OPTIMIZING A PARAMETRIC LINEAR FUNCTION OVER A NON-COMPACT REAL VARIETY

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Problem Statements

Let $h_1, \ldots, h_p$ be polynomials in $\mathbb{R}[X]$ which define the algebraic variety

$$\mathcal{V} = \{x \in \mathbb{C}^n \mid h_1(x) = \cdots = h_p(x) = 0\}.$$

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- How to compute a polynomial $\Phi \in \mathbb{R}[c_0, \mathbf{c}]$ s.t. c_0^* can be obtained by solving $\Phi(c_0, \gamma) = 0$ for a generic $\gamma \in \mathbb{R}^n$?

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- Can we compute a polynomial family $\{\Phi_i\} \in \mathbb{R}[c_0, \mathbf{c}]$, s.t. $\forall\, \gamma \in \mathbb{R}^n$, there exists k

$$\Phi_k(c_0, \gamma) \not\equiv 0,\ \ \Phi_k(c_0^*, \gamma) = 0?$$

State of the Art

Previous work on the problem:

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- For the specialized optimization problem, the algorithm based on modified polar varieties [Greuet, Guo, Safey El Din, Zhi], [Greuet, Safey El Din] allows us to compute a polynomial of degree singly exponential in n whose roots contain the maximal value. It works for noncompact cases.

Our Contributions

We generalize the results of Rostalski and Sturmfels and have the following conclusions:

- When $\mathcal{V}$ is nonsmooth and compact in $\mathbb{R}^n$, dual varieties of regular locus and singular locus give such a polynomial Φ for generic parameter's value γ.

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We compute finitely many pairs of polynomials (Φ_i, Z_i) where $\Phi_i \in \mathbb{Q}[c_0, \mathbf{c}]$ and $Z_i \in \mathbb{Q}[\mathbf{c}]$ such that

- for each γ, there exists k such that $\gamma \notin V(Z_k)$ and $\Phi_k(c_0, \gamma) \not\equiv 0$;
- if c_0^* is finite for γ, $\Phi_k(c_0^*, \gamma) = 0$.

Main Results for Smooth and Noncompact Case

Let $\mathcal{V}^* \subset \mathbb{P}^n$ be the dual variety to the projective closure of $\mathcal{V}$ and $C_{\mathbf{h}}$ the closure of the convex hull of $\mathcal{V} \cap \mathbb{R}^n$.
We extend the result of [Rostalski, Sturmfels] and have the following conclusions:

Theorem

Suppose that $\mathcal{V}$ is equidimensional and smooth, then

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Theorem

When $\mathcal{V}$ is irreducible, smooth and $C_{\mathbf{h}}$ contains no lines, we have

- $\mathcal{V}^*$ is an irreducible hypersurface,
- its defining polynomial is $\Phi(-c_0, c_1, \ldots, c_n)$, where $\Phi(c_0^*, \gamma) = 0$ for each $\gamma \in \mathbb{R}^n$.

Bad Parameters' Values Cases

$$\Phi = \Phi_0(c_1, \ldots, c_n)c_0^m + \Phi_1(c_1, \ldots, c_n)c_0^{m-1} + \cdots + \Phi_m(c_1, \ldots, c_n)$$

Example

We consider the optimization problem:

$$\sup \; c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \quad \text{s.t. } x \in \mathcal{V} \cap \mathbb{R}^4,$$

where $\mathcal{V} = \mathbf{V}(x_4 - (x_1 + x_1^2x_2^2 + x_1^4x_2x_3)^2)$. The dual variety $\mathcal{V}^*$ is defined by

$$\begin{aligned}\Phi := &1073741824c_0^{12}c_2^4c_4^2 + 268435456c_0^{11}c_1^2c_2^4c_4 - 134217728c_0^{11}c_2^2c_3^2c_4^3 \\ &- 33554432c_0^{10}c_1^2c_2^2c_3^2c_4^2 + \cdots + 520093696c_0^9c_1c_2^3c_3^2c_4^3.\end{aligned}$$

- When $\gamma \in V(c_2c_4, c_3c_4, c_3c_2c_1)$, $\Phi(c_0, \gamma) \equiv 0$.
- $\Phi(c_0, 0, 0, 0, -1) \equiv 0$ which gives no info on c_0^*.

