Near Optimal Subdivision Algorithms for Real Root Isolation

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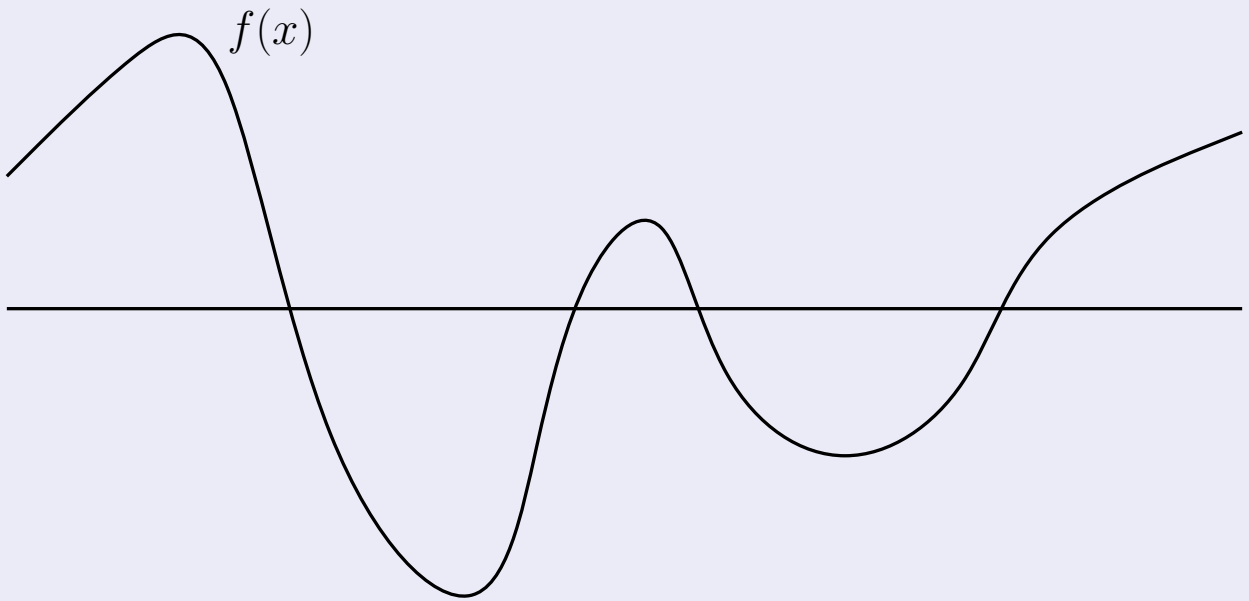
ISSAC, Bath, 2015

Real Root Isolation

The Problem

Input: $f(x) \in \mathbb{R}[x]$, degree n, and an interval I_0.

Output: Disjoint intervals with endpoints in $\mathbb{Q}$ containing roots of f.



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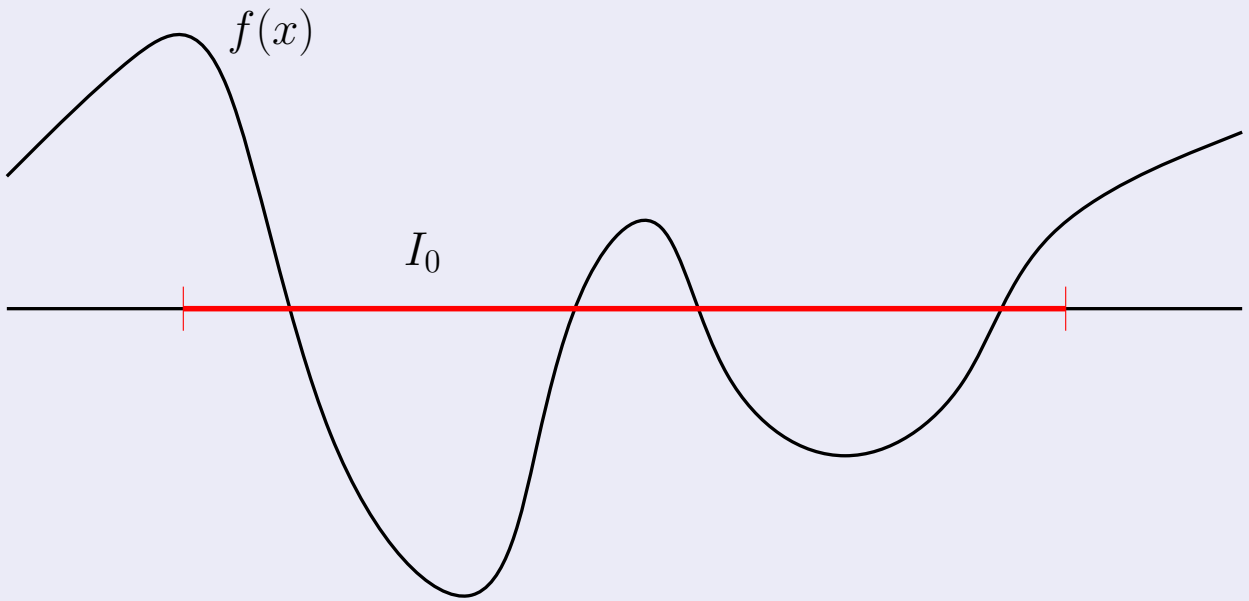

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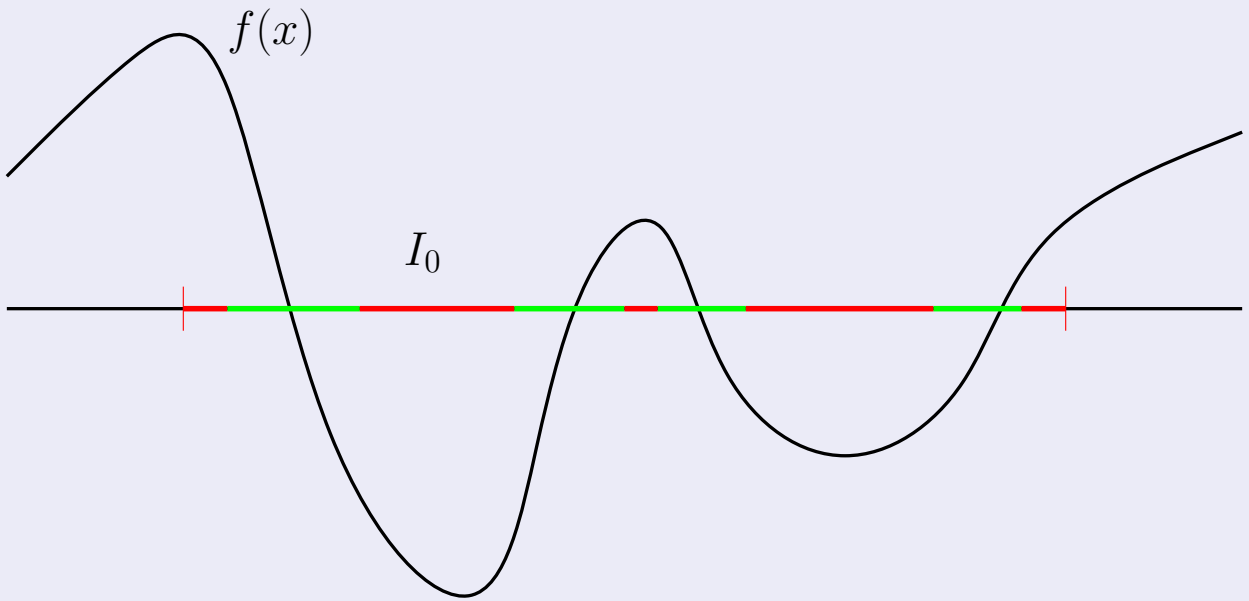

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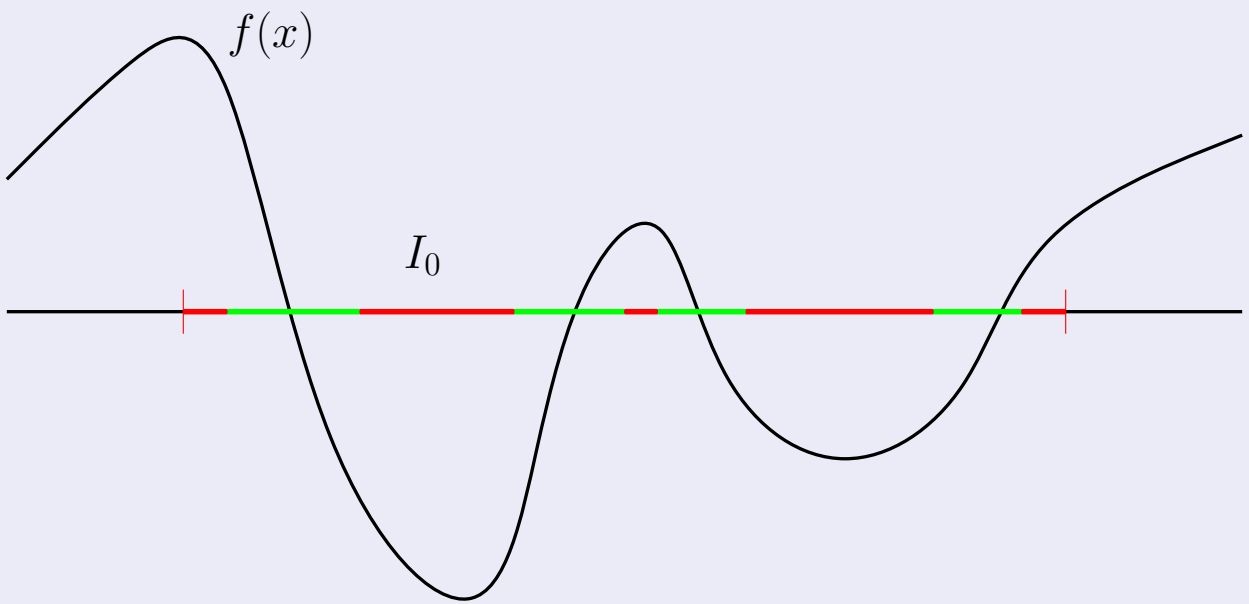

Assumption

All the roots are of multiplicity one, i.e., f is square-free.

Subdivision Methods

Predicates

- Exclusion $C_0(I)$: if true then I has no roots.
- Inclusion $C_1(I)$: if true then I has exactly one root.

E.g., Sturm sequences, Descartes's rule of signs, Interval arithmetic,...

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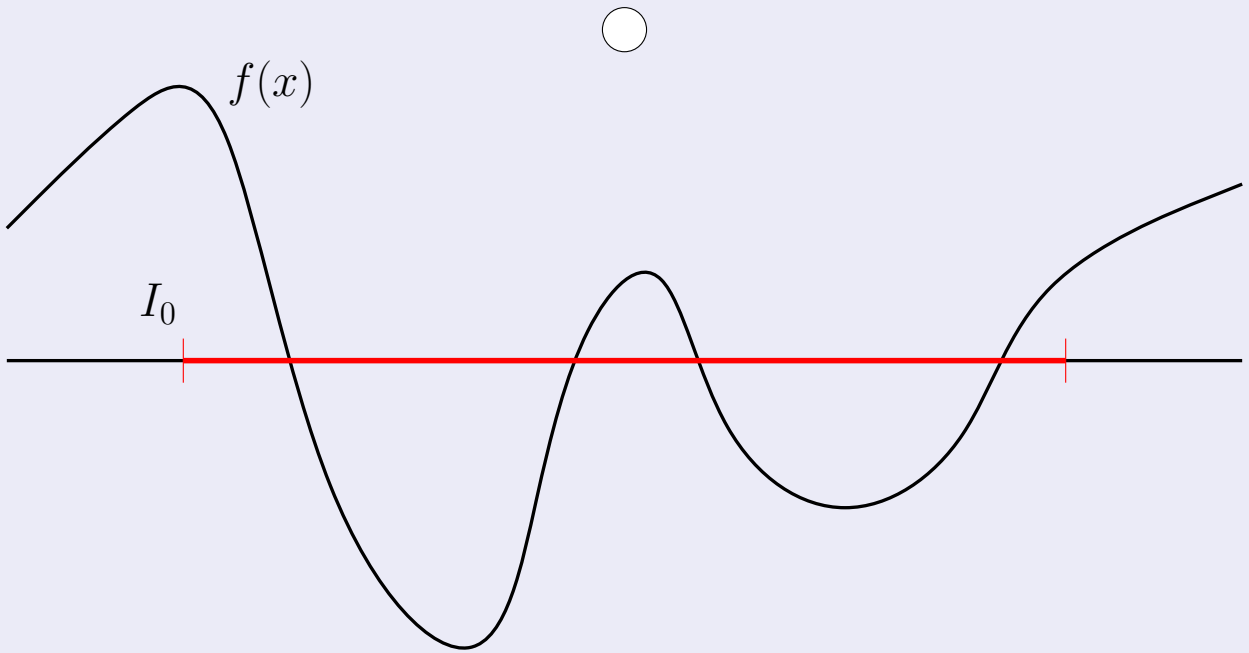

