Green's Functions for Stieltjes Boundary Problems

Markus Rosenkranz   Nitin Serwa

School of Mathematics, Statistics & Act. Sci.
University of Kent

ISSAC, 2015

Outline

Example of four point Boundary Problem.

Given $f \in C^\infty[a,b]$, find $u \in C^\infty[a,b]$ such that

$$\boxed{\begin{array}{l} -u'' = f, \\ u(0) + u(1/3) = u(1) + u(2/3) = 0. \end{array}}$$

Example of four point Boundary Problem.

Given $f \in C^\infty[a,b]$, find $u \in C^\infty[a,b]$ such that

$$\boxed{\begin{aligned}&-u'' = f,\\ &u(0) + u(1/3) = u(1) + u(2/3) = 0.\end{aligned}}$$

- Represented by pair $\left(-D^2,\ [\lfloor 0 \rfloor + \lfloor 1/3 \rfloor, \lfloor 1 \rfloor + \lfloor 2/3 \rfloor]\ \right)$.

Example of four point Boundary Problem.

Given $f \in C^\infty[a, b]$, find $u \in C^\infty[a, b]$ such that

$$\boxed{\begin{array}{l} -u'' = f, \\ u(0) + u(1/3) = u(1) + u(2/3) = 0. \end{array}}$$

- Represented by pair $\left(-D^2,\ [\lfloor 0 \rfloor + \lfloor 1/3 \rfloor, \lfloor 1 \rfloor + \lfloor 2/3 \rfloor]\ \right)$.
- Evaluation functionals: $\lfloor\ \rfloor$

Green's Operator

Similarly: Regular boundary problem $(T, \mathcal{B})$ for LODE.
Meaning $\exists$ unique solution $u \in C^\infty[a, b]$ for all $f \in C^\infty[a, b]$:

$$\boxed{\begin{array}{l} Tu = f, \\ \beta(u) = 0 \ (\beta \in \mathcal{B}). \end{array}}$$

Green's Operator

Similarly: Regular boundary problem $(T, \mathcal{B})$ for LODE.
Meaning $\exists$ unique solution $u \in C^\infty[a,b]$ for all $f \in C^\infty[a,b]$:

$$\boxed{\begin{aligned} &Tu = f, \\ &\beta(u) = 0 \ (\beta \in \mathcal{B}). \end{aligned}}$$

- Algorithm to compute Green's operator $G : f \mapsto u$.

Green's Operator

Similarly: Regular boundary problem $(T, \mathcal{B})$ for LODE.
Meaning $\exists$ unique solution $u \in C^\infty[a, b]$ for all $f \in C^\infty[a, b]$:

$$\boxed{\begin{array}{l} Tu = f, \\ \beta(u) = 0 \ (\beta \in \mathcal{B}). \end{array}}$$

- Algorithm to compute Green's operator $G : f \mapsto u$.
- However, sometimes we want Green's functions. Why?

Similarly: Regular boundary problem $(T, \mathcal{B})$ for LODE.
Meaning $\exists$ unique solution $u \in C^\infty[a, b]$ for all $f \in C^\infty[a, b]$:

$$\boxed{\begin{aligned} &Tu = f, \\ &\beta(u) = 0 \ (\beta \in \mathcal{B}). \end{aligned}}$$

- Algorithm to compute Green's operator $G : f \mapsto u$.
- However, sometimes we want Green's functions. Why?
  - Nice intuition (see below).
  - Standard form for solutions.
  - Useful for communicating with engineers.

Green's Operator

Similarly: Regular boundary problem $(T, \mathcal{B})$ for LODE.
Meaning $\exists$ unique solution $u \in C^\infty[a,b]$ for all $f \in C^\infty[a,b]$:

$$\boxed{\begin{aligned} &Tu = f, \\ &\beta(u) = 0 \ (\beta \in \mathcal{B}). \end{aligned}}$$

- Algorithm to compute Green's operator $G : f \mapsto u$.
- However, sometimes we want Green's functions. Why?
  - Nice intuition (see below).
  - Standard form for solutions.
  - Useful for communicating with engineers.
- How to extract Green's function from Green's operators?

Intuition behind Green's Function.

