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Towards a Direct Method for Finding Hypergeometric Solutions of Linear First Order Recurrence Systems

Johannes Middeke   Carsten Schneider
Johannes Kepler University
Research Institute for Symbolic Computation (RISC)
Altenbergerstraße 69, 4040 Linz, Austria
jmiddeke@risc.jku.at   cschneid@risc.jku.at

Abstract

We establish a connection between the hypergeometric solutions of a first order linear recurrence systems and the determinant of the system matrix. This enables us to find hypergeometric solutions for systems in a way similar to the scalar case. Our result works in the in the single basic and in the multibasic case.

1 Introduction

We consider a difference field $(\mathbb{K}(x_1,\dots,x_n),\sigma)$ where $\mathbb{K}$ is a field of characteristic 0, $\sigma\colon \mathbb{K}\to\mathbb{K}$ is an automorphism of $\mathbb{K}$ and where we extend σ to the rational function field in the n variables $x_1,\dots,x_n$ by letting $\sigma(x_j)=\alpha_j x_j+\beta_j$ with $\alpha_j,\beta_j\in\mathbb{K}$, $\alpha_j\neq 0$ and $(\alpha_j,\beta_j)\neq(1,0)$ for $j=1,\dots,n$. If $n\geq 2$, we will refer to this setting as the *multibasic* case; else, if $n=1$, we are in the *single basic* case.

Let $\mathbb{L}\supseteq\mathbb{K}(x_1,\dots,x_n)$ be a difference field extension. We say that $\gamma\in\mathbb{L}$ is *hypergeometric* over $\mathbb{K}(x_1,\dots,x_n)$ if $\sigma(\gamma)=h\gamma$ for some $h\in\mathbb{K}(x_1,\dots,x_n)^*$. A column vector $y\in\mathbb{L}^s$ is called hypergeometric if every component is either hypergeometric or zero.

We are looking for non-zero hypergeometric solutions y of the first order linear recurrence system

$$\sigma(y)=Ay \qquad \text{where} \quad A\in\mathbb{K}(x_1,\dots,x_n)^{s\times s},\quad \det(A)\neq 0 \tag{SYS}$$

and where the application of σ to the vector y is componentwise. A traditional approach for solving these systems is uncoupling [8, 4, 11, 7]; this yields a single higher order scalar equation whose hypergeometric solutions correspond to the hypergeometric solutions of the system. In fact, translating the solutions of the scalar equation back to the system will result in hypergeometric solutions of the shape $y=\gamma q$ where the scalar $\gamma\in\mathbb{L}$ is hypergeometric and $q\in\mathbb{K}(x_1,\dots,x_n)^s$ is a non-zero vector. Thus, it is sufficient to restrict ourselves to solutions of this form.

Our main result is the following theorem.

Theorem 1 *Let $d\in\mathbb{K}[x_1,\dots,x_n]\setminus\{0\}$ be such that $dA\in\mathbb{K}[x_1,\dots,x_n]^{s\times s}$ contains only polynomials. A non-zero vector $y=\gamma q$ is a hypergeometric solutions of* (SYS) *if and only if there exists $\lambda\in\mathbb{K}[x_1,\dots,x_n]$ with*

1. *$\lambda\mid\det(dA)$,*
2. *$\sigma(\gamma)=(\lambda/d)\gamma$ and*
3. *$q\in\mathbb{K}[x_1,\dots,x_n]^s$ is a polynomial vector with $\lambda\sigma(q)=(dA)q$.*

The theorem immediately suggests an algorithm to find hypergeometric solutions: For each divisor $\lambda \mid \det(dA)$ try to find the polynomial solutions q for the modified system $\lambda\sigma(q) = (dA)q$. Each such pair (λ, q) gives rise to a hypergeometric solution of the original system. Note that this is similar to the algorithm HYPER for scalar equations [10] in that we reduce the problem of finding hypergeometric solutions of a system to that of finding polynomial solutions of several related systems.