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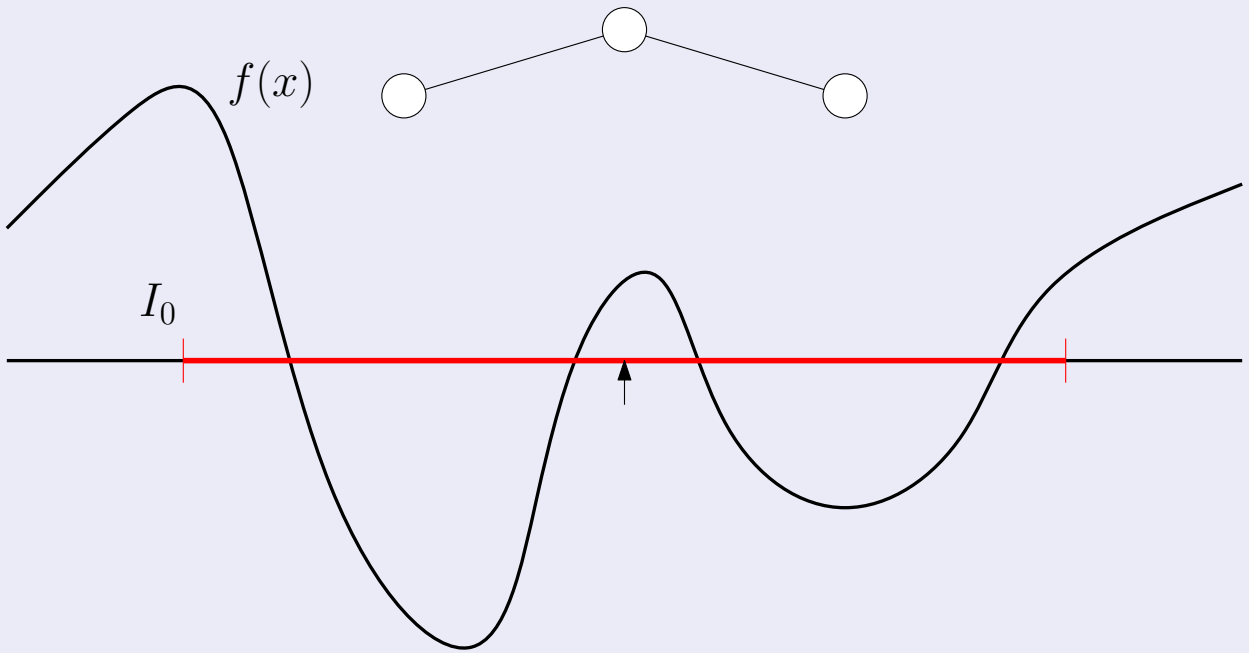

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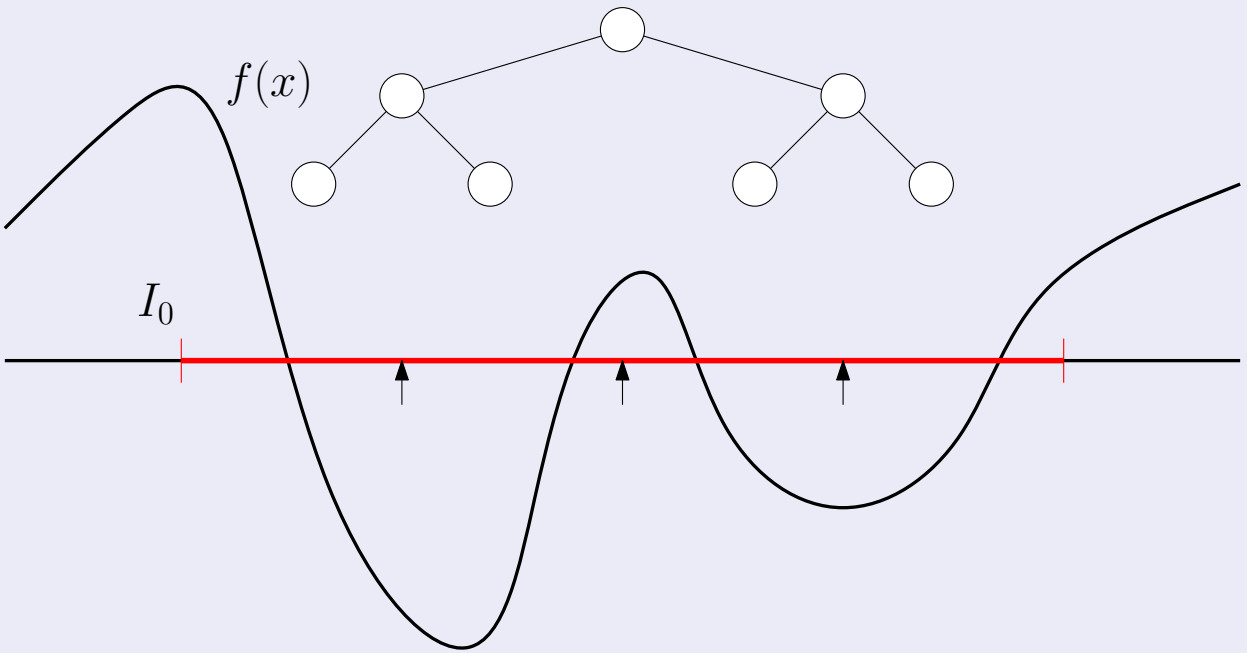

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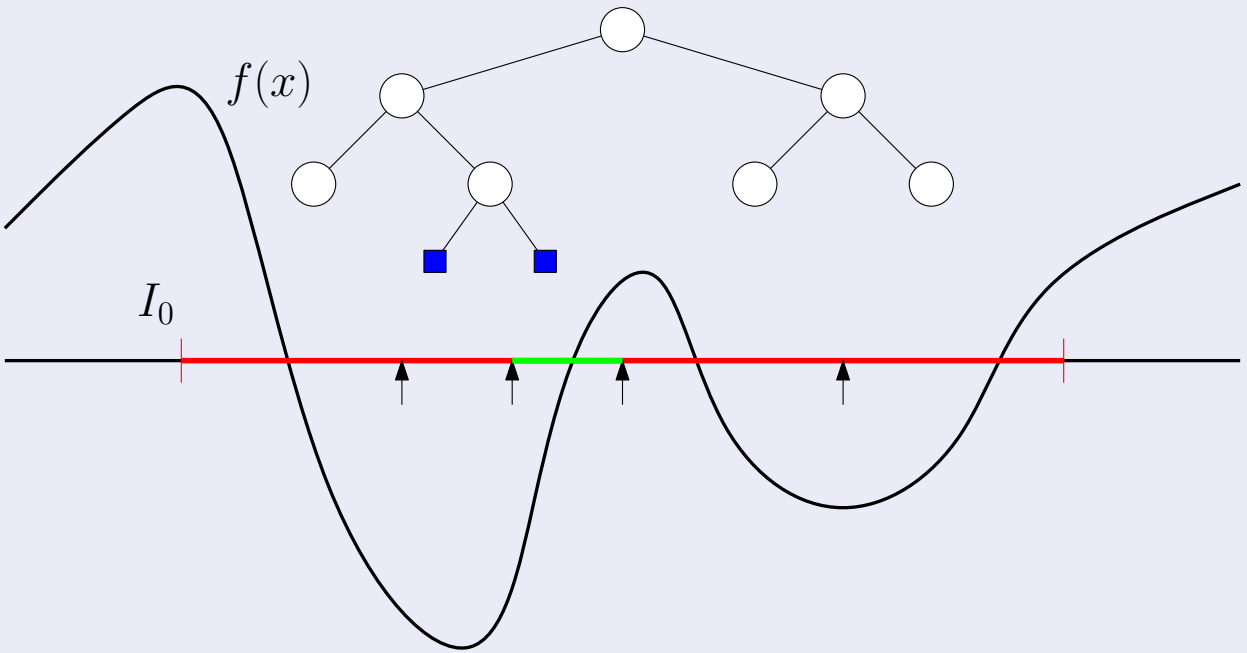

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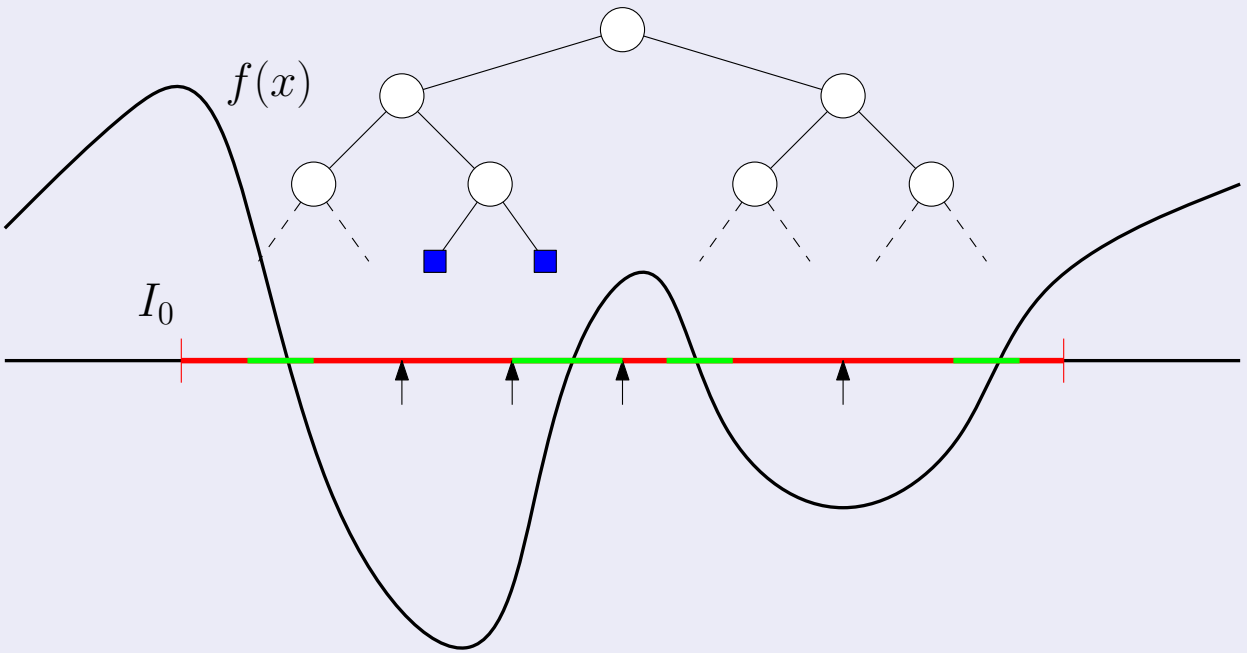

Subdivision Methods – Complexity Analysis

Two Components

- The size $\#T_{I_0}$ of the tree T_{I_0}.
- The worst case *arithmetic* complexity at a node. Usually $\widetilde{O}(n)$.

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Bounds on $\#T_{I_0}$

- $O(\log 1/\sigma)$, σ is the root separation bound for f.
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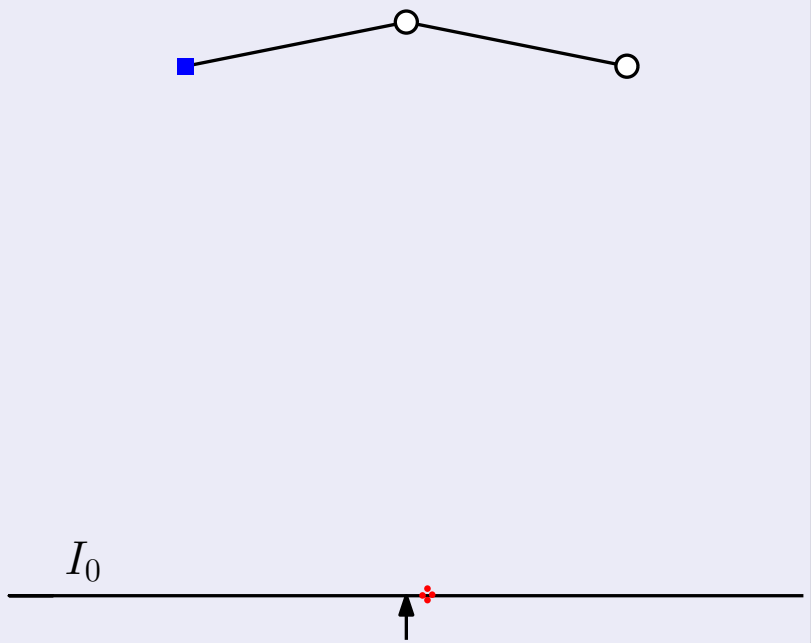

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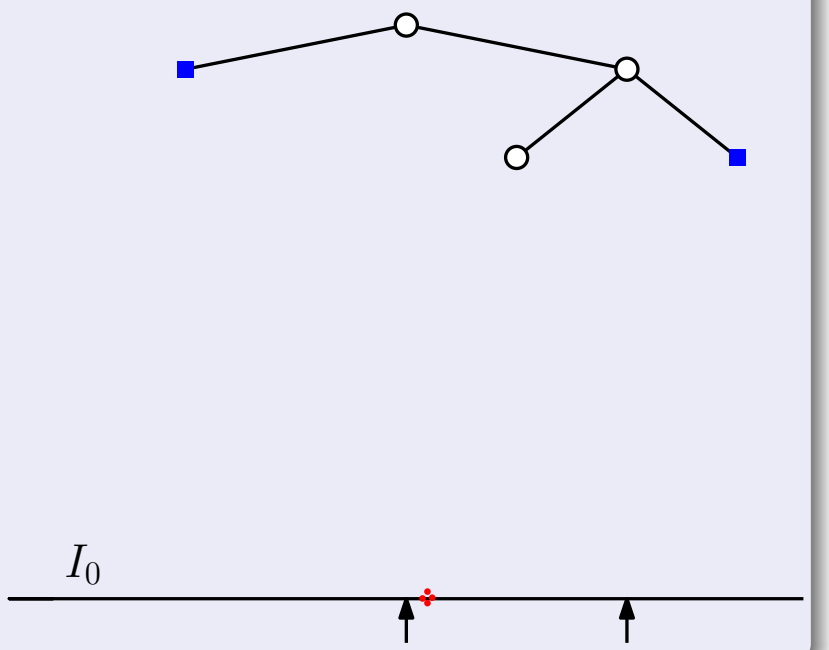

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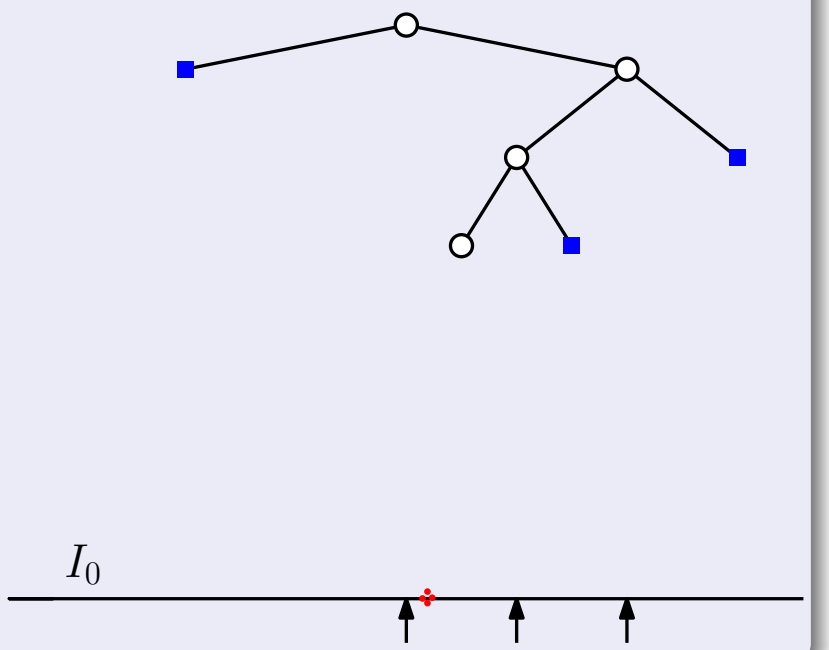

