Optimization Problems over Noncompact Semialgebraic Sets

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Problem Statements

Given a basic semialgebraic set

$$S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\},$$

where $g_i(X) \in \mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n],\ i = 1, \ldots, m$.

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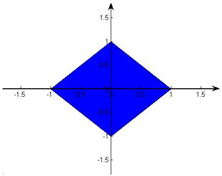
We consider the problem of optimizing a **linear function** over S:

$$c_0^* := \sup_{x \in S} \quad \mathbf{c}^T x = c_1 x_1 + \cdots + c_n x_n,$$

where $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$.

Linear Programming & Semidefinite Programming

- $S := \{x \in \mathbb{R}^n \mid Ax \geq b\}$ is a ***polyhedron*** $\longrightarrow$ Linear Programming



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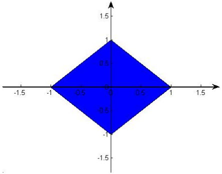

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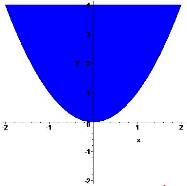
- $S := \{x \in \mathbb{R}^n \mid A_0 + \sum_{i=1}^n A_i x_i \succeq 0\}$ is a ***spectrahedron*** $\longrightarrow$ Semidefinite Programming



$$S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 - x_1^2 \geq 0\} = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid \begin{bmatrix} x_2 & x_1 \\ x_1 & 1 \end{bmatrix} \succeq 0\right\}.$$

Outlines

- Semidefinite representations of the ***closure of the convex hull*** of S:

$$\mathrm{cl}(\mathrm{co}(S)) := \bigcap_{p \in \mathbb{R}[X]_1, p|_S \geq 0} \{x \in \mathbb{R}^n \mid p(x) \geq 0\}.$$

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- Optimizing a **parametric** linear function over a real algebraic variety:
  - $c_0^* = \sup_{x \in S} \mathbf{c}^T x$ for ***unspecified parameters***;
  - $S = \mathcal{V} \cap \mathbb{R}^n$, $\mathcal{V} = \{v \in \mathbb{C}^n \mid h_1(v) = \cdots = h_p(v) = 0\}$.

Semidefinite Representations of Semialgebraic Sets

Let $S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$ be a semialgebraic set, then

$$\sup_{x \in S} \quad \mathbf{c}^T x \longleftrightarrow \sup_{x \in \mathrm{cl}(\mathrm{co}(S))} \quad \mathbf{c}^T x$$

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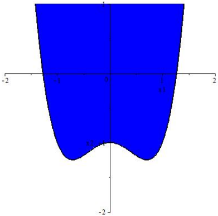
Example: $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 - (x_1^4 - x_1^2 - 1) \geq 0\}$:



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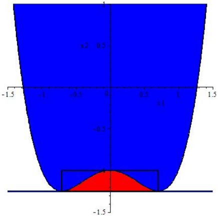

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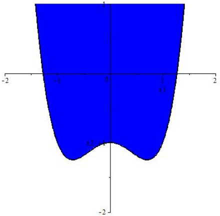

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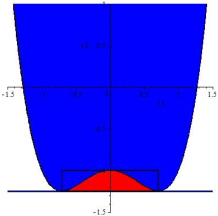

$\mathrm{cl}\,(\mathrm{co}(S))$

The Goal

is to characterize $\mathrm{cl}\,(\mathrm{co}(S))$ such that

optimization problems $\longrightarrow$ **semidefinite programs**.