Consider again general boundary problem for LODE

$$\boxed{\begin{array}{l} Tu = f, \\ \beta(u) = 0 \ (\beta \in \mathcal{B}). \end{array}}$$

Intuition behind Green's Function.

Consider again general boundary problem for LODE

$$\boxed{\begin{array}{l} Tu = f, \\ \beta(u) = 0 \ (\beta \in \mathcal{B}). \end{array}}$$

- Green's function g_ξ is solution for $f = \delta_\xi$.

Intuition behind Green's Function.

Consider again general boundary problem for LODE

$$\boxed{\begin{array}{l} Tu = f, \\ \beta(u) = 0 \ (\beta \in \mathcal{B}). \end{array}}$$

- Green's function g_ξ is solution for $f = \delta_\xi$.
- Note that $g(x, \xi) = g_\xi(x)$ and "$\delta(x, \xi) = \delta_\xi(x)$".

Intuition behind Green's Function.

Consider again general boundary problem for LODE

$$\boxed{\begin{array}{l} Tu = f, \\ \beta(u) = 0 \ (\beta \in \mathcal{B}). \end{array}}$$

- Green's function g_ξ is solution for $f = \delta_\xi$.
- Note that $g(x, \xi) = g_\xi(x)$ and "$\delta(x, \xi) = \delta_\xi(x)$".

$$\begin{aligned} u &= \int g_\xi \, f(\xi) \, d\xi \\ \Longrightarrow \qquad Tu &= \int Tg_\xi \, f(\xi) \, d\xi = \int \delta_\xi f(\xi) \, d\xi = f \\ \beta(u) &= \int \beta(g_\xi) \, f(\xi) \, d\xi = 0 \end{aligned}$$

Extraction of Green's functions: $G \rightsquigarrow g(x, \xi)$

Extraction of Green's functions: $G \rightsquigarrow g(x, \xi)$

Well known for "classical case" [3].

Green's Functions for Stieltjes Boundary Problems

Extraction of Green's functions: $G \rightsquigarrow g(x, \xi)$

Well known for "classical case" [3].

Sometimes one needs Stieltjes boundary conditions:

Extraction of Green's functions: $G \rightsquigarrow g(x, \xi)$

Well known for "classical case" [3].

Sometimes one needs Stieltjes boundary conditions:

- More than two evaluation points $\rightarrow$ multipoint BP.

Green's Functions for Stieltjes Boundary Problems

Extraction of Green's functions: $G \rightsquigarrow g(x, \xi)$

Well known for "classical case" [3].

Sometimes one needs Stieltjes boundary conditions:

- More than two evaluation points $\rightarrow$ multipoint BP.
- Derivatives of arbitrary order $\rightarrow$ ill-posed BP.

Green's Functions for Stieltjes Boundary Problems

Extraction of Green's functions: $G \rightsquigarrow g(x, \xi)$

Well known for "classical case" [3].

Sometimes one needs Stieltjes boundary conditions:

- More than two evaluation points $\rightarrow$ multipoint BP.
- Derivatives of arbitrary order $\rightarrow$ ill-posed BP.
- Global terms in the form of definite integrals $\rightarrow$ nonlocal BP.

Extraction of Green's functions: $G \rightsquigarrow g(x,\xi)$

Well known for "classical case" [3].

Sometimes one needs Stieltjes boundary conditions:

- More than two evaluation points $\rightarrow$ multipoint BP.
- Derivatives of arbitrary order $\rightarrow$ ill-posed BP.
- Global terms in the form of definite integrals $\rightarrow$ nonlocal BP.

Are there Green's functions $g(x,\xi)$ for such Stieltjes BPs?

Green's Functions for Stieltjes Boundary Problems

Extraction of Green's functions: $G \rightsquigarrow g(x, \xi)$

Well known for "classical case" [3].

Sometimes one needs Stieltjes boundary conditions:

- More than two evaluation points $\rightarrow$ multipoint BP.
- Derivatives of arbitrary order $\rightarrow$ ill-posed BP.
- Global terms in the form of definite integrals $\rightarrow$ nonlocal BP.

Are there Green's functions $g(x, \xi)$ for such Stieltjes BPs?

Can we extract it from G?

Integro-Differential Operators

Recall algebraic setting for boundary problems [3]:

Integro-Differential Operators

Recall algebraic setting for boundary problems [3]:

- Ordinary integro-differential K-algebra $(\mathcal{F}, \partial, \int)$.