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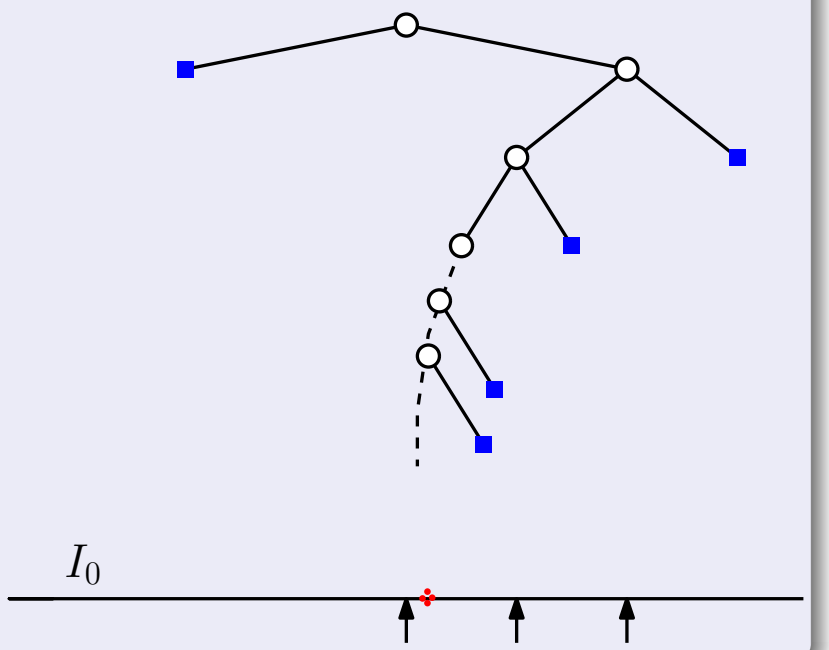

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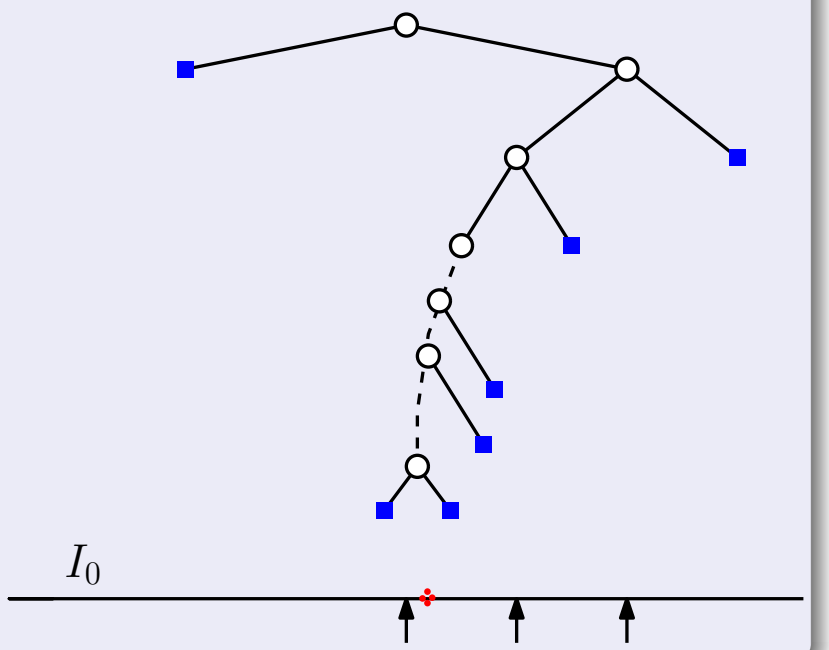

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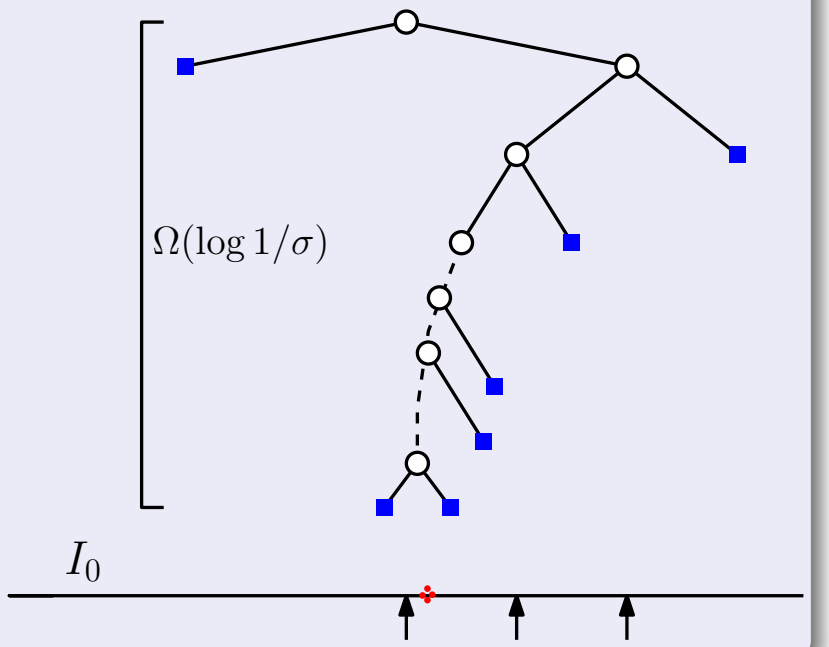

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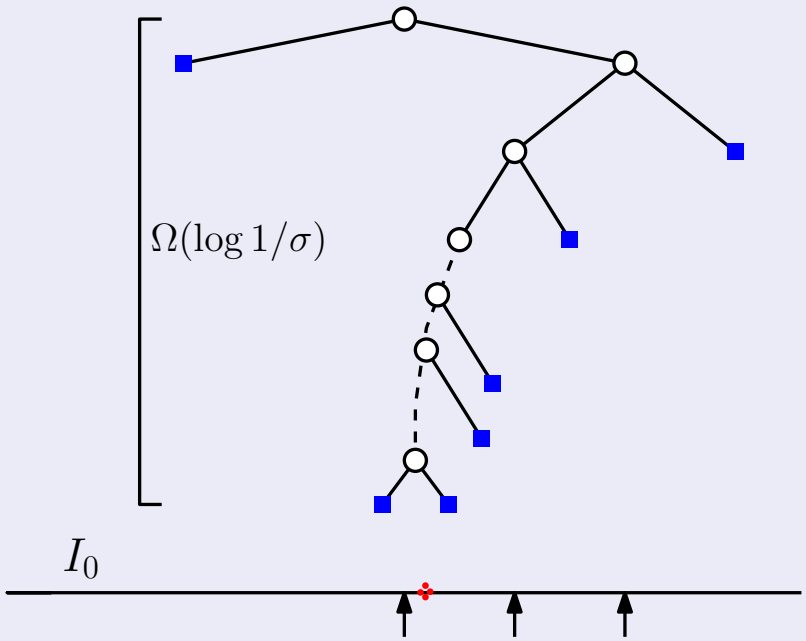

The Problem

Subdivision only gives linear convergence to clusters of roots.

The Remedy – Subdivision + Newton Iteration

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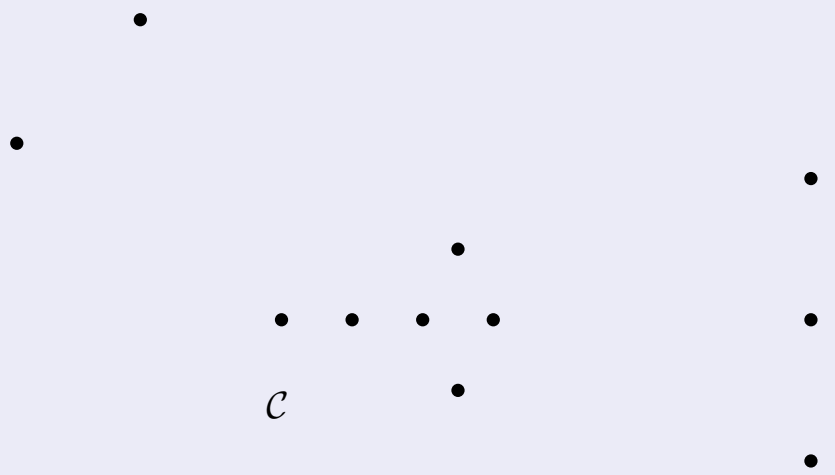

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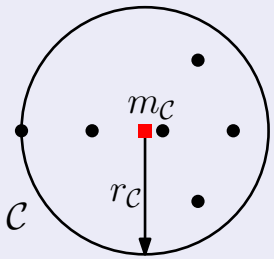

- $\mathcal{C}$ a subset of roots.
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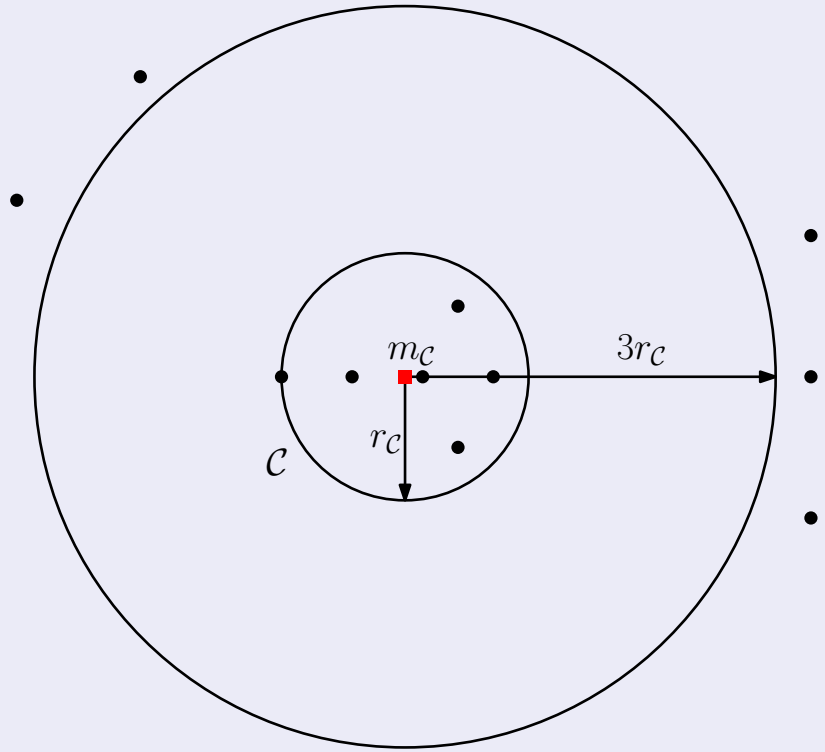

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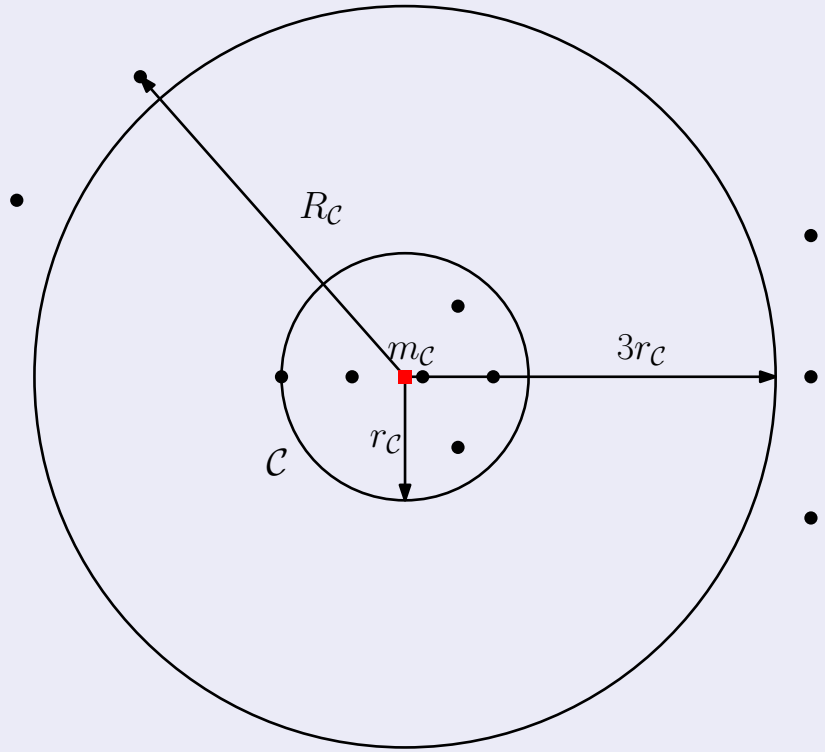

