermore, each pair R_m must be reduced by failing pairs in R' but $\#R' = \#S$.