Extending the GVW Algorithm to Local Ring

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Notations

- k : a field.
- $k[X]$: the polynomial ring in the variables $X = \{x_1, \ldots, x_n\}$.
- $R = \{f/(1+g) : f, g \in k[X], g(\mathbf{0}) = 0\}$: the local ring w.r.t. a local order $\succ$.
- $I = \langle f_1, \ldots, f_m \rangle$: an ideal.
- $\mathbf{e}_i$: the i-th unit vector of R^m.
- lm : leading monomial.

Definition 1 (Standard basis)

Let $\succ$ be a semigroup order, and I be an ideal in R or $k[X]$. A **standard basis** of I is a set $\{g_1, \ldots, g_s\}$ in I such that

$$\langle \mathrm{lm}(g_1), \ldots, \mathrm{lm}(g_s) \rangle = \langle \mathrm{lm}(I) \rangle.$$

$$I \subset k[X] \longleftrightarrow \text{Gröbner basis}$$
$$I \subset R \longleftrightarrow \text{standard basis}$$

Problem: How to find a new and efficient algorithm to compute the standard bases of ideals in a local ring?

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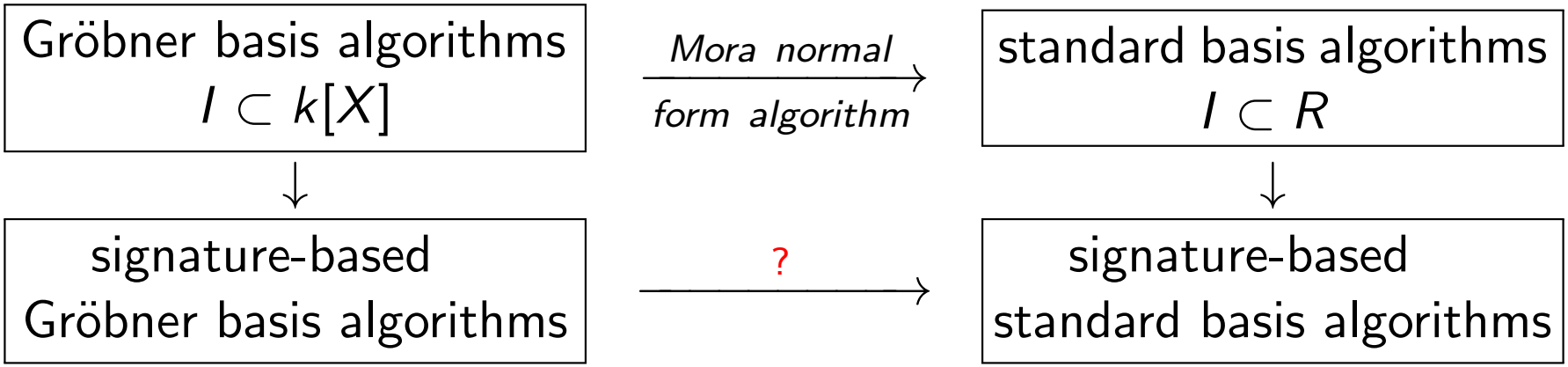

- **Original classical algorithm:**
  - H. Hironaka: Resolution of singularities of an algebraic variety over a field of characteristic zero, 1964. (standard basis)
  - B. Buchberger: Ein Algorithmus zum Auffinden der Basiselemente des Restklassenrings nach einem nulldimensionalen Polynomideal, 1965. (Buchberger's algorithm)
  - F. Mora: An algorithm to compute the equations of tangent cones, 1982. (Mora's algorithm)
  - D. Lazard: Gröbner bases, Gaussian elimination and resolution of systems of algebraic equations, 1983. (Lazard's homogenization approach)

- **Signature-based Gröbner basis algorithms**
  - J.-C. Faugère: A new efficient algorithm for computing Gröbner bases without reduction to zero (F5), 2002. (F5 algorithm)
  - S.H. Gao, F. Volny IV, and M.S. Wang : A new framework for computing Gröbner bases, 2010. (GVW algorithm)

signature-based standard basis algorithms

Define a subset in $R^m \times R$:

$$M = \{(\mathbf{u}, v) \in R^m \times R : \mathbf{u} \cdot \mathbf{f} = v, \mathbf{u} \in R^m\}$$

where $\mathbf{f} = (f_1, \ldots, f_m) \in (k[X])^m$.