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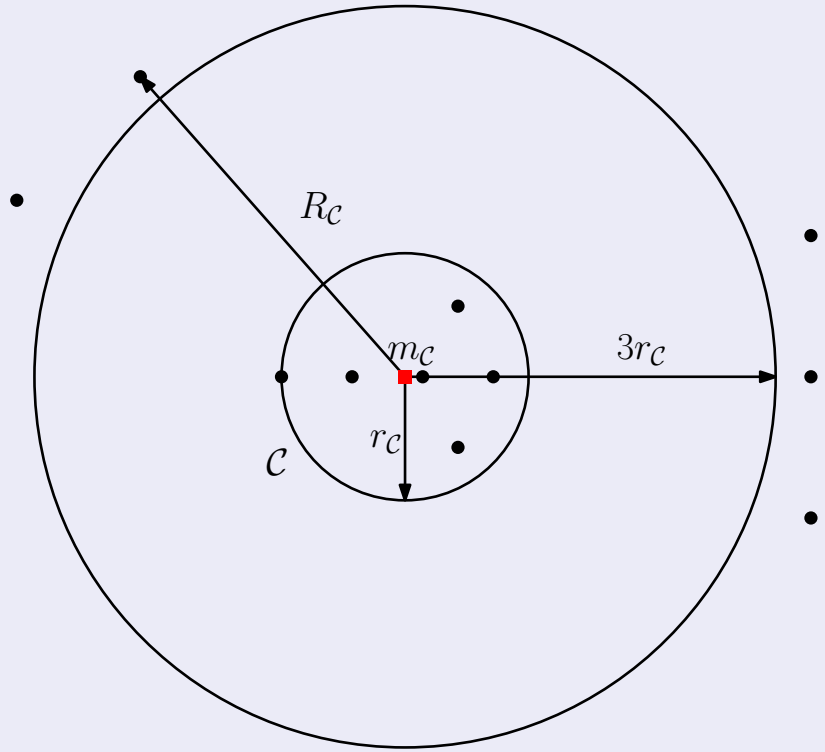

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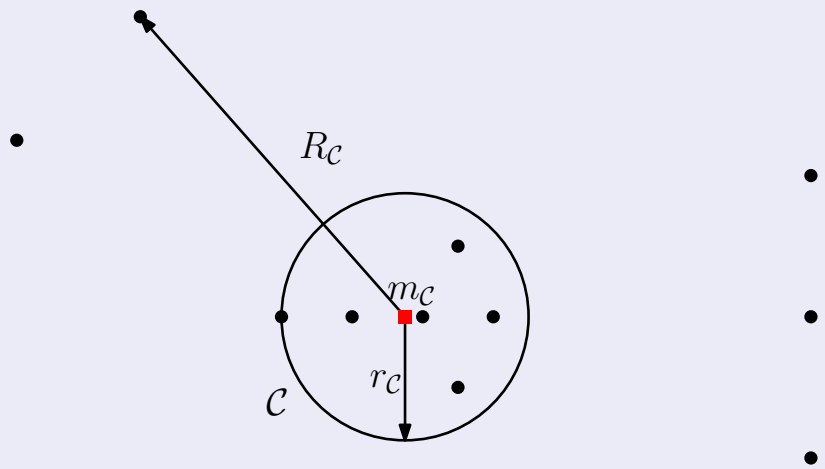

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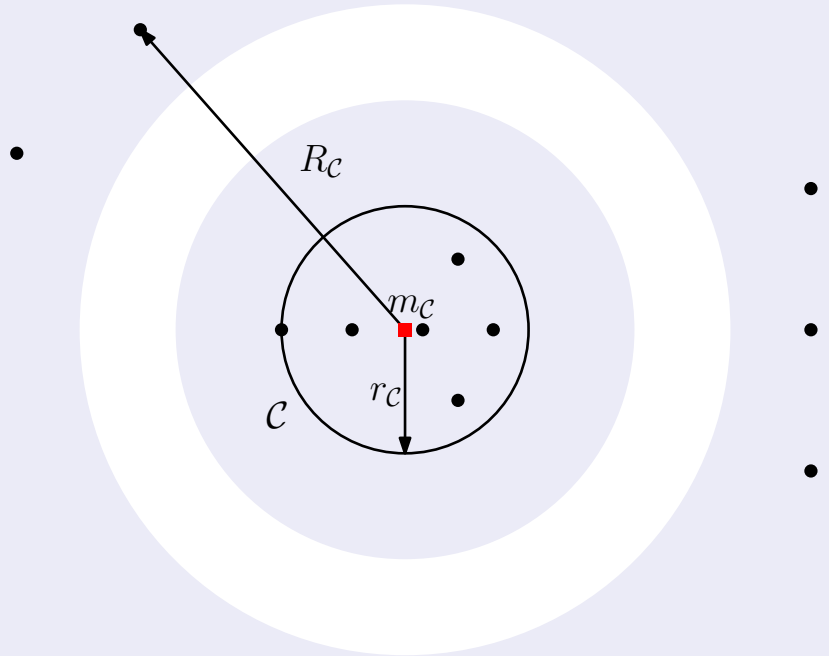

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If $R_{\mathcal{C}} \gtrsim n^2 r_{\mathcal{C}}$ then $\exists$ an annulus in which Newton iteration converges to $m_{\mathcal{C}}$.

Resulting Improvements

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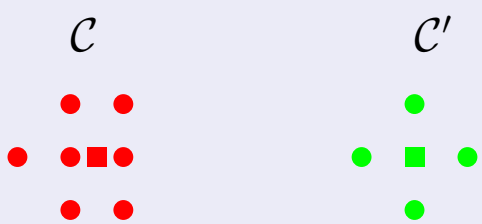
Our result

- Choice of any inclusion-exclusion predicate, Newton iteration.
- $O(n \log n)$, for Descartes's rule of signs or Sturm sequences.
- Analysis is independent of root bounds, uses geometry of clusters.
- Use the framework of continuous amortization.

Basic Idea

Cluster Tree



Claim

If $\mathcal{C} \cap \mathcal{C}' \neq \emptyset$ then either $\mathcal{C} \subseteq \mathcal{C}'$ or vice versa.

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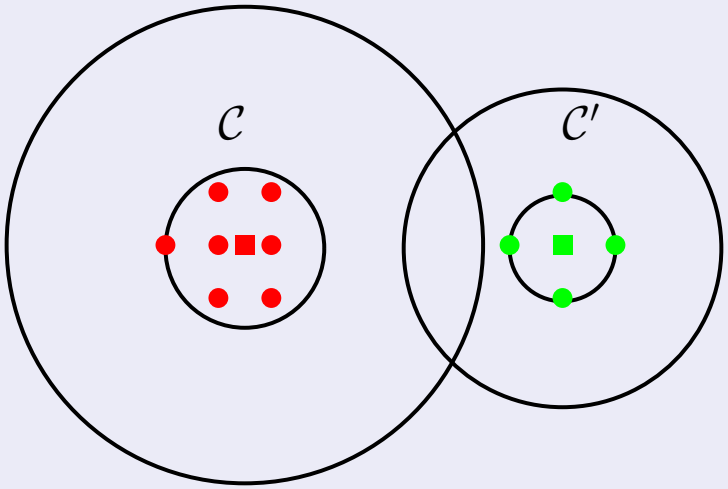

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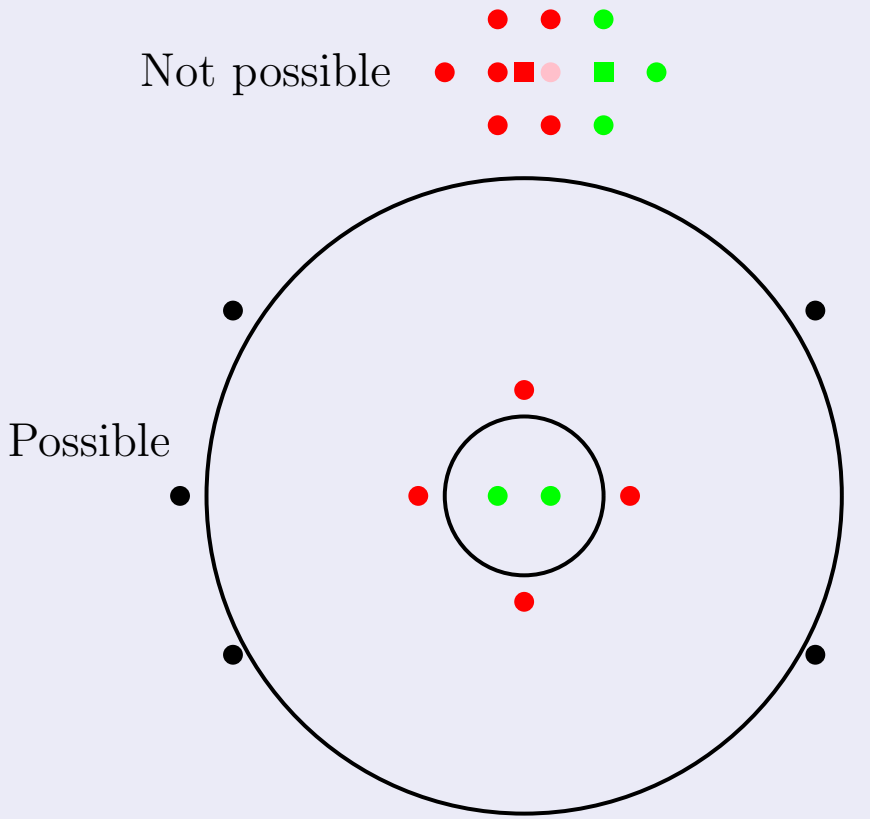

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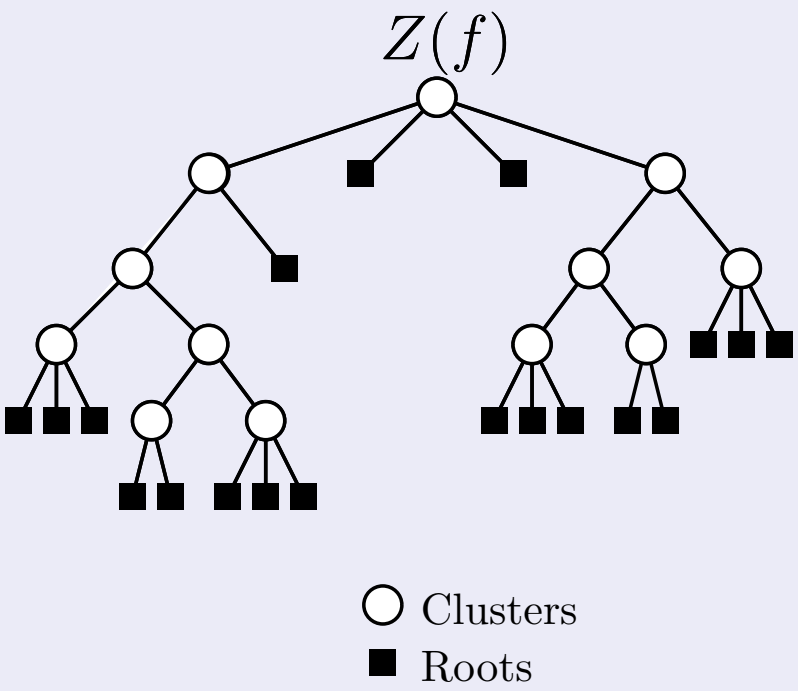

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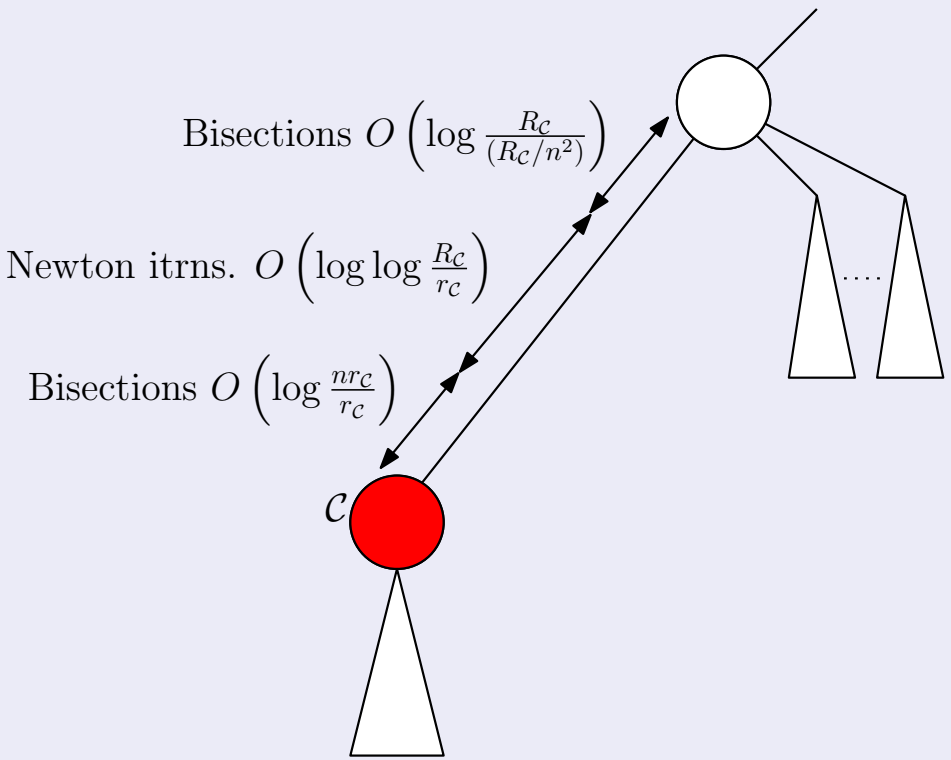
Lemma (Sagraloff-S.-Yap'13)

The set of clusters form a tree with root node as the set of all zeros $Z(f)$.

Basic Idea

Number of Bisections in Cluster Tree



Strongly Separated Clusters

- $R_C \gtrsim n^3 r_C$.
- Bisection $O(\log R_C/r_C)$ steps.
- Detect and converge using $O(\log\log R_C/r_C)$ Newton iterations.