The main difference between HYPER and the system case considered here is that we do not yet have a good way to determine the possible leading coefficients of λ in the theorem. This means that we have to iterate over infinitely many divisors or introduce a new variable c for the leading coefficient to be determined alongside q. According to [5], this c can be found by an unpublished methods of theirs if $\mathbb{K}$ is constant w. r. t. σ, i. e., if $\sigma|_{\mathbb{K}} = \mathrm{id}$. Alternatively, we can set a degree bound for q and make an ansatz with unknown c. If $\mathbb{K}$ is constant field, then the ansatz this leads to an equation $Mv = 0$ where M contains linear polynomials in $\mathbb{K}[c]$. This allows us find all solutions up to the given degree bound. The latter approach seems to work well in practise and we have implemented it as a Mathematica package. Our implementation can easily deal with small systems, but it gets slow for larger systems or systems with many variables since the current code relies on the Smith normal form to determine for which c the system $Mv = 0$ is solvable.

2 Conclusion

We have shown how for first order linear recurrence systems (SYS) in the multibasic case $\mathbb{K}(x_1, \ldots, x_n)$ where we have multiple indeterminates there is a connection between the hypergeometric solutions and the divisors of the system matrix A. This connection can be exploited to yield an algorithm for computing all hypergeometric solutions in a way similar to existing algorithms for scalar equations. We have implemented the algorithm in Mathematica.

In the case of a single variable $\mathbb{K}(x)$, hypergeometric solutions can be found by uncoupling the system using, e. g., one of [8, 4, 11, 7] and computing hypergeometric solutions of the resulting higher order scalar equations using, e. g., [10, 9, 1, 6]. Recently, an alternative algorithm was given in [2, 3] where the authors provide a way of deriving scalar equations from a system which is cheaper than the traditional uncoupling method.

The appealing new feature of the approach presented in this contribution is on the one hand its simplicity and on the other hand and perhaps more importantly the fact that it works with arbitrarily many variables $x_1, \ldots, x_n$.

References

[1] Sergei A. Abramov, Peter Paule, and Marko Petkovšek, *q-hypergeometric solutions of q-difference equations*, Discrete Mathematics (1998), no. 180, 3–22.

[2] Sergei A. Abramov, Marko Petkovšek, and Anna A. Ryabenko, *Hypergeometric solutions of first-order linear difference systems with rational-function coefficients*, pp. 1–14, Springer International Publishing, 2015.

[3] ———, *Resolving sequences of operator for linear ordinary differential and difference systems of arbitrary order*, Computational Mathematics and Mathematical Physics **56** (2016), no. 5, 894–910.

[4] Moulay Barkatou, *An algorithm for computing a companion block diagonal form for a system of linear differential equations*, Appl. Algebra Engrg. Comm. Comput. **4** (1993), no. 3, 185–195.

[5] Moulay Barkatou and Mark van Hoeij, personal communication.

[6] A. Bauer and M. Petkovšek, *Multibasic and mixed hypergeometric Gosper-type algorithms*, J. Symbolic Comput. **28** (1999), no. 4–5, 711–736.

[7] Alin Bostan, Frédéric Chyzak, and Élie de Panafieu, *Complexity estimates for two uncoupling algorithms*, Proceedings of ISSAC'13 (Boston), June 2013.

[8] M. Danilevski, A. *The numerical solutions of the secular equation (russian)*, Matem. Sbornik **44** (1937), no. 2, 169–171.

[9] M. van Hoeij, *Finite singularities and hypergeometric solutions of linear recurrence equations*, J. Pure Appl. Algebra **139** (1999), no. 1-3, 109–131.

[10] Marko Petkovšek, *Hypergeometric solutions of linear recurrences with polynomial coefficients*, Journal of Symbolic Computation (1992), no. 14, 243–264.

[11] B. Zürcher, *Rationale Normalformen von pseudo-linearen Abbildungen*, Master's thesis, ETH Zürich, 1994.