Algorithm for Non-parametric Optimization

Construct a one dimensional subvariety $\mathcal{C} \subseteq \mathcal{V}$ such that

$$\sup_{x \in \mathcal{V} \cap \mathbb{R}^n} f(x) = \sup_{x \in \mathcal{C} \cap \mathbb{R}^n} f(x).$$

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Algorithm: [Greuet and Safey El Din'14]

Input: $f(X) := \gamma^T X$, $\langle h_1, \ldots, h_p \rangle \in \mathbb{Q}[X]$ and $\gamma \in \mathbb{Q}^n$

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Output: $\Phi(c_0)$ s.t. $f(\mathsf{S}_1 \cup \mathsf{S}_2 \cup \mathsf{S}_3) \subseteq \mathbf{V}(\Phi)$

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Polar Varieties

Suppose $\mathcal{V}$ is an equidimensional and smooth variety. Let $\mathbf{H}$ be $\{h_1, \ldots, h_p\}$ and $\langle h_1, \ldots, h_p \rangle$ is radical. MaxMinors $(\mathrm{jac}(\mathbf{H}, \mathbf{X}_{\geq i+1}))$ is denoted to be the $(n-i) \times (n-i)$ minors of the Jacobian of $\mathbf{H}$ respect to $x_{i+1}, \ldots, x_n$.

Polar Varieties [Bank, Giusti, Heintz, Mbakop, Pardo, Safey El Din, Schost]

Polar varieties are defined to be a sequence of varieties $\{W_i\}$, where W_{n-i+1} is the **critical locus** of

$$\pi_i : (X_1, \ldots, X_n) \longrightarrow (X_1, \ldots, X_i)$$

restricted to $\mathcal{V}$.

W_{n-i+1} is the variety of $\mathbf{H}$ and MaxMinors $(\mathrm{jac}(\mathbf{H}, \mathbf{X}_{\geq i+1}))$.

Modified Polar Varieties

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Let W_{n-i+1} be the variety of $\mathbf{H}, \mathsf{MaxMinors}\,(\mathrm{Jac}([f, \mathbf{H}], \mathbf{X}_{\geq i+1}))$ and

$$W = \cup_{i=1}^{d} M_i, \text{ where } M_i = W_{n-i+1} \cap V(X_1, \ldots, X_{i-1}), 1 \leq i \leq d.$$

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After a generic linear change of coordinates,

- $f\left(V \cap \mathbb{R}^n\right) = f\left(W \cap \mathbb{R}^n\right)$;
- the set of asymptotic value of f on W is finite.
- $\dim(\mathcal{C}) = 1$ for $\mathcal{C} := \overline{W \setminus \mathsf{Crit}(f, \mathcal{V})}$.

Algorithm for Parametric Optimization

Input: f, $\mathcal{V}$

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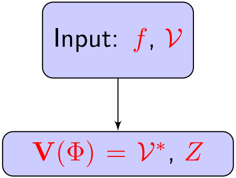

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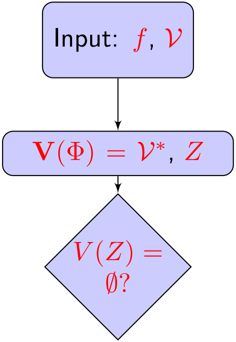

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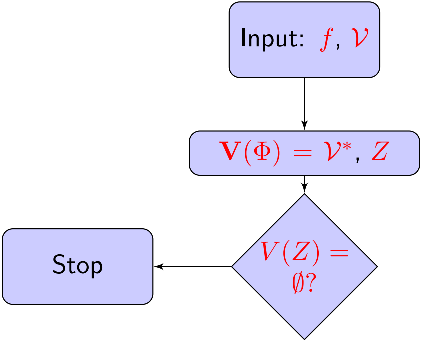

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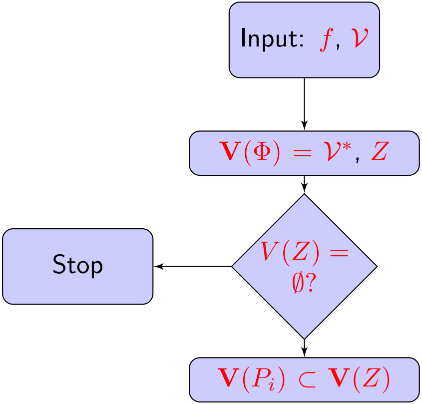