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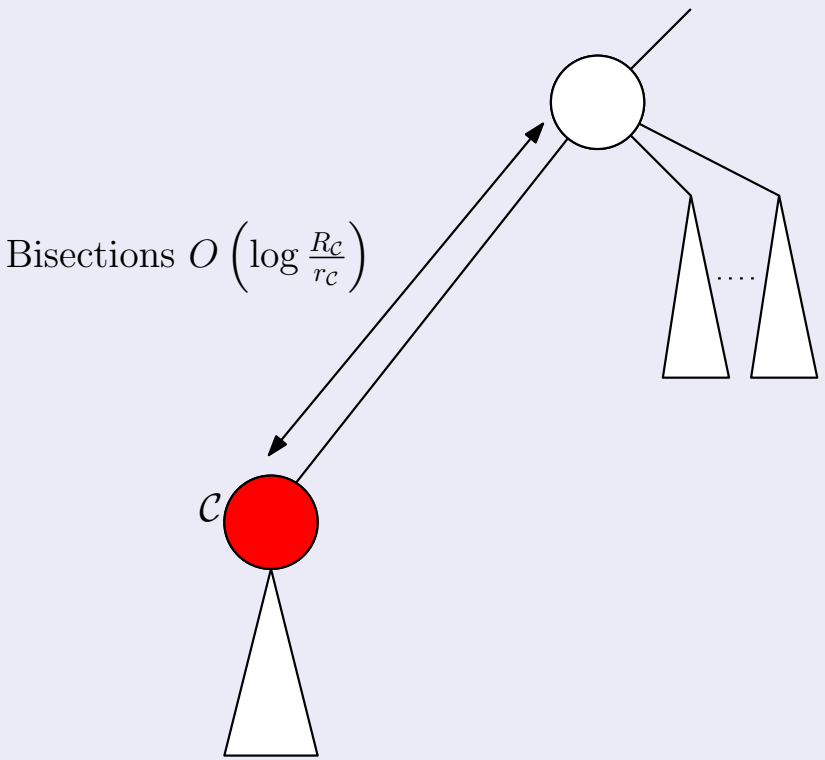

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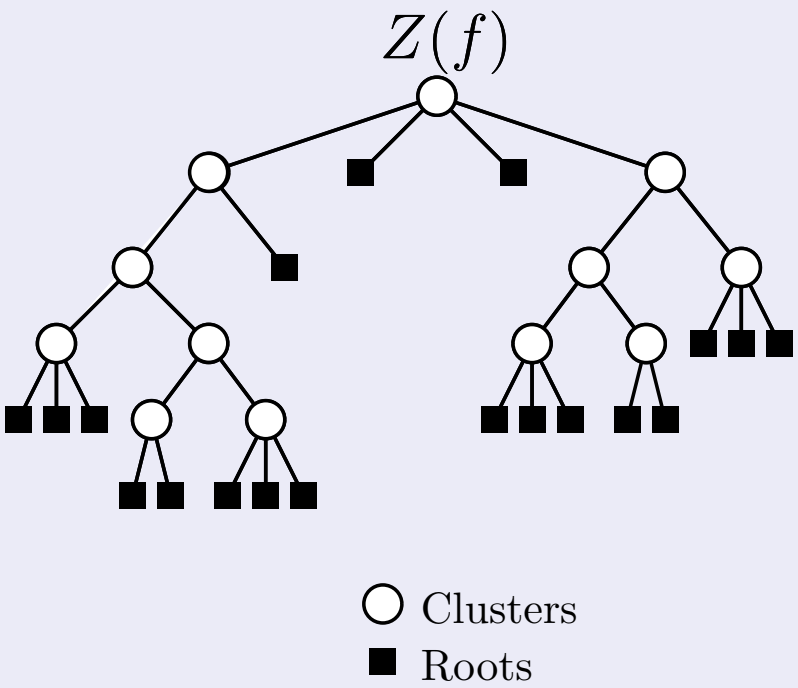

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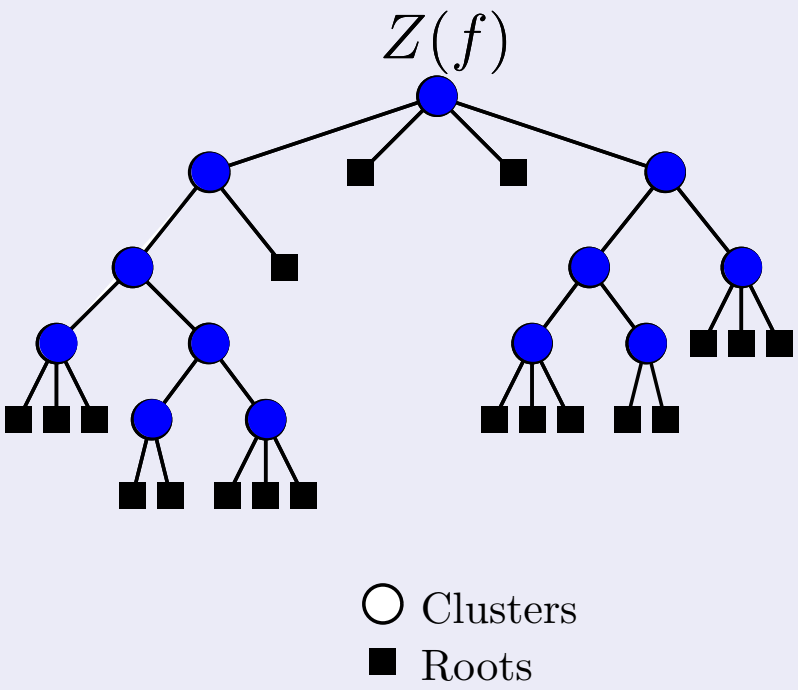

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The number of bisection steps in T_{I_0} is $O(n \log n)$.

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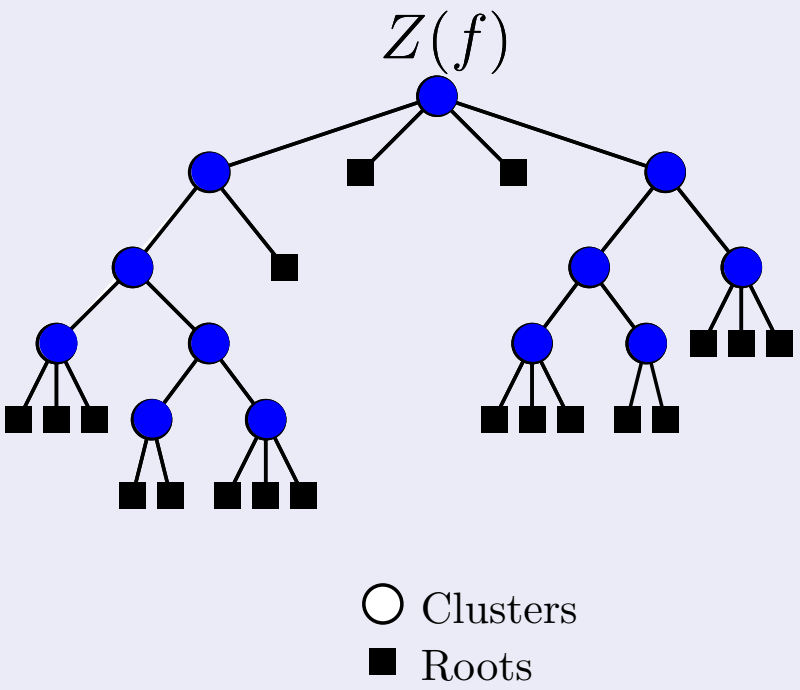
Number of Bisections in Cluster Tree



Strongly Separated Clusters

- $R_{\mathcal{C}} \gtrsim n^3 r_{\mathcal{C}}$.
- Bisection $O(\log R_{\mathcal{C}}/r_{\mathcal{C}})$ steps.
- Detect and converge using $O(\log\log R_{\mathcal{C}}/r_{\mathcal{C}})$ Newton iterations.

Ordinary Clusters

- $R_{\mathcal{C}} \lesssim n^3 r_{\mathcal{C}}$.
- Bisection steps $O(\log \frac{R_{\mathcal{C}}}{r_{\mathcal{C}}}) = O(\log n)$.

The number of bisection steps in T_{I_0} is $O(n \log n)$.

Detecting and Approximating Clusters

Ostrowski'34, S.-Batra'15

Let $f(x+z) := \sum_j f_j(z)x^j$, i.e., $f_j(z)$ is the jth Taylor coefficient at z.

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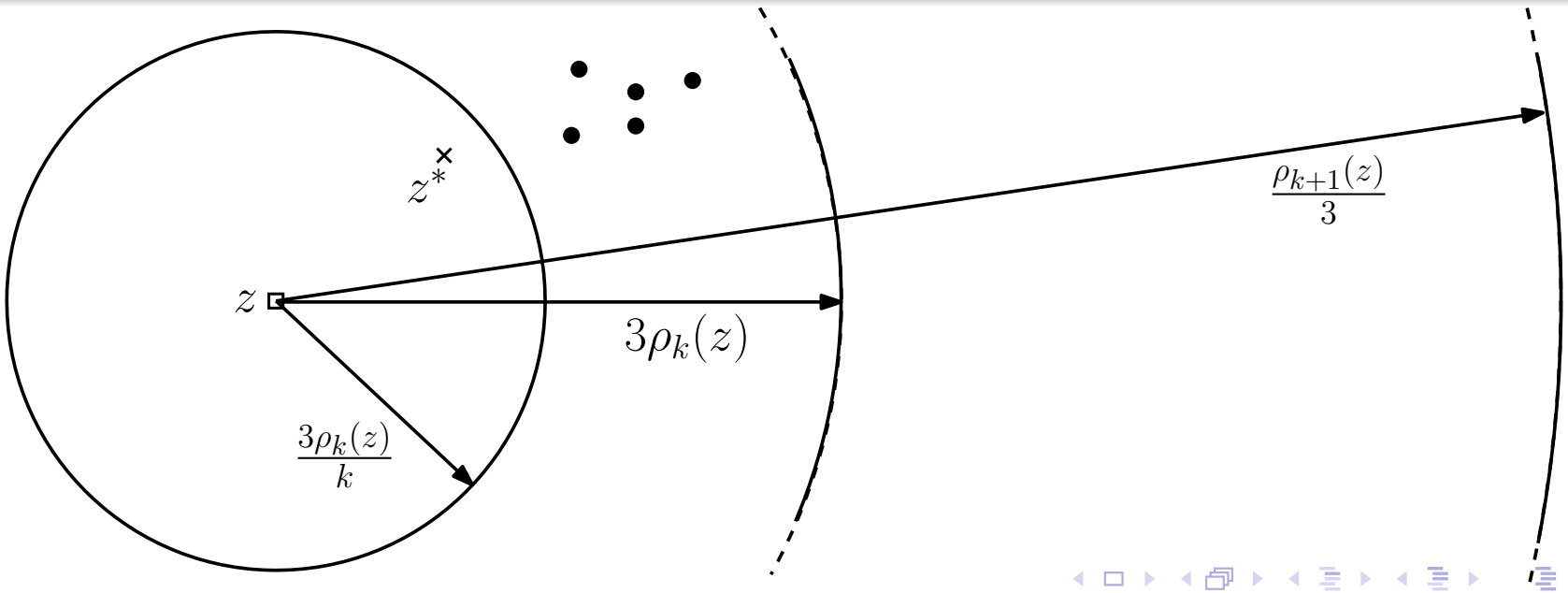

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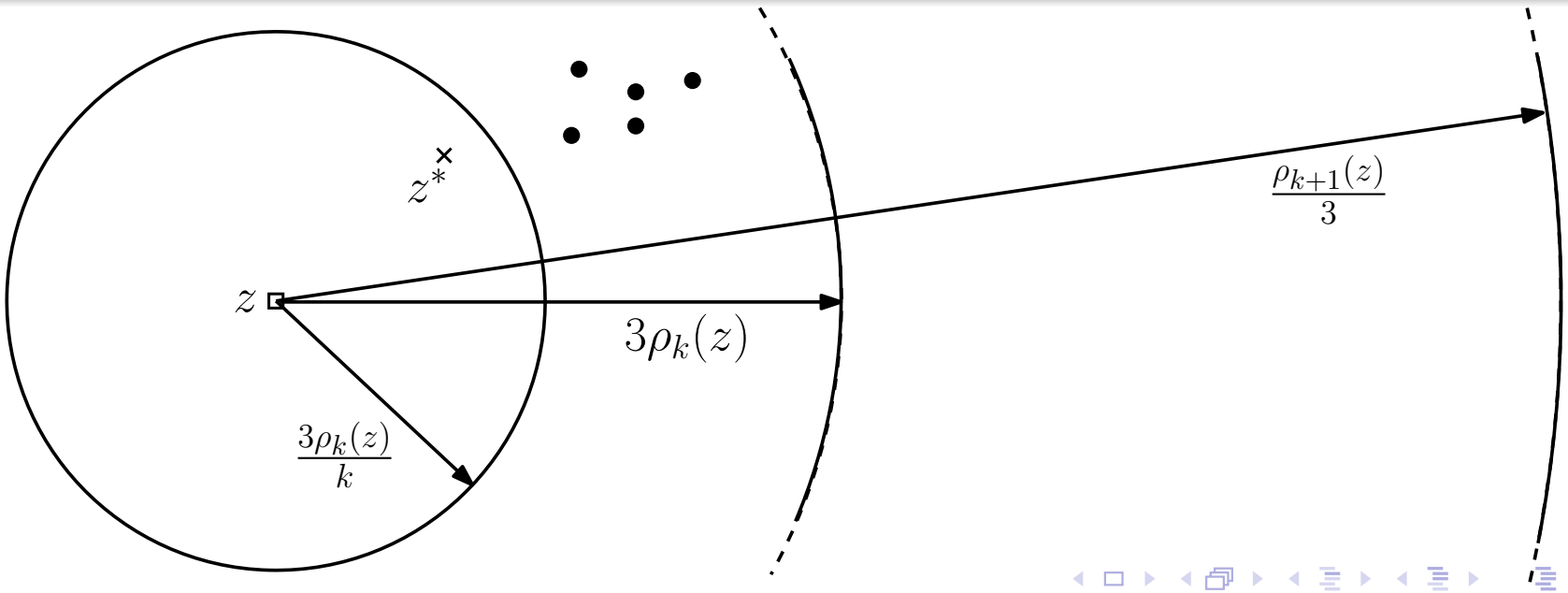