Integro-Differential Operators

Recall algebraic setting for boundary problems [3]:

- Ordinary integro-differential K-algebra $(\mathcal{F}, \partial, \int)$.
- Characters on $\mathcal{F}$: multiplicative linear functionals.

Integro-Differential Operators

Recall algebraic setting for boundary problems [3]:

- Ordinary integro-differential K-algebra $(\mathcal{F}, \partial, \int)$.
- Characters on $\mathcal{F}$: multiplicative linear functionals.
- Ring of integro-differential operators over $\mathcal{F}$ with characters Φ.

Integro-Differential Operators

Recall algebraic setting for boundary problems [3]:

- Ordinary integro-differential K-algebra $(\mathcal{F}, \partial, \int)$.
- Characters on $\mathcal{F}$: multiplicative linear functionals.
- Ring of integro-differential operators over $\mathcal{F}$ with characters Φ.
  - Standard integro-differential operator ring $\mathcal{F}_\Phi[\partial, \int]$

Integro-Differential Operators

Recall algebraic setting for boundary problems [3]:

- Ordinary integro-differential K-algebra $(\mathcal{F}, \partial, \int)$.
- Characters on $\mathcal{F}$: multiplicative linear functionals.
- Ring of integro-differential operators over $\mathcal{F}$ with characters Φ.
  - Standard integro-differential operator ring $\mathcal{F}_\Phi[\partial, \int]$
  - Equitable operator ring $\mathcal{F}[\partial, \int_\Phi]$.

Integro-Differential Operators

Recall algebraic setting for boundary problems [3]:

- Ordinary integro-differential K-algebra $(\mathcal{F}, \partial, \int)$.
- Characters on $\mathcal{F}$: multiplicative linear functionals.
- Ring of integro-differential operators over $\mathcal{F}$ with characters Φ.
  - Standard integro-differential operator ring $\mathcal{F}_\Phi[\partial, \int]$
  - Equitable operator ring $\mathcal{F}[\partial, \int_\Phi]$.
  - Isomorphic rings (alternative normal forms).

Integro-Differential Operators

Recall algebraic setting for boundary problems [3]:

- Ordinary integro-differential K-algebra $(\mathcal{F}, \partial, \int)$.
- Characters on $\mathcal{F}$: multiplicative linear functionals.
- Ring of integro-differential operators over $\mathcal{F}$ with characters Φ.
  - Standard integro-differential operator ring $\mathcal{F}_\Phi[\partial, \int]$
  - Equitable operator ring $\mathcal{F}[\partial, \int_\Phi]$.
  - Isomorphic rings (alternative normal forms).
- Later on will have $\mathcal{F} = C^\infty(\mathbf{R})$ again.

Stieltjes Boundary Conditions

Definition

The elements of right ideal $|\Phi) = \Phi \cdot \mathcal{F}_{\Phi}[\partial, \textstyle\int]$ are called Stieltjes boundary conditions.

Stieltjes Boundary Conditions

Definition

The elements of right ideal $|\Phi) = \Phi \cdot \mathcal{F}_\Phi[\partial, \int]$ are called Stieltjes boundary conditions.

Normal forms:

$$\beta(u) = \alpha_1(u^{k_1}) + \cdots + \alpha_r(u^{k_r}) + \int_{\beta_1}^{\gamma_1} + \cdots + \int_{\beta_s}^{\gamma_s}$$

Stieltjes Boundary Conditions

Definition

The elements of right ideal $|\Phi) = \Phi \cdot \mathcal{F}_\Phi[\partial, \int]$ are called Stieltjes boundary conditions.

Normal forms:

$$\beta(u) = \alpha_1(u^{k_1}) + \cdots + \alpha_r(u^{k_r}) + \int_{\beta_1}^{\gamma_1} + \cdots + \int_{\beta_s}^{\gamma_s}$$

Boundary operators:

Two-sided ideal (Φ)

Stieltjes Boundary Conditions

Definition

The elements of right ideal $|\Phi) = \Phi \cdot \mathcal{F}_\Phi[\partial, \textstyle\int]$ are called Stieltjes boundary conditions.