- M is a R-submodule in $R^m \times R$.
- M is generated by $(\mathbf{e}_1, f_1), \ldots, (\mathbf{e}_m, f_m)$.

Fix the local order $\prec_1$ in R, and the module order $\prec_2$ in R^m.

For convenience, denote by $\prec$ with no confusion.

signature: $p = (\mathbf{u}, v) \in M$, $s(p) = \mathrm{lm}(\mathbf{u})$.

Example 2

For local order $\prec$, $p = (\mathbf{u}, v) = ((x^2 + x^3, x^5 - 2x^7), x^4 + 2x^7)$

$\mathrm{lm}(v) = x^4, \quad s(p) = \mathrm{lm}(\mathbf{u}) = x^2\mathbf{e}_1 = (x^2, 0).$

signature-based Gröbner basis algorithms	$\xrightarrow{?}$	signature-based standard basis algorithms
global order($X^\alpha \succ 1$)	$\longrightarrow$	local order($X^\alpha \prec 1$)

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global order($X^\alpha \succ 1$) $\longrightarrow$ local order($X^\alpha \prec 1$)

?: local orders are not well-orderings.

- 1. there may not be a minimal element in an infinite set; (correctness)
- 2. the top-reduction steps may not terminate in the local ring. (termination)

1. a minimal element problem related to correctness

For any $(\mathbf{u}_0, v_0) \in M$, we consider the set

$$L(\mathrm{lm}(v_0)) = \{\mathrm{lm}(\mathbf{u}) : (\mathbf{u}, v) \in M \quad \text{and} \quad \mathrm{lm}(v) = \mathrm{lm}(v_0)\}.$$

$L(\mathrm{lm}(v_0))$ is a nonempty set.

Question: Does $L(\mathrm{lm}(v_0))$ have a minimal element ?

Example 3

$R = k[x_1, x_2]_{\langle x_1, x_2\rangle}$, $\mathbf{f} = (x_1, x_2)$, $M = \{(\mathbf{u}, v) : \mathbf{u} \cdot \mathbf{f} = v\}$;
$\prec_1$: *an anti-graded lex order with* $x_2 \prec_1 x_1$ *in* R;
$\prec_2$: *a POT order with* $\mathbf{e}_2 \prec_2 \mathbf{e}_1$ *in* R^2;

$p_0 = (\mathbf{u}_0, v_0) = ((x_1, x_1 + 1),\ x_1^2 + x_1x_2 + x_2) \in M$;
$p_1 = (\mathbf{u}_1, v_1) = ((x_1^2, x_1 + 1),\ x_1^3 + x_1x_2 + x_2) \in M$;
.....
$p_i = (\mathbf{u}_i, v_i) = ((x_1^{1+i}, x_1 + 1),\ x_1^{2+i} + x_1x_2 + x_2) \in M$;
.....

$L(\mathrm{lm}(v_0)) = L(x_2) \supseteq \{x_1^i \mathbf{e}_1 : i \in \mathbb{Z}_{\geq 1}\}$ has not a minimal element.

But, at the following case:

Lemma 4

Let $\prec_1$ be an anti-graded order in R, and $\prec_2$ be a TOP order in R^m, where $\prec_2$ is compatible with $\prec_1$. Then for any $(\mathbf{u}_0, v_0) \in M$, $L(\mathrm{lm}(v_0))$ has a minimal element.

2. the reduction problem related to termination

the top-reduction steps may not terminate in the local ring.