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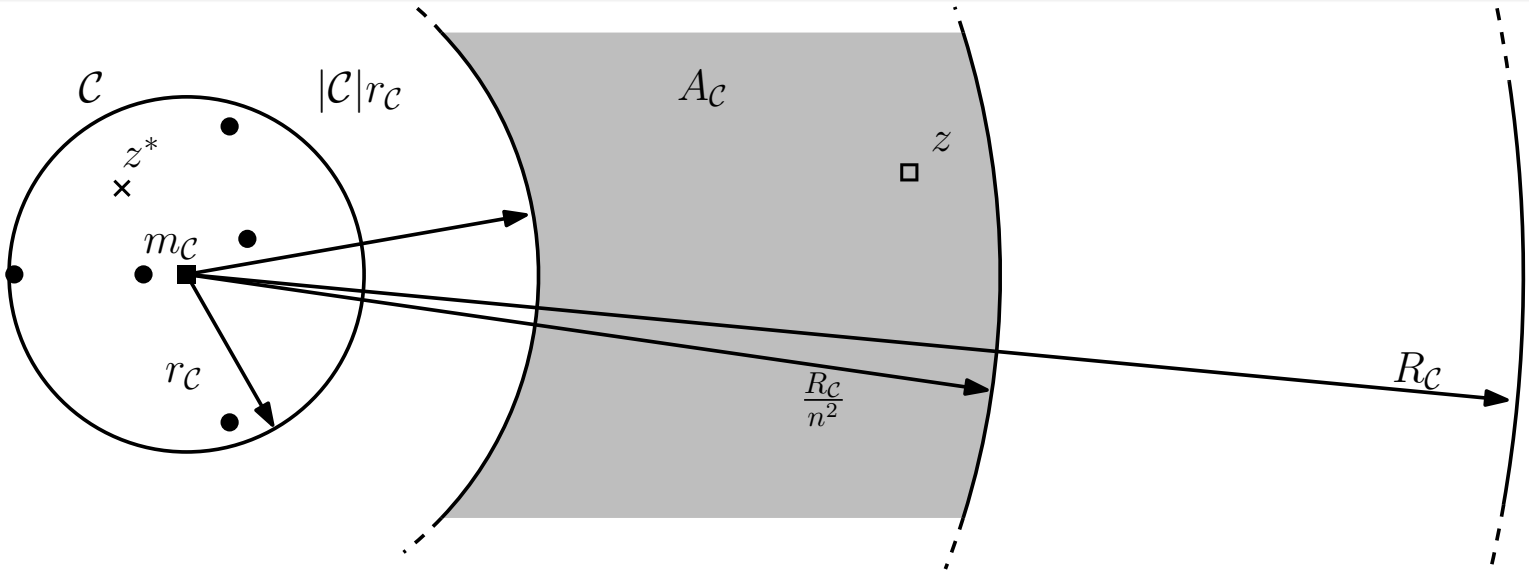

The algorithm NewtonIsol

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Bounding the size of the tree T_{I_0}

T_{I_0} is a binary tree. Let $L(T_{I_0})$ be its set of leaves. Bound $|L(T_{I_0})|$.

Intervals I in $L(T_{I_0})$ – Two Types

1. Where $C_0(I)$ or $C_1(I)$ holds.
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Observation: If J is the parent of I then both $C_0(J)$ and $C_1(J)$ failed.

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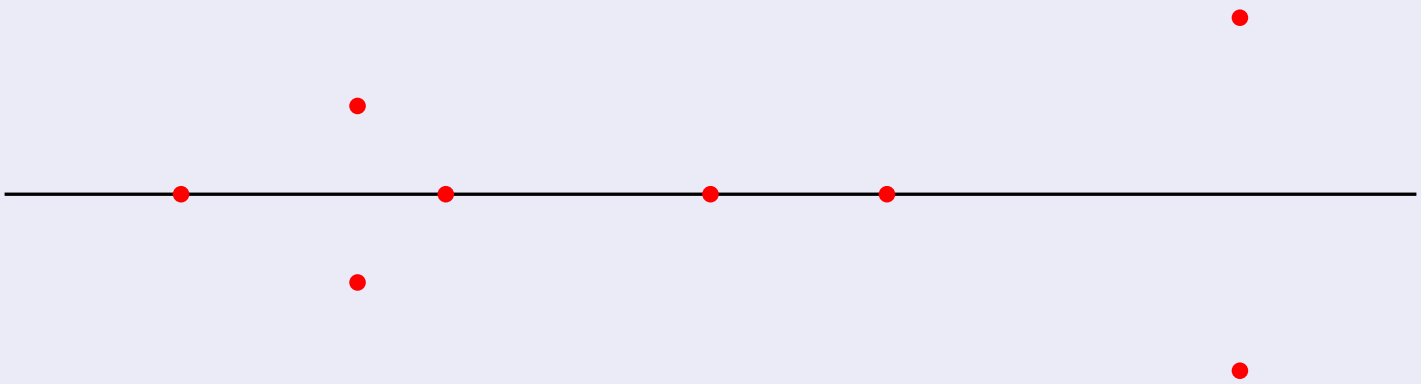

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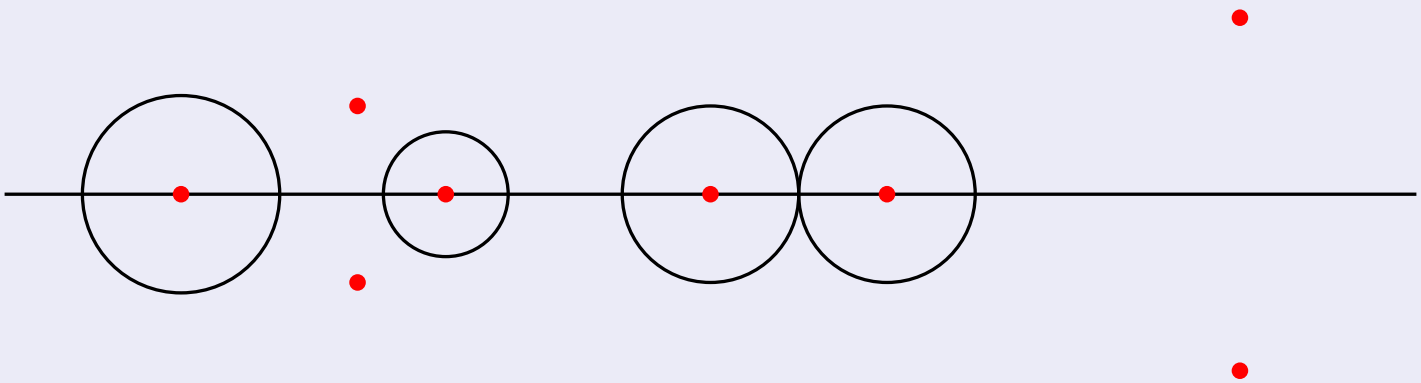

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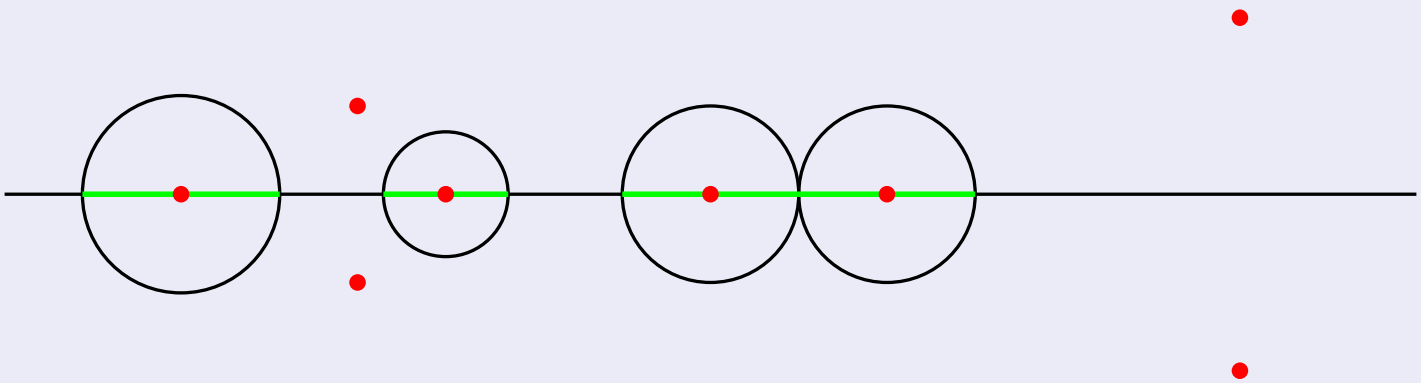

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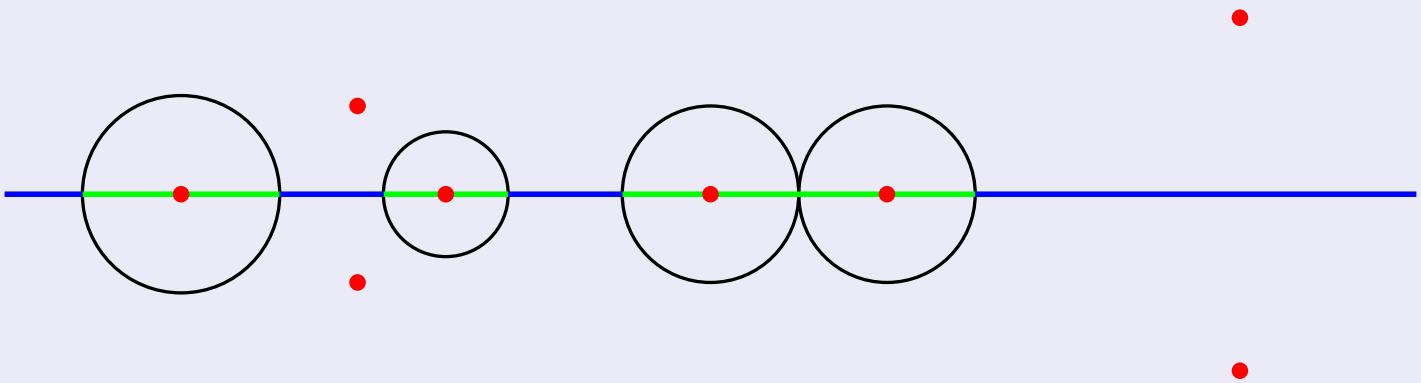

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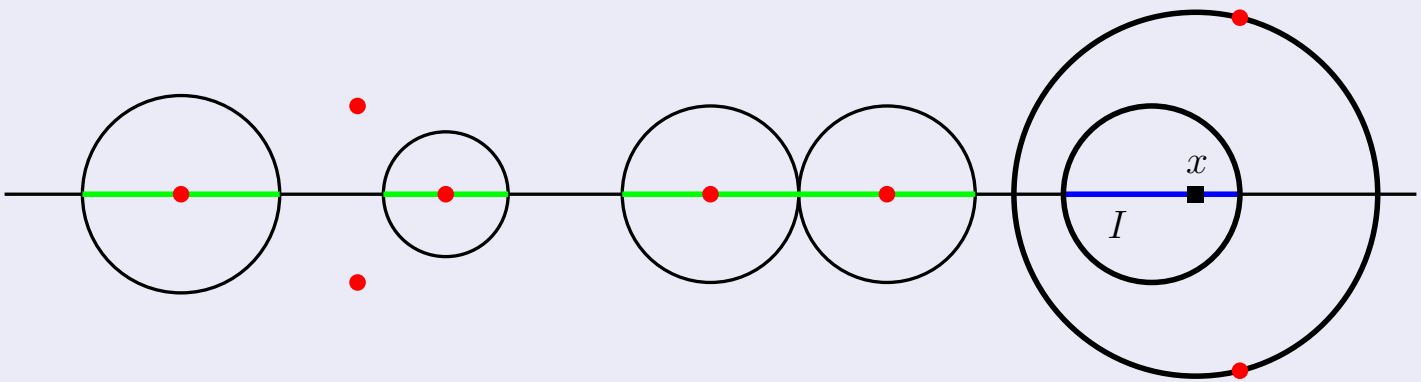

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$\int_{I_0 \setminus \cup_C A_C} G(x)dx = O(|T| \log n)$, where C are ssc, T is cluster tree?

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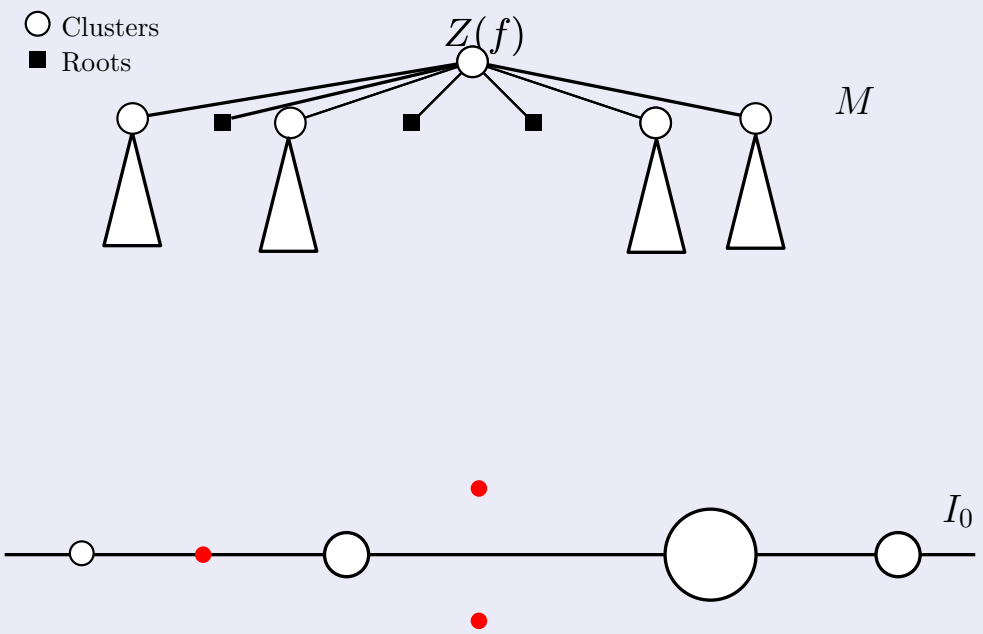

– Cluster C, $I_C := [m_C \pm 2r_C]$.