Semidefinite Representation of $\mathrm{cl}\,(\mathrm{co}(S))$

- A set $S \subset \mathbb{R}^n$ is a ***spectrahedron*** if it has the form

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$$S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (-1)^{(i+\delta)} m_i \geq 0,\ i = 0, \ldots, \delta - 1\},$$

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- A set $S \subset \mathbb{R}^n$ is a ***projected spectrahedron*** if it has the form

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \exists (y_1, \dots, y_m) \in \mathbb{R}^m, A_0 + \sum_{i=1}^{n} A_i x_i + \sum_{j=1}^{m} B_j y_j \succeq 0\},$$

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The TV Screen

defined by $\{(x_1, x_2) \mid 1 - x_1^4 - x_2^4 \geq 0\}$ is a **projected spectrahedron**:

$$\left\{(x_1, x_2) : \exists (y_1, y_2), \mathrm{diag}\left(\begin{bmatrix} 1 + y_1 & y_2 \\ y_2 & 1 - y_1 \end{bmatrix}, \begin{bmatrix} 1 & x_1 \\ x_1 & y_1 \end{bmatrix}, \begin{bmatrix} 1 & x_2 \\ x_2 & y_2 \end{bmatrix}\right) \succeq 0\right\}.$$

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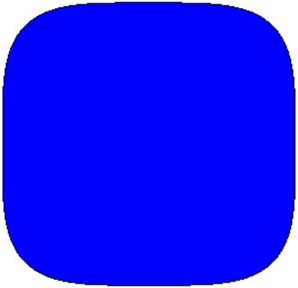
But it is not a **spectrahedron** [Helton and Vinnikov].



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Theoretical Results

- The Helton & Nie conjecture: **every** convex semialgebraic set has a **semidefinite representation** [Helton, Nie].

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- **Theta body** approximations of $\mathrm{cl}\,(\mathrm{co}(S))$ [Gouveia, Parrilo, Thomas].
- **Lasserre's semidefinite** relaxations of $\mathrm{cl}\,(\mathrm{co}(S))$ [Lasserre].
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Characterizing $p|_S \geq 0$ by Sums of Squares of Polynomials

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Definition

Given multivariate polynomials $G = \{g_1, \ldots, g_m\}$, the ***quadratic module*** generated by the g_i is the set

$$\mathcal{Q}(G) := \left\{\sigma_0 + \sigma_1 g_1 + \cdots + \sigma_m g_m, \text{ where } \sigma_0, \ldots, \sigma_m \in \Sigma^2\right\}.$$

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- we have $p \in \mathcal{Q}(G) \Longrightarrow p|_S \geq 0$;
- when do we have $\mathbf{p|_S \geq 0 \Longrightarrow p \in \mathcal{Q}(G)}$?

Putinar's Positivstellensatz

- The k***-th quadratic module*** of G is defined as

$$\mathcal{Q}_k(G) := \left\{ \sum_{j=0}^{m} \sigma_j g_j \;\middle|\; g_0 = 1,\ \sigma_j \in \Sigma^2,\ \deg(\sigma_j g_j) \le 2k, 0 \le j \le m \right\}.$$

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Theta Bodies: $p|_S \geq 0 \longleftarrow p \in \mathcal{Q}_k(G)$

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See [Gouveia, Parrilo, Thomas] for $V_{\mathbb{R}}(I)$ being **compact**.

Example [Gouveia, Thomas]

Given two curves cut out by $g_1 = X_1^4 - X_2^2 - X_3^2$, $g_2 = X_1^4 + X_1^2 + X_2^2 - 1$.
Its first theta body is an ellipsoid

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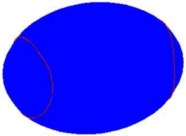

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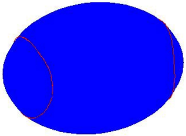
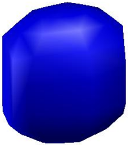
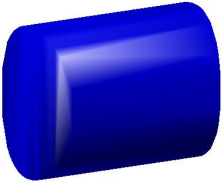


The second and third theta bodies:





Lasserre's Semidefinite Relaxations of $\mathrm{cl}\,(\mathrm{co}(S))$

Given $y = \{y_\alpha\}$, let $\mathscr{L}_y : \mathbb{R}[X] \to \mathbb{R}$ be the linear functional

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Moment matrix $M_k(y)$

with rows and columns indexed in the basis X^α

$$M_k(y)(\alpha,\beta) := \mathscr{L}_y(\mathbf{X}^\alpha \mathbf{X}^\beta) = \mathbf{y}_{\alpha+\beta},\ \ \alpha,\beta \in \mathbb{N}_k^n, |\alpha|, |\beta| \le k.$$