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Example 5

Given the anti-graded order and $\mathbf{e}_2 \prec \mathbf{e}_1$;
$p_1 = (\mathbf{e}_1,\ x)$; $p_2 = (\mathbf{e}_2,\ x - x^2)$;

the top-reduction steps:

$$p_3 = Red(p_1, p_2) = p_1 - p_2 = (\mathbf{e}_1 - \mathbf{e}_2,\ x^2);$$
$$p_4 = Red(p_3, p_2) = p_3 - xp_2 = (\mathbf{e}_1 - (1+x)\mathbf{e}_2,\ x^3);$$
$$p_5 = Red(p_4, p_2) = p_4 - x^2p_2 = (\mathbf{e}_1 - (1+x+x^2)\mathbf{e}_2,\ x^4);$$
$$p_6 = Red(p_5, p_2) = p_5 - x^3p_2 = (\mathbf{e}_1 - (1+x+x^2+x^3)\mathbf{e}_2,\ x^5);$$
$$\cdots\cdots$$

Theorem 6

Let $G = \{p_1 = (\mathbf{u}_1, f_1), \ldots, p_s = (\mathbf{u}_s, f_s)\} \subset (k[X])^m \times k[X]$ *and* $p = (\mathbf{u}, f)$. *Then there is an algorithm for producing polynomials* $h, a_1, \ldots, a_s$ *in* $k[X]$ *and* $r = (\mathbf{w}, v)$ *in* $(k[X])^m \times k[X]$ *such that*

$$hp = a_1 p_1 + \cdots + a_s p_s + r,$$

where $\mathrm{lm}(h) = 1$, $\mathrm{lm}(a_i f_i) \preceq \mathrm{lm}(f)$, $\mathrm{lm}(a_i \mathbf{u}_i) \preceq \mathrm{lm}(\mathbf{u})$, $\mathrm{lm}(\mathbf{w}) = \mathrm{lm}(\mathbf{u})$, *and either* $v = 0$ *or* $\mathrm{lm}(f_i) \nmid \mathrm{lm}(v)$.

$r = \overline{p}^G$: **the remainder** of p regularly top-reduced by G

Definition 7 (Strong standard bases)

Let G be a finite subset of M. If for any nonzero $(\mathbf{u}, v) \in M$, it is top-reducible by some element in G, Then G is called a **strong standard basis** *for M.*

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$$G \text{ is a strong standard basis for } M$$
$$\Downarrow$$
$$V = \{v : (\mathbf{u}, v) \in G\} \text{ is a standard basis for ideal } I \text{ in } R$$

J-pair: (similar to S-polynomial or S-pair) $\longrightarrow$ Jonit pair

$p_1 = (\mathbf{u}_1, v_1), p_2 = (\mathbf{u}_2, v_2) \in M$, and $v_1 v_2 \neq 0$.

$t = \mathrm{lcm}(\mathrm{lm}(v_1), \mathrm{lm}(v_2))$, $t_1 = t/\mathrm{lm}(v_1)$, $t_2 = t/\mathrm{lm}(v_2)$

$T = \max\{t_1 \mathrm{lm}(\mathbf{u}_1), t_2 \mathrm{lm}(\mathbf{u}_2)\} = t_1 \mathrm{lm}(\mathbf{u}_1)$.

$$t_1 p_1 - c t_2 p_2 = (t_1 \mathbf{u}_1 - c t_2 \mathbf{u}_2, t_1 v_1 - c t_2 v_2),$$

If

$$\mathrm{lm}(t_1 \mathbf{u}_1 - c t_2 \mathbf{u}_2) = T, \qquad (\textit{regular})$$

then $t_1 p_1$ is called the **J-pair** of p_1 and p_2.

cover:

a pair $(\mathbf{u}, v) \in M$ is covered by $G \subset M$, if $\exists\ (\mathbf{u}_i, v_i) \in G$, s.t. $\mathrm{lm}(\mathbf{u}_i) \mid \mathrm{lm}(\mathbf{u})$, $t = \mathrm{lm}(\mathbf{u})/\mathrm{lm}(\mathbf{u}_i)$, and $t\mathrm{lm}(v_i) \prec_1 \mathrm{lm}(v)$

$$p = (\mathbf{u}, v)$$
$$tp_i = (t\mathbf{u}_i, tv_i)$$

Example 8

$p = (\mathbf{u}, v) = ((x_1^2 x_2,\ x_1^4 x_2),\ x_2^2)$;
$p_i = (u_i, v_i) = ((x_1 x_2,\ x_2^3),\ x_1^2 + x_2^3)$.