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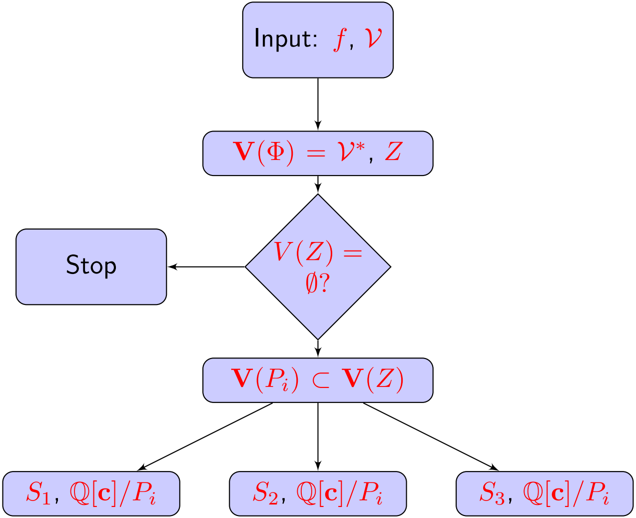

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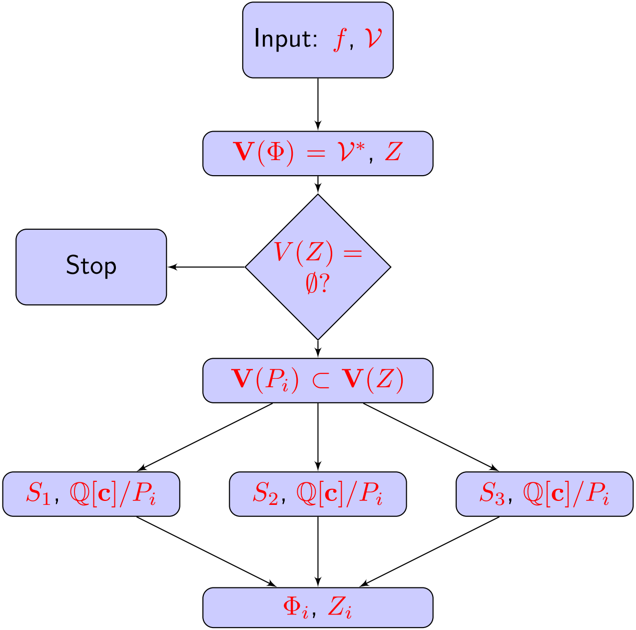

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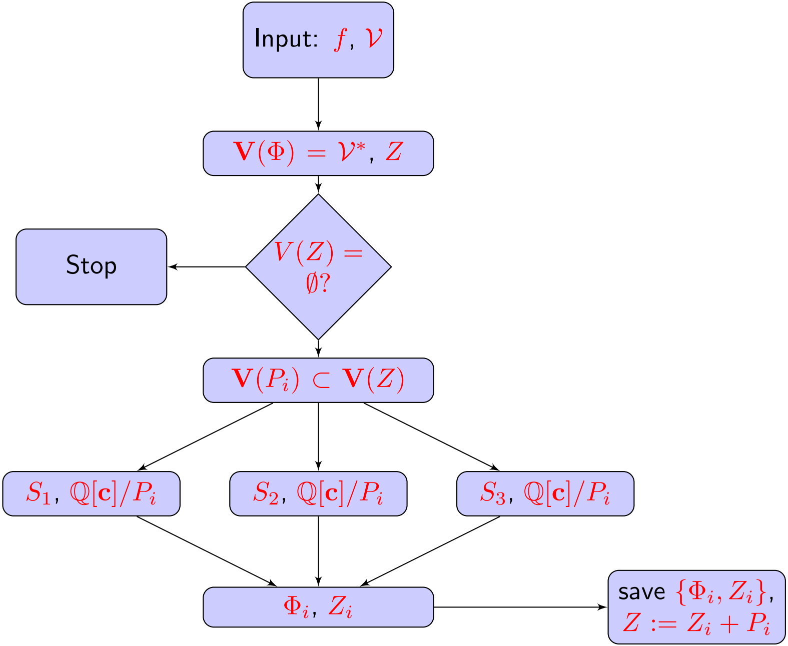