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For instance, in $\mathbb{R}^2$

$$\begin{bmatrix} 1 \\ X_1 \\ X_2 \end{bmatrix} \begin{bmatrix} 1 & X_1 & X_2 \end{bmatrix} = \begin{bmatrix} 1 & X_1 & X_2 \\ X_1 & X_1^2 & X_1X_2 \\ X_2 & X_1X_2 & X_2^2 \end{bmatrix} \mapsto M_1(y) = \begin{pmatrix} y_{00} & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{pmatrix}$$

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$$\begin{bmatrix} 1 \\ X_1 \\ X_2 \end{bmatrix} \begin{bmatrix} 1 & X_1 & X_2 \end{bmatrix} = \begin{bmatrix} 1 & X_1 & X_2 \\ X_1 & X_1^2 & X_1 X_2 \\ X_2 & X_1 X_2 & X_2^2 \end{bmatrix} \mapsto M_1(y) = \begin{pmatrix} y_{00} & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{pmatrix}$$

- We have $\mathbf{M_k(y)} \succeq \mathbf{0} \iff \mathscr{L}(\mathbf{h^2}) \ge \mathbf{0},\ \forall\ \mathbf{h} \in \mathbb{R}[\mathbf{X}]_\mathbf{k}$

Lasserre's Semidefinite Relaxations of $\mathrm{cl}\,(\mathrm{co}(S))$

Given a polynomial $p(X) = \sum_\gamma p_\gamma \mathbf{X}^\gamma$, let $d_p = \lceil \deg(p)/2 \rceil$.

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with rows and columns indexed in the basis X^α

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We have

- $\mathbf{M_{k-d_p}(py) \succeq 0 \iff \mathscr{L}_y(h^2 p) \ge 0,\ \forall\ h \in \mathbb{R}[X], \deg(h) \le k - d_p}$.

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Let $G = \{g_1, \ldots, g_m\}$, $s(k) := \binom{n+k}{n}$ and $k_j := \lceil \deg g_j / 2 \rceil$.

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The k-***th Lasserre's relaxation*** is defined as:

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- $\mathrm{co}(S) \subseteq \Omega_k(G) \subseteq \mathrm{TH}_k(G)$. If $\mathcal{Q}_k(G)$ is **closed**, $\mathrm{TH}_k(G) = \mathrm{cl}(\Omega_k(G))$.

When S is not Compact

Consider the basic semialgebraic set

$$S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_1^2 - x_2^3 \geq 0\}.$$

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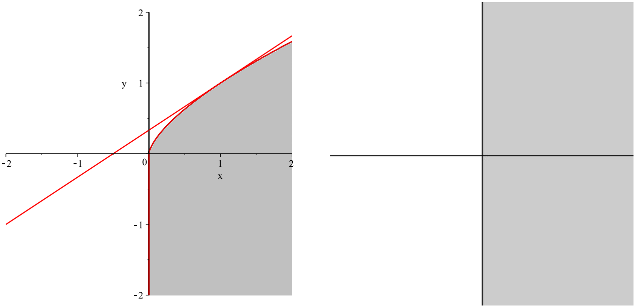

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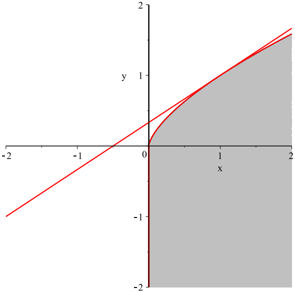

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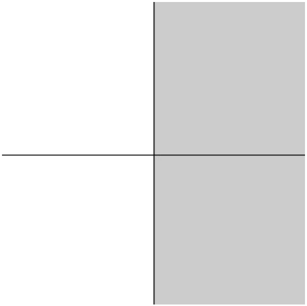

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Semidefinite Representation of a Noncompact Set S