Normal forms:

$$\beta(u) = \alpha_1(u^{k_1}) + \cdots + \alpha_r(u^{k_r}) + \int_{\beta_1}^{\gamma_1} + \cdots + \int_{\beta_s}^{\gamma_s}$$

Boundary operators:

Two-sided ideal $(\Phi) =$ left $\mathcal{F}$-module generated by $|\Phi)$

Stieltjes Boundary Conditions

Definition

The elements of right ideal $|\Phi) = \Phi \cdot \mathcal{F}_\Phi[\partial, \int]$ are called Stieltjes boundary conditions.

Normal forms:

$$\beta(u) = \alpha_1(u^{k_1}) + \cdots + \alpha_r(u^{k_r}) + \int_{\beta_1}^{\gamma_1} + \cdots + \int_{\beta_s}^{\gamma_s}$$

Boundary operators:

Two-sided ideal $(\Phi) =$ left $\mathcal{F}$-module generated by $|\Phi)$

Standard decomposition:

$$\mathcal{F}_\Phi[\partial, \textstyle\int] = \mathcal{F}[\partial] \dotplus \mathcal{F}[\int] \dotplus (\Phi)$$

Equitable Operators

Equitable operator ring used for extracting Green's function via

$$\iota\colon \mathcal{F}_\Phi[\partial, \textstyle\int] \to \mathcal{F}[\partial, \textstyle\int_\Phi],$$

the translation isomorphism defined as follows:

Equitable Operators

Equitable operator ring used for extracting Green's function via

$$\iota\colon \mathcal{F}_{\Phi}[\partial, \textstyle\int] \to \mathcal{F}[\partial, \textstyle\int_{\Phi}],$$

the translation isomorphism defined as follows:

- Fixes $\mathcal{F}$ and ∂.

Equitable Operators

Equitable operator ring used for extracting Green's function via

$$\iota\colon \mathcal{F}_{\Phi}[\partial, \textstyle\int] \to \mathcal{F}[\partial, \textstyle\int_{\Phi}],$$

the translation isomorphism defined as follows:

- Fixes $\mathcal{F}$ and ∂.
- Translate characters by $\iota(\varphi) = \mathrm{id} - \int_{\varphi} \partial$.

Equitable Operators

Equitable operator ring used for extracting Green's function via

$$\iota\colon \mathcal{F}_\Phi[\partial, {\textstyle\int}] \to \mathcal{F}[\partial, {\textstyle\int}_\Phi],$$

the translation isomorphism defined as follows:

- Fixes $\mathcal{F}$ and ∂.
- Translate characters by $\iota(\varphi) = \mathrm{id} - \int_\varphi \partial$.
- Translate back integrals by $\iota^{-1}(\int_\varphi) = (\mathrm{id} - \varphi)\int$.

Extracting Green's Function

Interval $J \subset \mathbf{R}$ containing all evaluation points.

Extracting Green's Function

Interval $J \subset \mathbf{R}$ containing all evaluation points.

Theorem

The Green's function of any regular Stieltjes boundary problem with m evaluations $\alpha_1, \ldots, \alpha_m$ has the form $g(x,\xi) = \tilde{g}(x,\xi) + \hat{g}(x,\xi)$, where the functional part $\tilde{g} \in C(J^2)$ is defined by the $2(m-1)$ case branches

$$\xi \in [\alpha_i, \alpha_{i+1}] \; x \leq \xi,$$
$$\xi \in [\alpha_i, \alpha_{i+1}] \; \xi \leq x,$$

while the distributional part $\hat{g}(x,\xi)$ is an $\mathcal{F}$-linear combination of the $\delta(\xi - \alpha_i)$ and their derivatives.

Illustration of Proof

Consider an example:

- Green's operator
  $G = x\int - \int x + x\lfloor 1 \rfloor \int x - x\lfloor 1 \rfloor \int + e^x \lfloor -1 \rfloor \partial \in \mathcal{F}_\Phi[\partial, \int]$.

Illustration of Proof

Consider an example:

- Green's operator
  $G = x\int - \int x + x\lfloor 1 \rfloor \int x - x\lfloor 1 \rfloor \int + e^x \lfloor -1 \rfloor \partial \in \mathcal{F}_\Phi[\partial, \int]$.
- Translate to equitable ring by $\lfloor \alpha \rfloor \int \to \int_0 - \int_\alpha$.