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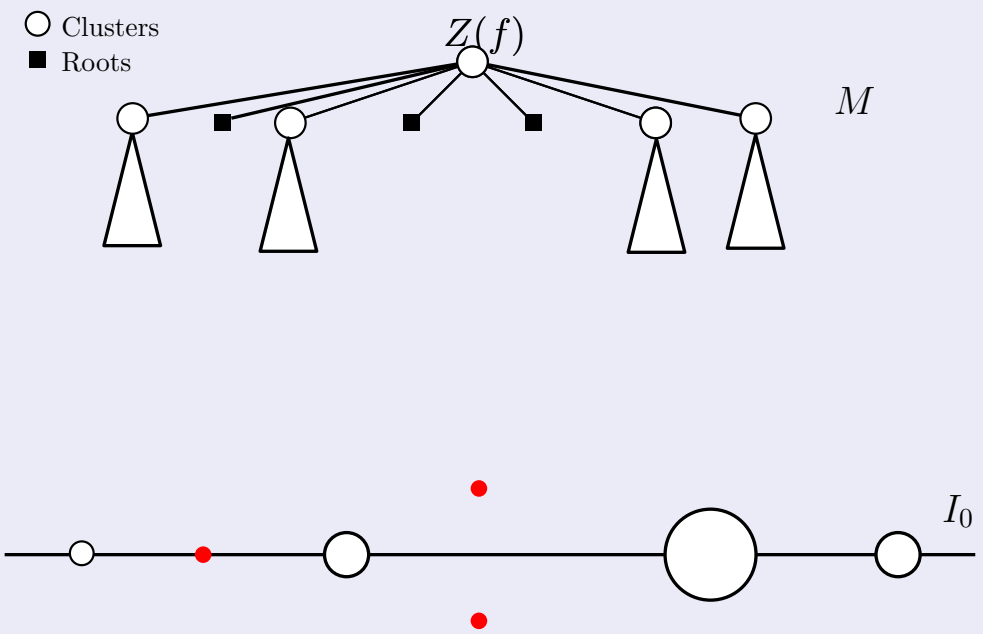

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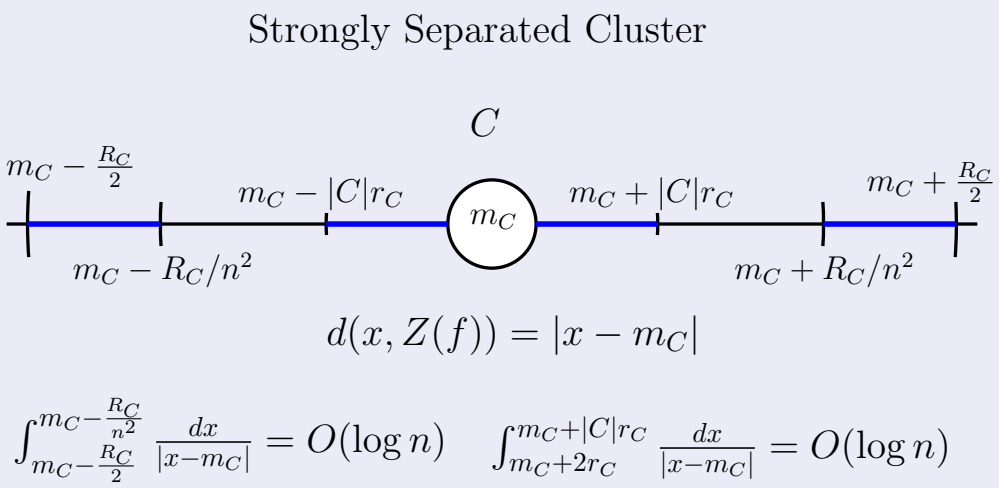

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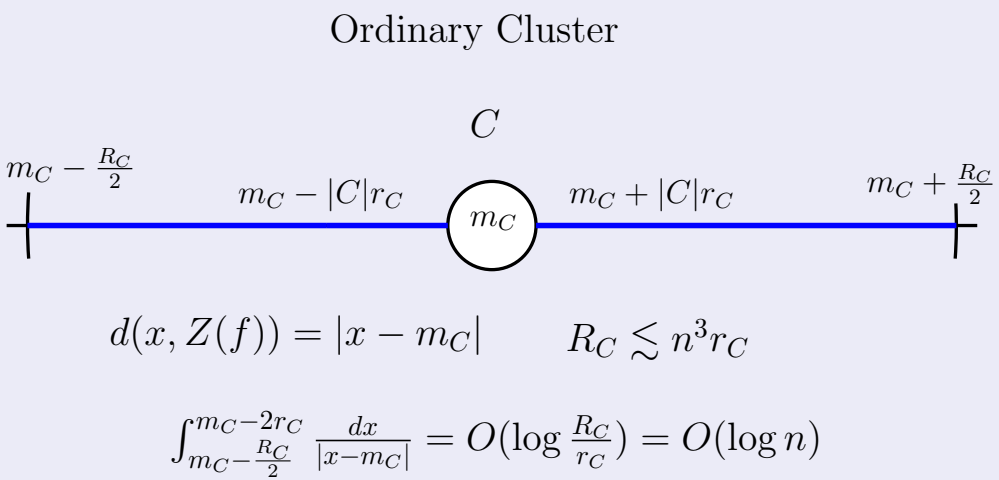

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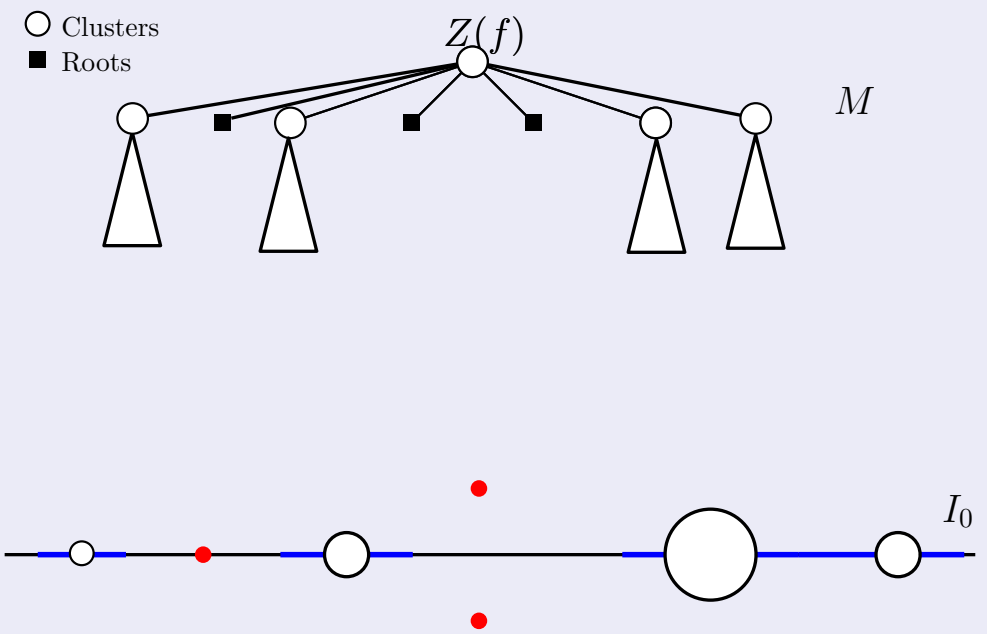

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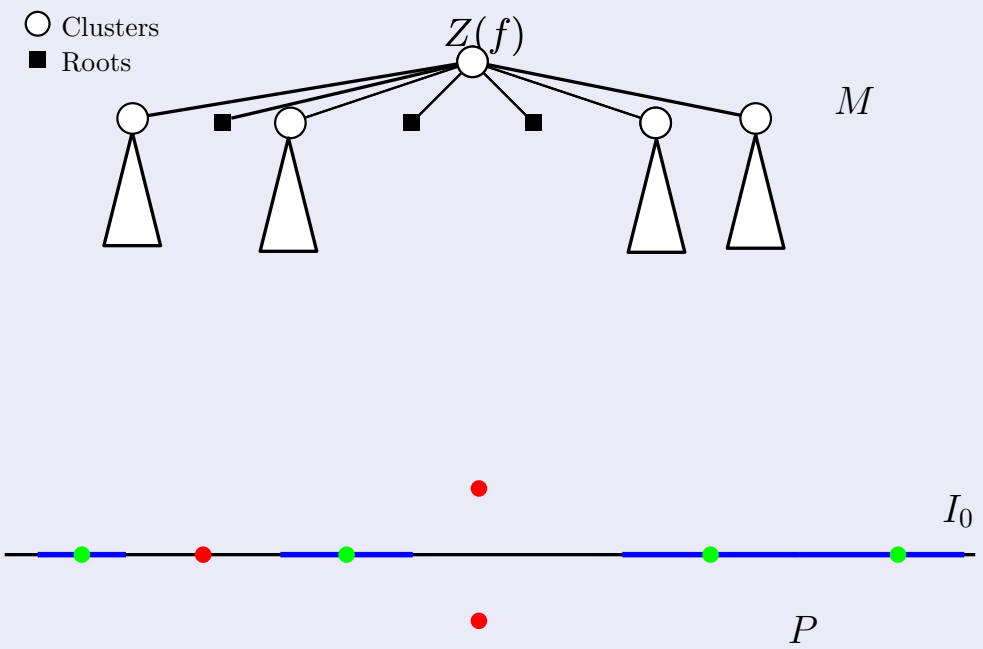

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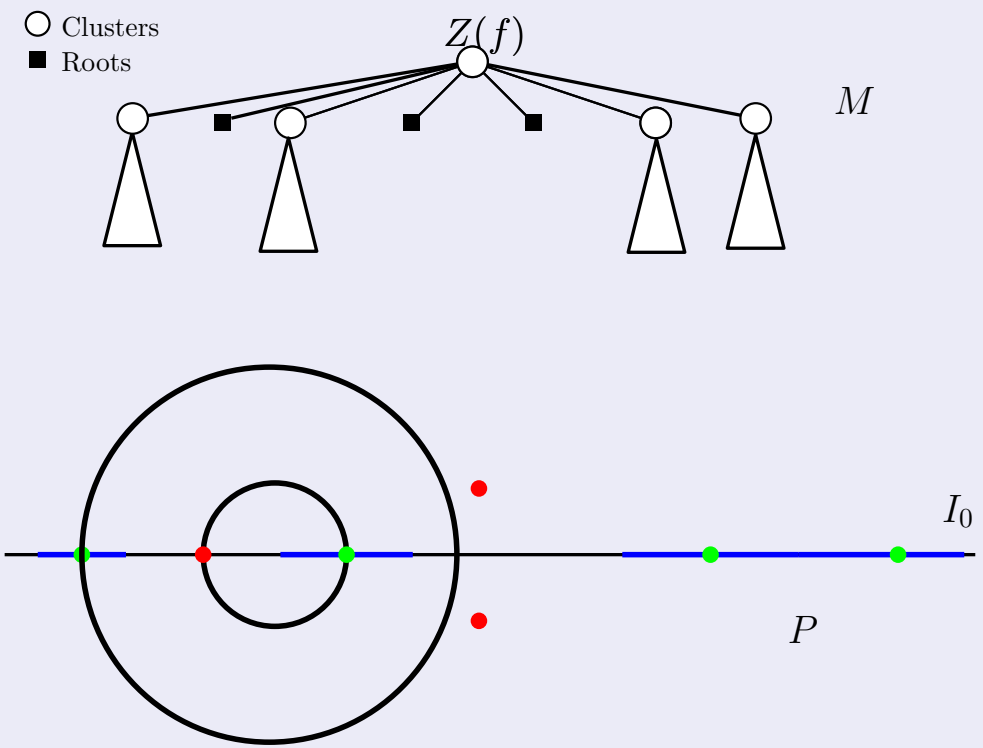

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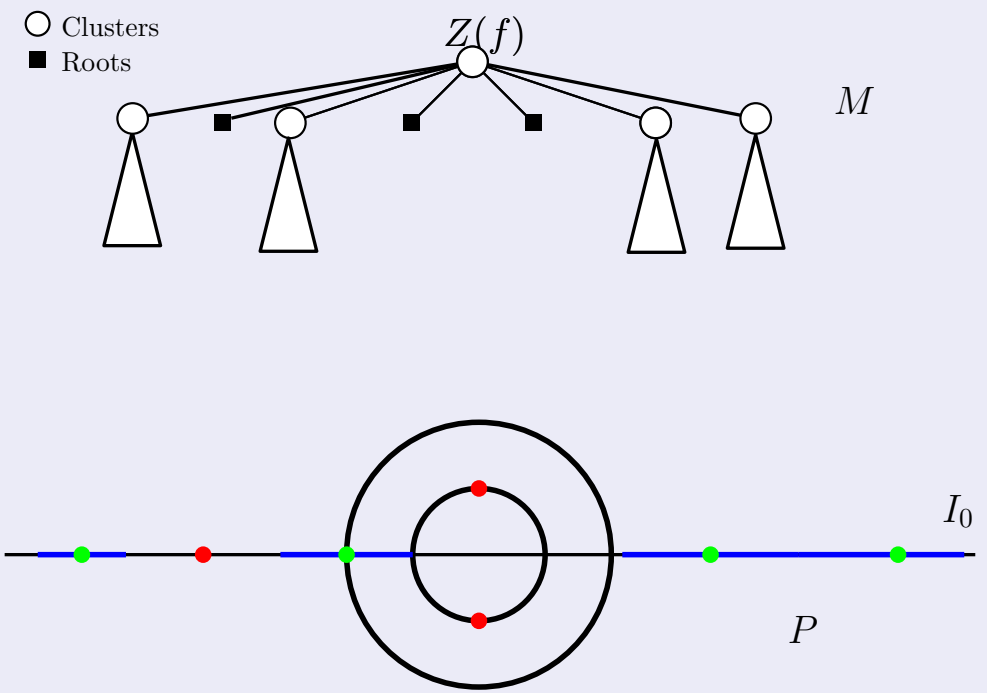