- Homogenization

$$\tilde{f}(\widetilde{X}) = X_0^{\deg(f)} f(X/X_0) \in \mathbb{R}[X_0, X_1, \ldots, X_n] = \mathbb{R}[\widetilde{X}].$$

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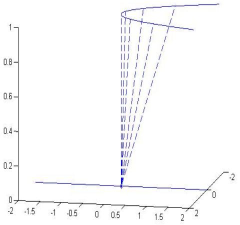
$\widetilde{S}_2 \backslash \mathrm{cl}\left(\widetilde{S}_1\right) = \{(0, 0, x_2) \in \mathbb{R}^3 \mid \mathbf{x_2 < 0}\} \Longrightarrow S$ is **not closed** at ∞.



Modified Lasserre's Hierarchy and Theta Body

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Modified Theta Body

$$\widetilde{\mathrm{TH}}_k(\widetilde{G}) := \{x \in \mathbb{R}^n \mid \tilde{l}(1, x) \geq 0, \quad \forall\, \tilde{l} \in \mathcal{Q}_k(\widetilde{G}) \cap \mathbb{P}[\widetilde{X}]_1\}.$$

where $\mathbb{P}[\widetilde{X}]_1$ is a set of homogeneous polynomials of degree one in $\mathbb{R}[\widetilde{X}]$.

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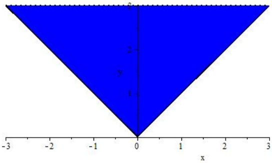

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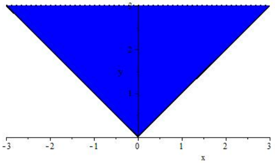

- K is pointed $\iff$ $\{\, \exists \mathbf{c} \in \mathbb{R}^n$ s.t. $\langle \mathbf{c}, x \rangle > 0$ for all $x \in K \backslash \{0\} \,\}$,

Remark: $\langle \mathbf{c}, x \rangle = \mathbf{c}^T x = c_1 x_1 + \cdots + c_n x_n$.

Modified Lasserre's Hierarchy and Theta Body

Assumptions [Guo, Wang, Zhi]

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Consider the set $S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_1^2 - x_2^3 \geq 0\}$, we have

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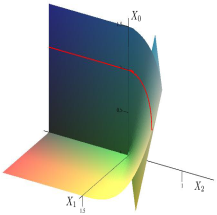

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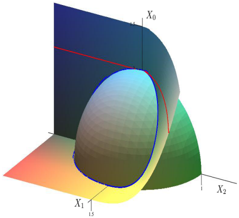

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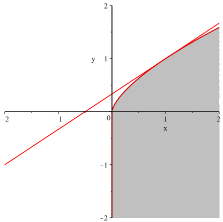

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The convergence might fail

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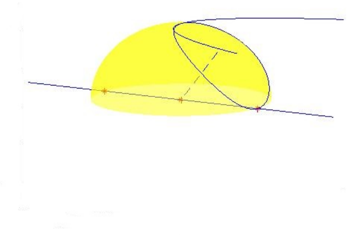

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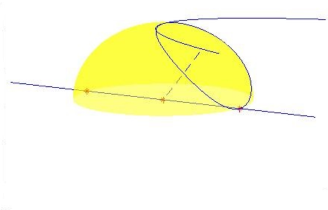

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We notice that the property of closedness at ∞ depends not only on S but also on its **generators**.

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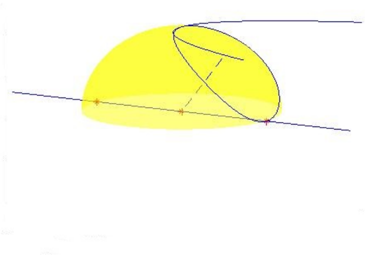

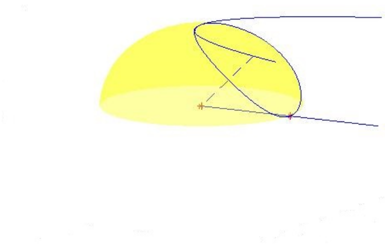

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- the assumptions of **pointedness** and **closedness at infinity** are essential.