Illustration of Proof

Consider an example:

- Green's operator
  $G = x\int - \int x + x\lfloor 1 \rfloor \int x - x\lfloor 1 \rfloor \int + e^x \lfloor -1 \rfloor \partial \in \mathcal{F}_\Phi[\partial, \int]$.
- Translate to equitable ring by $\lfloor \alpha \rfloor \int \to \int_0 - \int_\alpha$.
- Yields $G = x\int_0 x - x\int_1 x - \int_0 x + x\int_1 + e^x \lfloor -1 \rfloor \partial \in \mathcal{F}[\partial, \int_\Phi]$.

Illustration of Proof

Consider an example:

- Green's operator
  $G = x\int - \int x + x\lfloor 1\rfloor\int x - x\lfloor 1\rfloor\int + e^x\lfloor -1\rfloor\partial \in \mathcal{F}_\Phi[\partial, \int]$.
- Translate to equitable ring by $\lfloor\alpha\rfloor\int \to \int_0 - \int_\alpha$.
- Yields $G = x\int_0 x - x\int_1 x - \int_0 x + x\int_1 + e^x\lfloor -1\rfloor\partial \in \mathcal{F}[\partial, \int_\Phi]$.
- Extract $\tilde{g}$ by $f\int_\alpha g = f(x)\,g(\xi)\,[\alpha \le \xi][\xi \le x]$
  $- f(x)\,g(\xi)\,[\xi \le \alpha][x \le \xi]$.

Illustration of Proof

Consider an example:

- Green's operator
  $G = x\int - \int x + x\lfloor 1 \rfloor \int x - x\lfloor 1 \rfloor \int + e^x \lfloor -1 \rfloor \partial \in \mathcal{F}_\Phi[\partial, \int]$.
- Translate to equitable ring by $\lfloor \alpha \rfloor \int \to \int_0 - \int_\alpha$.
- Yields $G = x\int_0 x - x\int_1 x - \int_0 x + x\int_1 + e^x \lfloor -1 \rfloor \partial \in \mathcal{F}[\partial, \int_\Phi]$.
- Extract $\tilde{g}$ by $f\int_\alpha g = f(x)\, g(\xi)\, [\alpha \leq \xi][\xi \leq x]$
  $- f(x)\, g(\xi)\, [\xi \leq \alpha][x \leq \xi]$.
- Extract $\hat{g}$ by $f\, \lfloor \alpha \rfloor \partial^i = (-1)^i f(x)\, \delta^{(i)}(\xi - \alpha)$.

Illustration of Proof

Consider an example:

- Green's operator
  $G = x{\textstyle\int} - {\textstyle\int} x + x\lfloor 1 \rfloor {\textstyle\int} x - x \lfloor 1 \rfloor {\textstyle\int} + e^x \lfloor -1 \rfloor \partial \in \mathcal{F}_\Phi[\partial, {\textstyle\int}]$.
- Translate to equitable ring by $\lfloor \alpha \rfloor {\textstyle\int} \to {\textstyle\int}_0 - {\textstyle\int}_\alpha$.
- Yields $G = x{\textstyle\int}_0 x - x{\textstyle\int}_1 x - {\textstyle\int}_0 x + x{\textstyle\int}_1 + e^x \lfloor -1 \rfloor \partial \in \mathcal{F}[\partial, {\textstyle\int}_\Phi]$.
- Extract $\tilde{g}$ by $f{\textstyle\int}_\alpha g = f(x)\, g(\xi)\, [\alpha \le \xi][\xi \le x]$
  $\qquad - f(x)\, g(\xi)\, [\xi \le \alpha][x \le \xi]$.
- Extract $\hat{g}$ by $f\, \lfloor \alpha \rfloor \partial^i = (-1)^i f(x)\, \delta^{(i)}(\xi - \alpha)$.
- Resulting Green's function:

$$\tilde{g}(x,\xi) = \begin{cases} (x-1)\xi & \text{for } 0 \le \xi \le x \le 1, \\ x(\xi - 1) & \text{for } 0 \le x \le \xi \le 1. \end{cases}$$

$$\hat{g}(x,\xi) = -e^x \delta'(\xi + 1)$$

Four Point Boundary Problem

Back to our first example:

$$\boxed{\begin{array}{l} -u'' = f, \\ u(0) + u(1/3) = u(1) + u(2/3) = 0 \end{array}}$$