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Bound for dense pointset P

1. For a $q \in P \cap \mathbb{R}$, $J_q := [q \pm d_2(q, P)/2]$.
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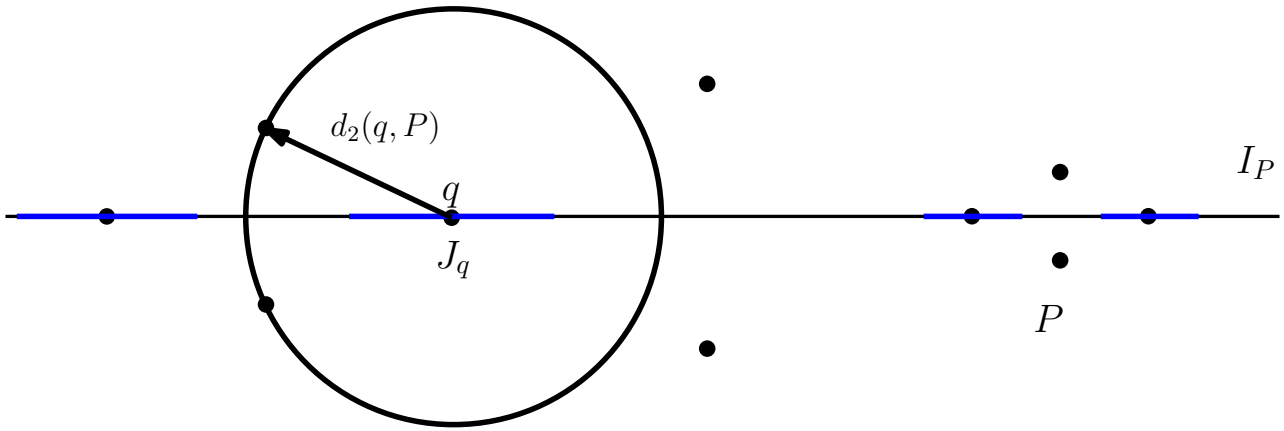

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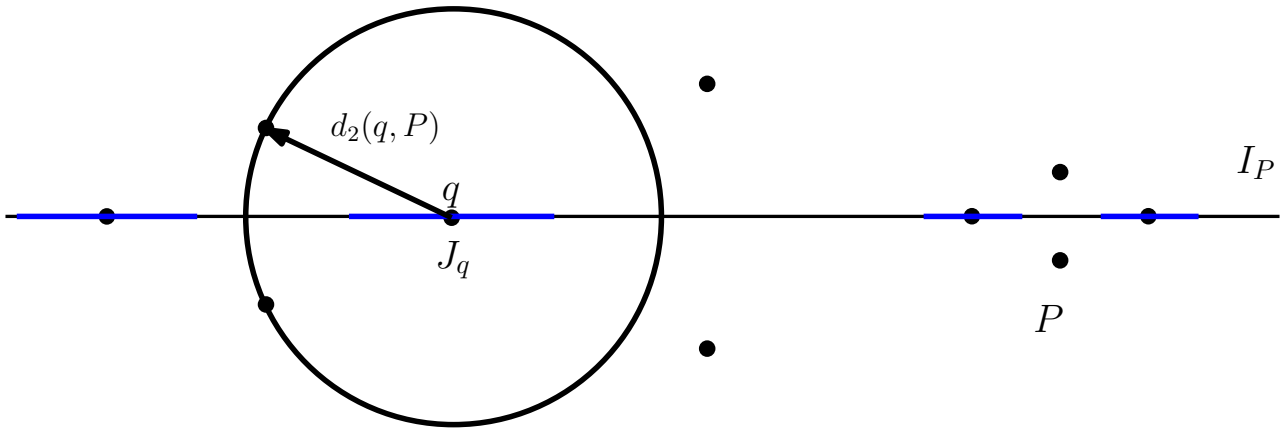

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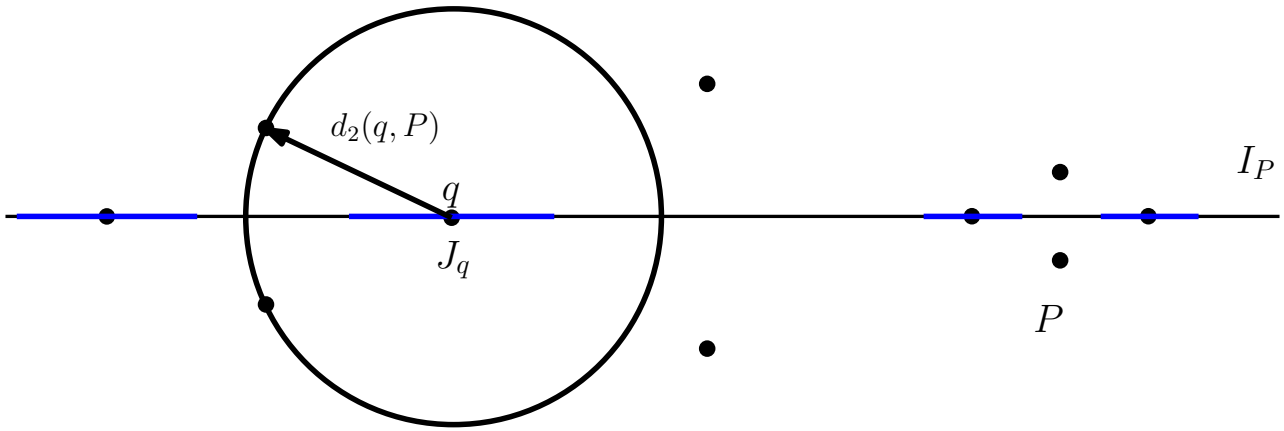

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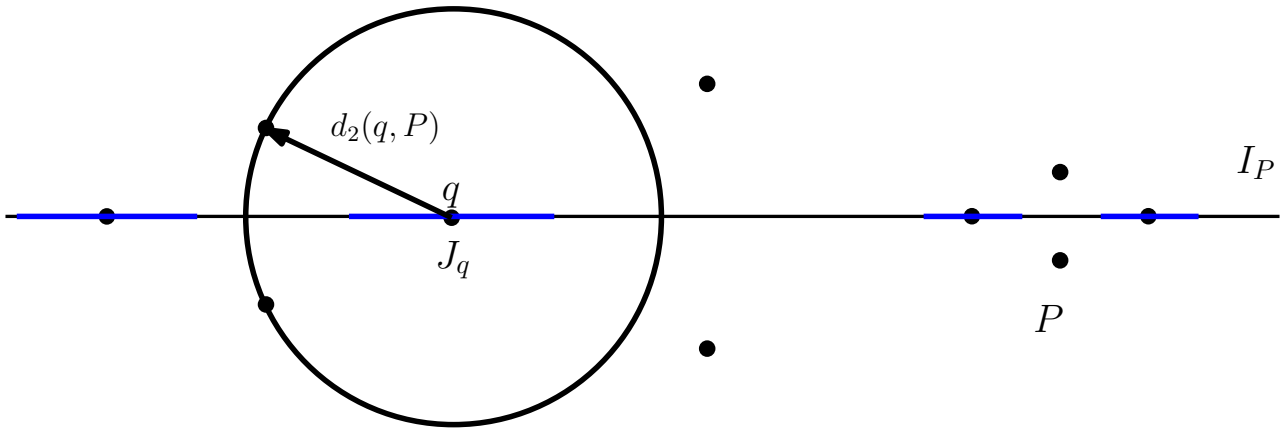

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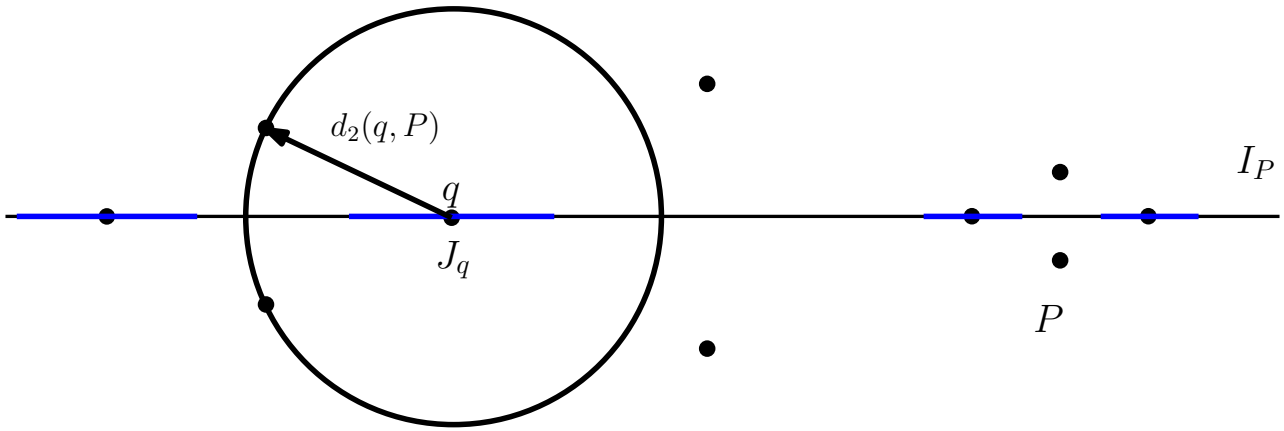

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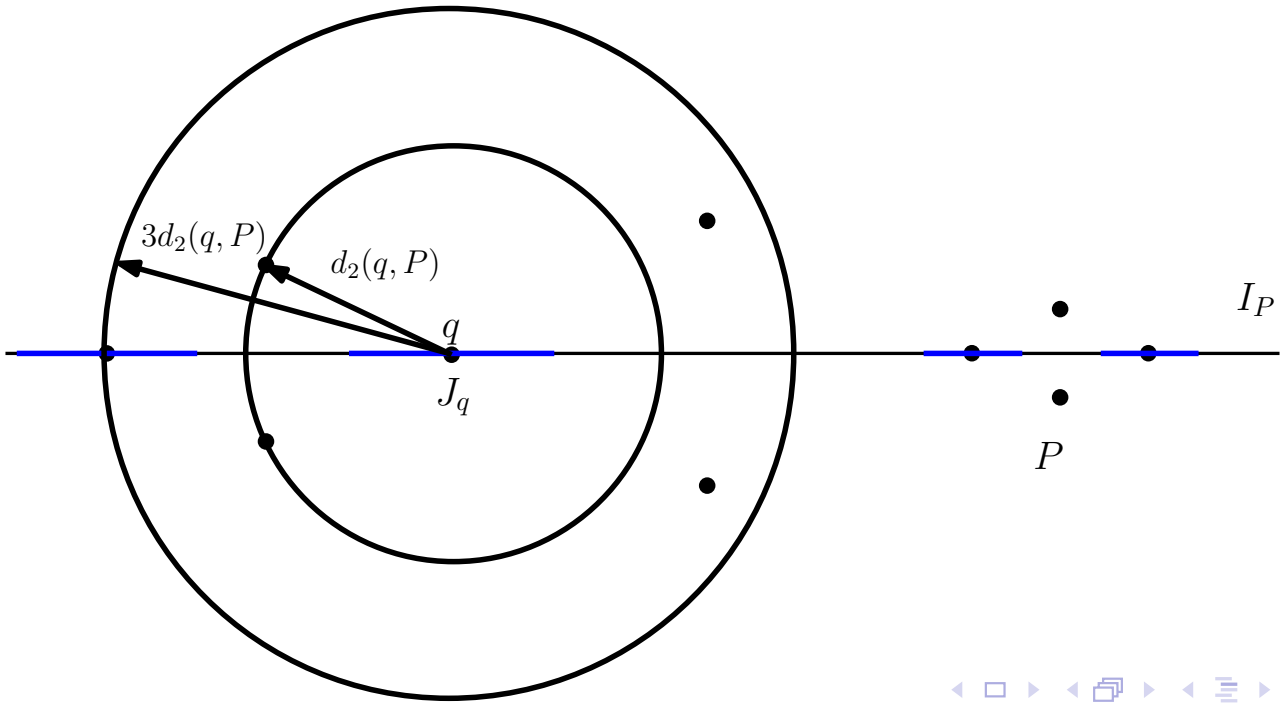

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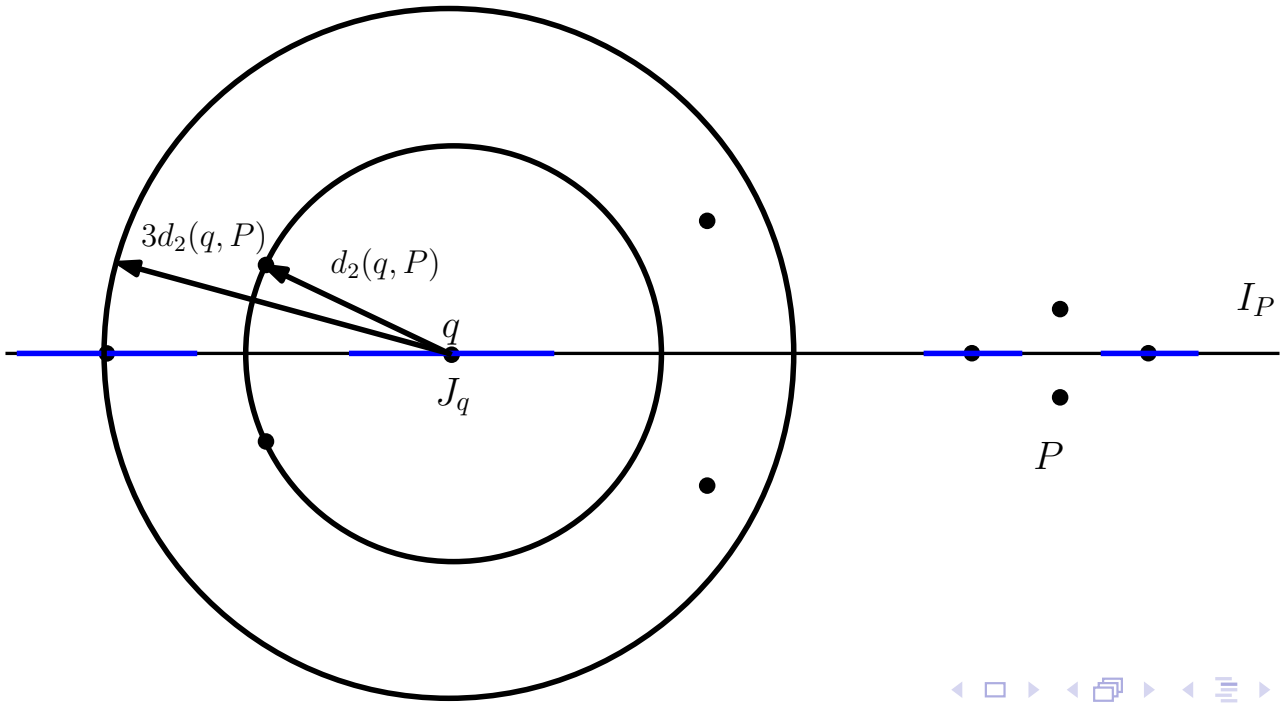

Conclusion and Further directions

Summary

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