Outlines

- Semidefinite representations of the closure of the convex hull of S:

$$\mathrm{cl}(\mathrm{co}(S)) := \bigcap_{p \in \mathbb{R}[X]_1, p|_S \geq 0} \{x \in \mathbb{R}^n \mid p(x) \geq 0\}.$$

- Optimizing a **parametric** linear function over a real algebraic variety:
  - $c_0^* = \sup_{x \in S} \mathbf{c}^T x$ for ***unspecified parameters***;
  - $S = \mathcal{V} \cap \mathbb{R}^n$, $\mathcal{V} = \{v \in \mathbb{C}^n \mid h_1(v) = \cdots = h_p(v) = 0\}$.

Optimizing a Parametric Linear Function

We consider the optimization problem:

$$c_0^* := \sup_{x \in \mathcal{V} \cap \mathbb{R}^n} \quad \mathbf{c}^T x = c_1 x_1 + \cdots + c_n x_n.$$

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The problem

is how to compute a polynomial $\Phi \in \mathbb{R}[c_0, \mathbf{c}]$ s.t. c_0^* can be obtained by solving $\Phi(c_0, \gamma) = 0$ for a generic $\gamma \in \mathbb{R}^n$?

Previous Work

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Our goal

is to compute Φ for $\mathcal{V} \cap \mathbb{R}^n$ being **nonsmooth** or **noncompact**.

Compact Cases

The ***dual variety*** $\mathcal{V}^*$ is the **Zariski closure** of the set

$$\{u \in \mathbb{P}^n \mid u \text{ lies in the row space of } \mathsf{Jac}(\mathcal{V}) \text{ at } x \in V_{\mathsf{reg}}\}.$$

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Suppose $\mathcal{V} = \{v \in \mathbb{C}^n \mid h_1(v) = \cdots = h_p(v) = 0\}$ is smooth and J is the ideal generated by using ***KKT conditions***:

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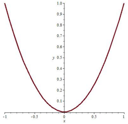
- If $\mathcal{V}$ is **irreducible**, **smooth** and **compact** in $\mathbb{R}^n$, then $\mathcal{V}^*$ is defined by an **irreducible** polynomial $\Phi(-c_0, c_1, \ldots, c_n) = 0$.

Noncompact Cases

- The optimal value c_0^* could be **infinite**, e.g. $h_1 = X_2 - X_1^2$, $c_0 = c_1X_1 + c_2X_2$ has a **finite maximum** value only when $c_2 < 0$.



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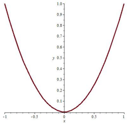
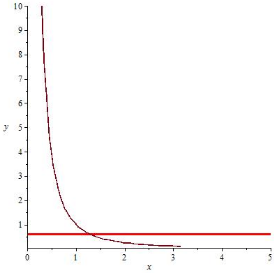


- The maximal value c_0^* could be **unattainable**, e.g. $h_1 = X_2X_1^2 - 1$, $c_0 = -X_2$ has maximum value 0 which is **not attainable**.



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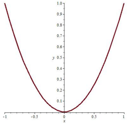
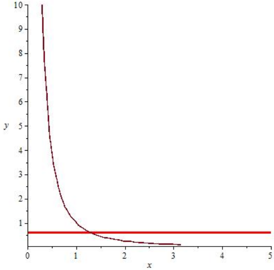
- The optimal value c_0^* could be **infinite**, e.g. $h_1 = X_2 - X_1^2$, $c_0 = c_1 X_1 + c_2 X_2$ has a **finite maximum** value only when $c_2 < 0$.



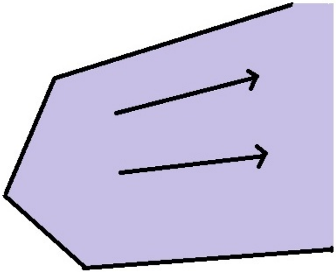
- The maximal value c_0^* could be **unattainable**, e.g. $h_1 = X_2 X_1^2 - 1$, $c_0 = -X_2$ has maximum value 0 which is **not attainable**.