Four Point Boundary Problem

Back to our first example:

$$-u'' = f,$$
$$u(0) + u(1/3) = u(1) + u(2/3) = 0$$

- Green's operator (computed using INTDIFFOP by A. Korporal):

$$\begin{aligned} G = x{\textstyle\int} - {\textstyle\int} x + (-5/24 + x/4)\lfloor 1/3 \rfloor {\textstyle\int} \\ + (5/8 - 3x/4)\lfloor 1/3 \rfloor {\textstyle\int} x + (1/8 - 3x/4)\lfloor 1 \rfloor {\textstyle\int} x \\ + (1/12 - x/2)\lfloor 2/3 \rfloor {\textstyle\int} + (-1/8 + 3x/4)\lfloor 2/3 \rfloor {\textstyle\int} x \end{aligned}$$

Four Point Boundary Problem

Back to our first example:

$$\begin{aligned} &-u'' = f, \\ &u(0) + u(1/3) = u(1) + u(2/3) = 0 \end{aligned}$$

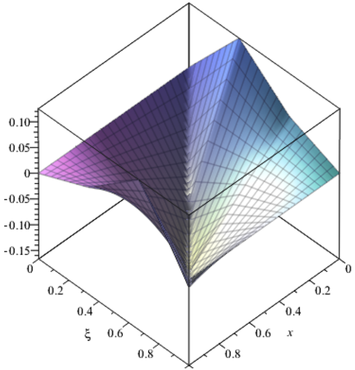
- Green's operator (computed using INTDIFFOP by A. Korporal):

$$\begin{aligned} G = {} & x{\textstyle\int} - {\textstyle\int} x + (-5/24 + x/4)\lfloor 1/3 \rfloor {\textstyle\int} \\ & + (5/8 - 3x/4)\lfloor 1/3 \rfloor {\textstyle\int} x + (1/8 - 3x/4)\lfloor 1 \rfloor {\textstyle\int} x \\ & + (1/12 - x/2)\lfloor 2/3 \rfloor {\textstyle\int} + (-1/8 + 3x/4)\lfloor 2/3 \rfloor {\textstyle\int} x \end{aligned}$$

- Green's function (computed using extra code)

$$g(x,\xi) = \begin{cases} (3/4)x\xi - (5/8)\xi & : 0 \le \xi \le 1/3, \xi \le x \\ (3/4)x\xi + (3/8)\xi - x & : 0 \le \xi \le 1/3, x \le \xi \\ (3/2)x\xi - (5/4)\xi - (1/4)x + 5/24 & : 1/3 \le \xi \le 2/3, \xi \le x \\ (3/2)x\xi - (1/4)\xi - (5/4)x + 5/24 & : 1/3 \le \xi \le 2/3, x \le \xi \\ (3/4)x\xi - (9/8)\xi + (1/4)x + 1/8 & : 2/3 \le \xi \le 1, \xi \le x \\ (3/4)x\xi - (1/8)\xi - (3/4)x + 1/8 & : 2/3 \le \xi \le 1, x \le \xi \end{cases}$$

Graph of its Green's Function



Nonclassical Boundary Problem

Example with three evaluations, nonlocal part and higher-order derivative:

$$\boxed{\begin{aligned} &u'' - u = f, \\ &u'''(-1) - \textstyle\int_0^1 u(\xi)\,\xi\,d\xi = 0, \\ &u'(-1) - u''(1) + \textstyle\int_{-1}^1 u(\xi)\,d\xi = 0, \end{aligned}}$$

Nonclassical Boundary Problem

Example with three evaluations, nonlocal part and higher-order derivative:

$$\begin{aligned} &u'' - u = f, \\ &u'''(-1) - \textstyle\int_0^1 u(\xi)\,\xi\,d\xi = 0, \\ &u'(-1) - u''(1) + \textstyle\int_{-1}^1 u(\xi)\,d\xi = 0, \end{aligned}$$