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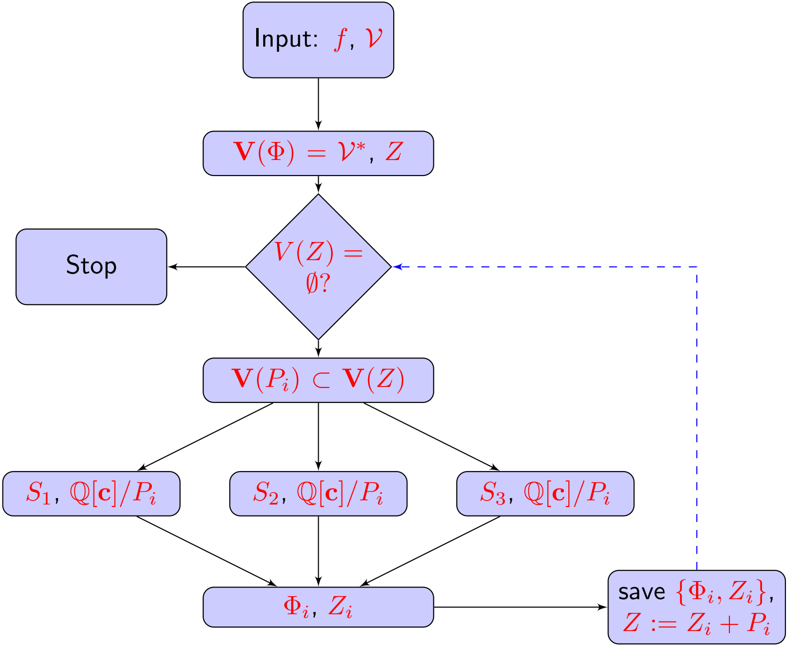

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Identify Bad Parameters' Values

Bad parameters' values

γ is a bad parameter's value of Φ_i if

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- Some parameters' values are not in generic position by one random linear change.
- The computation of Gröbner basis may not be commutative for the specialization operation.

Example (continued)

$I = \langle c_2c_4, c_3c_4, c_1c_2c_3 \rangle$: $\mathbf{V}(I)$ contains the bad parameters' values.

The primary decomposition of I:

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Let us consider $\mathbf{P} = \langle c_2, c_3\rangle$ and solve the following optimization problem:

$$\begin{aligned} &\textsf{sup} \quad c_1x_1 + c_4x_4 \\ &s.t. \quad x_4 - (x_1 + x_1^2x_2^2 + x_1^4x_2x_3)^2 = 0. \end{aligned}$$

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- Consider the following optimization problem:

$$\begin{aligned} &\text{sup } x_4 \\ &s.t. \;\; x_4 - (x_1 + x_1^2 x_2^2 + x_1^4 x_2 x_3)^2 = 0. \end{aligned}$$

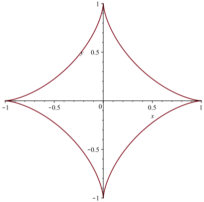
We get $\Phi = c_0$ and $V(Z) = \emptyset$.

Main Results for Singular Case

Let $\mathcal{V}$ be defined by

$$\left(X_1^2 + X_2^2 - 1\right)^3 + 27X_1^2X_2^2.$$

The defining polynomial of $\mathcal{V}^*$ is $\Phi_1 := -c_1^2c_2^2 + c_0^2c_1^2 + c_2^2c_0^2$.



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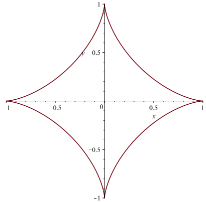

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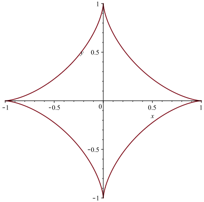

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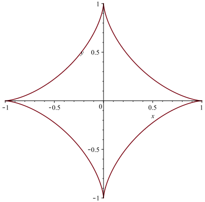

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Step 3 Compute the singular locus $\widetilde{\mathcal{V}}_{k,i}$ of each $\mathcal{V}_{k,i}$. Let $\mathcal{V}_{k+1}=\cup_i\widetilde{\mathcal{V}}_{k,i}$ and $k=k+1$. Go to Step 1.

Main Results for Singular Case

Theorem

The algorithm terminates in a finite number k *steps and we have*

$$(-c_0^* : \gamma_1 : \cdots : \gamma_n) \subseteq \cup_{i=1}^{k} (\mathcal{V}^{(k)})^*.$$

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In this example, $\mathrm{Sing}(\mathcal{V})$ is equidimensional. Its dual variety $(\mathcal{V}^{(2)})^*$ is defined by

$$\begin{aligned}\Phi_2 = &(c_0 - c_2)(c_0 + c_2)(c_0 - c_1)(c_0 + c_1)(c_0^2 + c_1^2 - 2c_1c_2 + c_2^2) \\ &(c_0^2 + c_1^2 + 2c_1c_2 + c_2^2).\end{aligned}$$

Then we have $(-c_0^* : \gamma_1 : \gamma_2) \in \mathbf{V}(\Phi_1\Phi_2)$.

Conclusions and Future Work

- How to compute a polynomial Φ when the feasible set is a real variety which is noncompact and nonsmooth?
- How to compute a polynomial Φ when the feasible set is a semialgebraic set?