- Do we still have similar results as in [Rostalski, Sturmfels]?

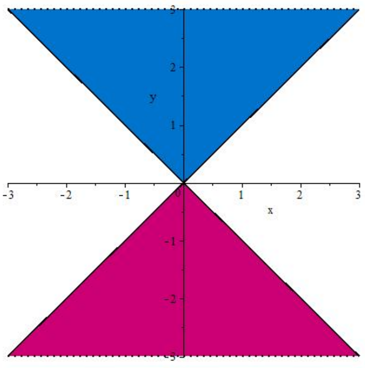
The Recession Cone

The ***recession cone*** 0^+C of a convex set C is the collection of all vectors y satisfying $x + \lambda y \in C$ for every $\lambda > 0$ and $x \in C$.



The Polar of a Convex Cone

Let K be a convex cone, then $K^o = \{\mathbf{c} \in \mathbb{R}^n \mid \langle \mathbf{c}, x\rangle \leq 0$ for all $x \in K\}$,



K and K^o

Noncompact Pointed Cases

Let $\operatorname{dom}(c_0^*)$ be the collection of $\gamma \in \mathbb{R}^n$ such that $c_0^*(\gamma)$ is **finite** on C.

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(a) $(0^+C)^\circ$ is an **n-dimensional** convex set;

(b) $\operatorname{int}((0^+C)^\circ) \subseteq \operatorname{dom}(c_0^*) \subseteq (0^+C)^\circ$. Moreover, $c_0 = c^T X$ can **attain** its maximum value on C for **every** vector $c \in \operatorname{int}((0^+C)^\circ)$.

Example

Consider $C = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq x_1^2\}$ and $c_0 = c_1 X_1 + c_2 X_2$.

$$0^+C = \{(x_1, x_2) \mid x_1 = 0, x_2 \geq 0\},$$
$$(0^+C)^\circ = \{(c_1, c_2) \mid c_2 \leq 0\},$$
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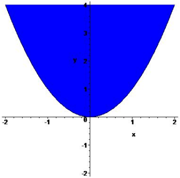
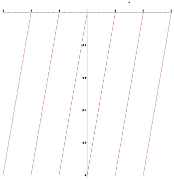
$\Longrightarrow$ c_0 has a finite maximum value on C when $c_2 < 0$.



Graph of C



Graph of $\operatorname{dom}(c_0^*)$

Extending Rostalski-Sturmfels' Results for Pointed Cases

Theorem [Guo, Safey El Din, Wang, Zhi]

Let $\mathcal{V}^* \subset (\mathbb{P}^n)^*$ be the dual variety to the projective closure of $\mathcal{V}$ and $C = \mathrm{cl}\,(\mathrm{co}(\mathcal{V} \cap \mathbb{R}^n))$. If $\mathcal{V}$ is **irreducible**, **smooth** and 0^+C is **pointed**, then

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- $(-c_0^* : \gamma_1 : \cdots : \gamma_n) \in \mathcal{V}^*$ for every $\gamma \in \mathrm{dom}\,(c_0^*)$, hence
$$\dim(\mathcal{V}^*) = n - 1.$$

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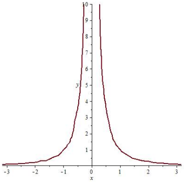

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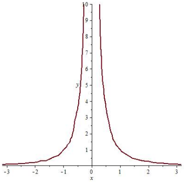

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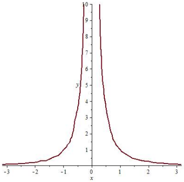

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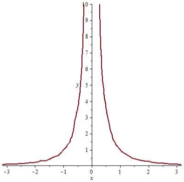

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$$\Phi = \Phi_0(c_1, \ldots, c_n)c_0^m + \Phi_1(c_1, \ldots, c_n)c_0^{m-1} + \cdots + \Phi_m(c_1, \ldots, c_n).$$

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Guo, Safey El Din, Wang, Zhi, ISSAC'2015, July 8, 11:00, Room CB1.11

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- find the **smallest** m such that C has a K-lift for $K \subseteq \mathbb{R}^m$?