Green's operator (with $\sigma := 2(2e-3)(e-1)$ for brevity):

$$\begin{aligned} \sigma G = {} & \sigma/2\,(e^x \textstyle\int e^{-x} - e^{-x} \textstyle\int e^x) \\ & + 2(-e^{x+3} + e^{x+2} - e^{x+1} + e^{-x+2} - e^{-x+1})(\lfloor -1 \rfloor \partial + \lfloor 1 \rfloor \textstyle\int x) \\ & + (e-1)(-e^{x+2} - 2e^{x+1} + e^{-x+1})(\lfloor -1 \rfloor \textstyle\int + \lfloor 1 \rfloor \textstyle\int) \\ & + (3e^{x+2} - e^{x+1} - 3e^{-x+1} + 3e^{-x}) \lfloor 1 \rfloor \textstyle\int e^x \\ & + (2e^{x+2} - 3e^{x+1})(e^{-1} \lfloor -1 \rfloor \textstyle\int e^{-x} + e \lfloor -1 \rfloor \textstyle\int e^x) \\ & + (-e^{x+3} - e^{x+2} + 2e^{x+1} + e^{-x+2} - e^{-x+1}) \lfloor 1 \rfloor \end{aligned}$$

Green's Function for Nonclassical Problem

Green's function

Green's Function for Nonclassical Problem

Green's function

- Distributional part

$$\sigma\,\hat{g}(x,\xi) = (-e^{x+3} - e^{x+2} + 2e^{x+1} + e^{-x+2} - e^{-x+1})\,\delta(\xi-1)$$
$$+\,2\,(-e^{x+3} + e^{x+2} - e^{x+1} + e^{-x+2} - e^{-x+1})\,\delta'(\xi-1)$$

Green's Function for Nonclassical Problem

Green's function

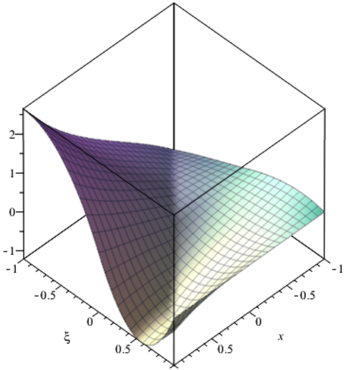
- Distributional part

$$\sigma\,\hat{g}(x,\xi) = \left(-e^{x+3} - e^{x+2} + 2e^{x+1} + e^{-x+2} - e^{-x+1}\right)\delta(\xi-1) + 2\left(-e^{x+3} + e^{x+2} - e^{x+1} + e^{-x+2} - e^{-x+1}\right)\delta'(\xi-1)$$

- Functional part

$$\tilde{g}(x,\xi) = \begin{cases} \begin{array}{l} -1 \le \xi \le 0 \\ \xi \le x \end{array} & \begin{array}{l} 3e^{x+2+\xi} + 3e^{x-\xi} - 2e^{x+1-\xi} - 2e^{3+x+\xi} \\ \quad + e^{3+x} + e^{-x+1} + e^{x+2} - e^{-x+2} - 2e^{x+1} \end{array} \\ \begin{array}{l} -1 \le \xi \le 0 \\ x \le \xi \end{array} & \begin{array}{l} -2e^{x+1} + 2e^{-x+2+\xi} - 5e^{-x+1+\xi} - 2e^{x+2-\xi} \\ \quad -2e^{3+x+\xi} + 3e^{-x+\xi} + e^{-x+1} + e^{x+2} \\ \quad + e^{3+x} + 3e^{x+1-\xi} + 3e^{x+2+\xi} - e^{-x+2} \end{array} \\ \begin{array}{l} 0 \le \xi \le 1 \\ \xi \le x \end{array} & \begin{array}{l} -2e^{3+x}\xi - 2e^{-x+1}\xi + 2e^{x+2}\xi + 2e^{-x+2}\xi \\ \quad -2e^{x+1}\xi + 3e^{x+2+\xi} + 3e^{x-\xi} - 5e^{x+1-\xi} \\ \quad +2e^{-x+1+\xi} - e^{x+1+\xi} - 2e^{-x+2+\xi} + 2e^{x+2-\xi} \\ \quad -e^{3+x} - e^{-x+1} - e^{x+2} + e^{-x+2} + 2e^{x+1} \end{array} \\ \begin{array}{l} 0 \le \xi \le 1 \\ x \le \xi \end{array} & \begin{array}{l} -2e^{3+x}\xi - 2e^{-x+1}\xi + 2e^{x+2}\xi + 2e^{-x+2}\xi \\ \quad -2e^{x+1}\xi + 3e^{-x+\xi} + 3e^{x+2+\xi} - e^{3+x} \\ \quad -e^{-x+1} - e^{x+2} + e^{-x+2} + 2e^{x+1} \\ \quad -3e^{-x+1+\xi} - e^{x+1+\xi} \end{array} \end{cases}$$