$x_1 p_i = (u_i, v_i) = ((x_1^2 x_2,\ x_1 x_2^3),\ x_1^3 + x_1 x_2^3)$ and $x_1^3 \prec x_2^2$.

main theorem

Theorem 9 (Cover Theorem)

Let $G \subset (k[X])^m \times k[X]$ be a finite subset of M such that, for any term $T \in R^m$, there is a pair $(\mathbf{u}, v) \in G$ and a monomial t such that $T = t\,\mathrm{lm}(\mathbf{u})$. Then the following are equivalent:

1. *G is a strong standard basis for M; (the corresponding v part is a standard basis)*
2. *every J-pair of G is covered by G.*

Priori criterion with "Signature"

Cover Criterion:

Any J-pair that is covered by G can be discarded without performing any reductions

Cover Criterion includes:

Syzygy Criterion, Signature Criterion and Rewrite Criterion

▶ GVW algorithm in local ring

Input: $F = \{f_1, \ldots, f_m\} \subset k[X]$
Output: V — a standard basis for $\langle f_1, \ldots, f_m \rangle \subset R$

1 $G := \{(\mathbf{e}_1, f_1), \ldots, (\mathbf{e}_m, f_m)\}$; $H := \{\mathrm{lm}(f_i\mathbf{e}_j - f_j\mathbf{e}_i)\}$; $JP := \{\text{J-pairs of } G\}$;
2 **while** $JP \neq \emptyset$ **do**
3 choose $(T, v) \in JP$, and $JP := JP \setminus \{(T, v)\}$;
4 **if** (T, v) is covered by G or H **then** next; $\longrightarrow$Cover Criterion
5 **else**
6 $(T_0, v_0) := \overline{(T, v)}^G$;
7 **if** $v_0 = 0$ **then** $H := H \cup \{T_0\}$;
8 **else**
9 $JP := JP \cup \{\text{J-pairs between } (T_0, v_0) \text{ and } G\}$; $\longrightarrow$Cover Criterion
$H := H \cup \{\mathrm{lm}(v_0 T_j - v_j T_0)\}$;
$G := G \cup \{(T_0, v_0)\}$;
10 **end if**
11 **end while**
12 **return** $V := \{v \mid (T, v) \in G\}$

An Example

Example 10

Given $\prec_1$ be the anti-graded reverse lex order, and $\prec_2$ is a TOP order in R^3 and compatible with $\prec_1$,

$x_1 \succ x_2 \succ x_3, \quad \mathbf{e}_1 \succ \mathbf{e}_2 \succ \mathbf{e}_3$

$R = \mathbb{C}[x_1, x_2, x_3]_{\langle x_1, x_2, x_3\rangle}$, $I = \langle f_1, f_2, f_3\rangle \subset R$,

$f_1 = x_1^2 - 5x_2x_3 - 2x_2^2x_3, f_2 = 2x_1x_2 + 2x_2^3 - x_3^3, f_3 = -x_1x_2 + x_2x_3^2.$

Compute a standard basis for I.