When C is a Polytope and $K = \mathbb{R}^m_+$

- Given a nonnegative matrix $A \in \mathbb{R}^{n\times m}_+$, a nonnegative factorization is

$$A = UV, \quad U \in \mathbb{R}^{n\times k}_+, V \in \mathbb{R}^{k\times m}_+.$$

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Theorem [Yannikakis]

The minimal m such that C has a $\mathbb{R}_+^m$-lift is equal to the **nonnegative rank** of its slack matrix.

Nonnegative Matrix Factorization, ISSAC'2015 Tutorial by Ankur Moitra.

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More results on cone lifts and factorizations:

- When C is a **convex body** [Gouveia,Parrilo,Thomas], [Fiorini et. al.].
- When C is a **noncompact** convex set and 0^+C is **pointed** [Wang, Zhi].

Thanks to

- All my collaborators on these work
  - University Paris 06: Mohab Safey El Din
  - Dalian University of Technology: Feng Guo
  - Ph.D student: Chu Wang
- Steve Linton, Kazuhiro Yokoyama and PCs, James Davenport.

Thank You!

When C is a Convex Body

A convex set is called a ***convex body*** if it is **full dimensional**, **compact**, and contains the **origin** as its **interior**.

- The ***slack operator*** $S_C : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$S_C(x, y) := 1 - \langle x, y \rangle \text{ for } (x, y) \in \mathrm{ext}(C) \times \mathrm{ext}(C^o).$$

- The slack operator S_C is ***K-factorizable*** if there exist maps

$$A : \mathrm{ext}(C) \to K \text{ and } B : \mathrm{ext}(C^o) \to K^*$$

such that $S_C(x, y) = \langle A(x), B(y) \rangle$ for all $(x, y) \in \mathrm{ext}(C) \times \mathrm{ext}(C^o)$.

Theorem [Gouveia et al]

If C has a proper K-lift, then S_C is K-factorizable. Conversely, if S_C is K-factorizable, then C has a K-lift.

When C is a Noncompact Convex Sets and 0^+C is Pointed

- We set the ***slack operator*** S_C to be

$$S_C = \begin{cases} S_C^i(x,y) = i - \langle x, y\rangle, \ (x,y) \in \text{ext}(C) \times D_i, i = 1, 0, -1, \\ S_{0^+C}^i(x,y) = -\langle x, y\rangle, \ (x,y) \in \text{ext}_2(0^+C) \times D_i, i = 1, 0, -1, \end{cases}$$

$$D_1 = \text{ext}(C^o), \ D_0 = \text{ext}_2(0^+C^o) \cap \{x \mid \delta^*(x, C) = 0\},$$

$$D_{-1} = \text{ext}(\delta^*(x, C) \leq -1), \ \delta^*(x, C) := \sup\{\langle x, y\rangle \mid y \in C\}.$$

- The slack operator S_C is K-***factorizable*** if there exist maps

$$A_1 : \text{ext}(C) \to K, \ A_2 : \text{ext}_2(0^+C) \to K,$$

$$B_1 : D_1 \to K^*, \ B_0 : D_0 \to K^*, \ B_{-1} : D_{-1} \to K^* \text{ s.t.}$$

$$S_c^i(x,y) = \langle A_1(x), B_i(y)\rangle, \ \forall \ (x,y) \in \text{ext}(C) \times D_i, \ i = 1, 0, -1;$$
$$S_{0^+C}^i(x,y) = \langle A_2(x), B_i(y)\rangle, \ \forall \ (x,y) \in \text{ext}_2(0^+C) \times D_i, \ i = 1, 0, -1.$$

Theorem [Wang, Zhi]

Assume C is not a translated cone, if C has a proper K-lift, S_C is K-factorizable. Conversely, if S_C is K-factorizable, then C has a K-lift.