Graph of Functional Part of its Green's Function



Summary

From Green's operator to Green's function:

Summary

From Green's operator to Green's function:

- Generalised Green's functions for Stieltjes BPs.

Summary

From Green's operator to Green's function:

- Generalised Green's functions for Stieltjes BPs.
- Extraction algorithm from Green's operators.

Summary

From Green's operator to Green's function:

- Generalised Green's functions for Stieltjes BPs.
- Extraction algorithm from Green's operators.

Form of Green's functions:

- Decomposition $g(x, \xi) = \tilde{g}(x, \xi) + \hat{g}(x, \xi)$.

Summary

From Green's operator to Green's function:

- Generalised Green's functions for Stieltjes BPs.
- Extraction algorithm from Green's operators.

Form of Green's functions:

- Decomposition $g(x, \xi) = \tilde{g}(x, \xi) + \hat{g}(x, \xi)$.
- Functional part $\tilde{g}(x, \xi)$ defined by $2(m-1)$ case branches.

Summary

From Green's operator to Green's function:

- Generalised Green's functions for Stieltjes BPs.
- Extraction algorithm from Green's operators.

Form of Green's functions:

- Decomposition $g(x,\xi) = \tilde{g}(x,\xi) + \hat{g}(x,\xi)$.
- Functional part $\tilde{g}(x,\xi)$ defined by $2(m-1)$ case branches.
- Distributional part $\hat{g}(x,\xi)$ is $\mathcal{F}$-linear combination of $\delta(\xi - \alpha_i)$ and their derivatives.

Future work

In this paper: $\mathcal{F} = C^\infty(\mathbf{R})$.

Future work

In this paper: $\mathcal{F} = C^\infty(\mathbf{R})$. How about general $(\mathcal{F}, \partial, \int)$?

Future work

In this paper: $\mathcal{F} = C^\infty(\mathbf{R})$. How about general $(\mathcal{F}, \partial, \int)$?

Need algebraic structures where these Green's functions "live":

Future work

In this paper: $\mathcal{F} = C^\infty(\mathbf{R})$. How about general $(\mathcal{F}, \partial, \int)$?

Need algebraic structures where these Green's functions "live":

- Functional part $g(x, \xi)$: Ring $\mathcal{F} \otimes \mathcal{F}$ is sufficient.

Future work

In this paper: $\mathcal{F} = C^\infty(\mathbf{R})$. How about general $(\mathcal{F}, \partial, \int)$?

Need algebraic structures where these Green's functions "live":

- Functional part $g(x, \xi)$: Ring $\mathcal{F} \otimes \mathcal{F}$ is sufficient.
- Distributional part $\hat{g}(x, \xi)$: Integro-differential module generated over $\mathcal{F} \otimes \mathcal{F}$ by "algebraic Diracs".

Thank you

For Further Reading I

A. Korporal, G. Regensburger, and M. Rosenkranz.
A Maple package for integro-differential operators and boundary problems, 2010.
Also presented as a poster at ISSAC '10.

M. Rosenkranz and G. Regensburger.
Integro-differential polynomials and operators.
In D. Jeffrey, editor, *ISSAC'08: Proceedings of the 2008 International Symposium on Symbolic and Algebraic Computation*. ACM Press, 2008.

M. Rosenkranz and G. Regensburger.
Solving and factoring boundary problems for linear ordinary differential equations in differential algebras.
Journal of Symbolic Computation, 43(8):515–544, 2008.

For Further Reading II

M. Rosenkranz.
A new symbolic method for solving linear two-point boundary value problems on the level of operators.
J. Symbolic Comput., 39(2):171–199, 2005.

I. Stakgold and M. Holst.
Green's functions and boundary value problems.
Pure and Applied Mathematics (Hoboken). John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2011.