An Example

Initial:

$G_0 := \{(\mathbf{e}_1, f_1), (\mathbf{e}_2, f_2), (\mathbf{e}_3, f_3)\}$;

$H_0 := \{x_1^2\mathbf{e}_2, x_1^2\mathbf{e}_3, x_1x_2\mathbf{e}_2\}$; the signature of principle syzygies

$JP_0 := \{(T_1, v_1), (T_2, v_2), (T_3, v_3)\}$

$= \{(x_1\mathbf{e}_3, x_1f_3), (x_1\mathbf{e}_2, x_1f_2), (\mathbf{e}_2, f_2)\}$; the J-pairs set of G_0

An Example

First cycle:

- Select the J-pair $(T_1, v_1) = (x_1\mathbf{e}_3, x_1 f_3)$ from JP_0;
- (T_1, v_1) is not covered by G_0 or H_0;
- $p_4 = (T_1, \widetilde{v_1}) = \overline{(T_1, v_1)}^{G_0} = (x_1\mathbf{e}_3, -5x_2^2x_3 + x_1x_2x_3^2 - 2x_2^3x_3)$;
- $JP_1 := \{(T_2, v_2), (T_3, v_3)\}$
  $= \{(x_1\mathbf{e}_3, x_1 f_3), (x_1\mathbf{e}_2, x_1 f_2), (\mathbf{e}_2, f_2)\}$ Cover Criterion
  $H_1 := H_0 = \{x_1^2\mathbf{e}_2, x_1^2\mathbf{e}_3, x_1x_2\mathbf{e}_2\}$ Cover Criterion
  $G_1 := G_0 \cup \{p_4\}$.

An Example

Sixth cycle:

$$JP_5 := \{(T_7, v_7), (T_5, v_5)\} = \{(x_1x_3\mathbf{e}_2, x_1\tilde{v}_4), (x_1\mathbf{e}_2, x_1\tilde{v}_3)\}$$
$$G_5 := G_4 \cup \{p_8\} = \{(\mathbf{e}_1, f_1), (\mathbf{e}_2, f_2), (\mathbf{e}_3, f_3), p_4, p_5, p_6, p_7, p_8\}$$

- Select $(T_7, v_7) = (x_1x_3\mathbf{e}_2, 2x_1x_2x_3^3 + *)$ from JP_5 ;
- (T_7, v_7) is covered by $p_5 = (x_1\mathbf{e}_2, -x_1x_3^3 + *)$; Cover Criterion

$$(T_7, v_7) = (x_1x_3\mathbf{e}_2, 2x_1x_2x_3^3 + *)$$
$$x_3p_5 = (x_1x_3\mathbf{e}_2, -x_1x_3^4 + *)$$

Discard (T_7, v_7) , and reselect another J-pair to continue the cycle.

An Example

Continue: $\cdots$

Discard: 23 J-pairs by using Cover Criterion (including: Syzygy Criterion, Signature Criterion and Rewrite Criterion)

Do: 5 regular top-reductions

the standard basis of I in $R = \{f_1, f_2, f_3, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4, \tilde{v}_5\}$.

Implementation

Table: examples

ideal	signature-based method(our)			classical method		
	J-pairs	discard	discard/J-pairs	S-polys	discard	discard/S-polys
I_1	21	14	67%	28	6	21%
I_2	21	14	67%	21	9	43%
I_3	15	12	80%	15	8	53%
I_4	20	16	80%	21	10	48%
I_5	15	9	60%	15	4	27%
I_6	20	16	80%	21	6	29%
I_7	14	11	79%	15	4	27%
I_8	35	29	83%	28	9	32%
I_9	10	7	70%	15	6	40%
I_{10}	21	17	81%	66	28	43%

- We implement the two algorithms in *Maple*, and the codes and examples are available on the web:
  `http://www.mmrc.iss.ac.cn/~dwang/software.html`.

Conclusion

- Propose an efficient algorithm to compute the standard bases in local ring. (signature-based algorithm)
- Solve two key problems:
  - an infinite set may have not a minimal element in local ring. $\longrightarrow$ the signature set $L(\mathrm{lm}(v_0))$ w.r.t. v_0 has a minimal element (anti-graded order and TOP order)
  - the general division algorithm may not terminate in local ring. $\longrightarrow$ extend Mora normal form algorithm to do regular top-reduction (signature-based case)

Thanks for your attention!