Graph-coloring ideals

Nullstellensatz certificates, Gröbner bases for chordal graphs, and hardness of Gröbner bases

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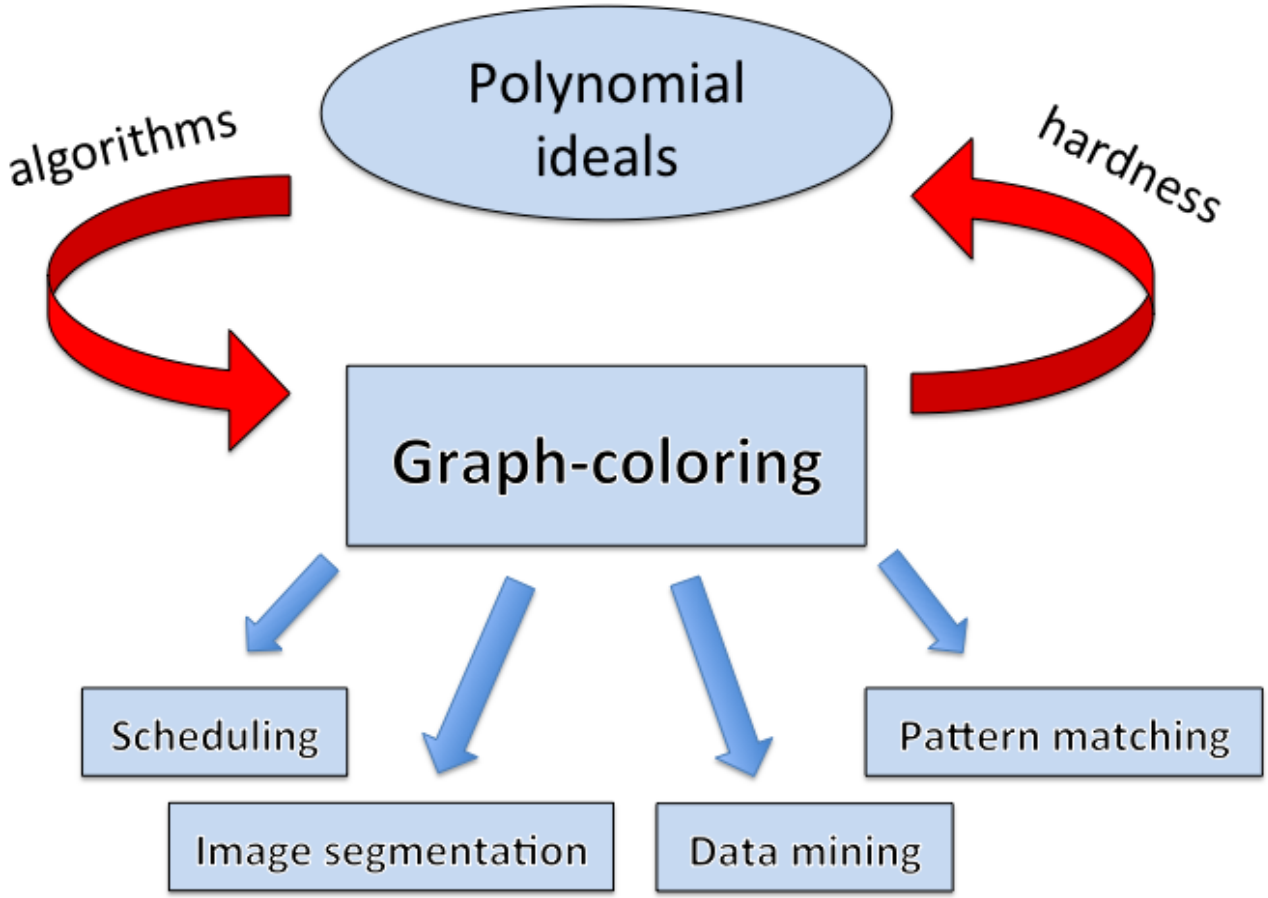
Bath, July 2015

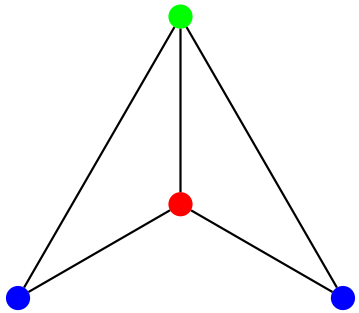
Overview


Polynomial ideals
algorithms
hardness
Graph-coloring
Scheduling
Image segmentation
Data mining
Pattern matching

Graph-coloring

Graph-coloring problem:

- Proper coloring: no two neighboring vertices the same color
- Is there a proper coloring with k colors?



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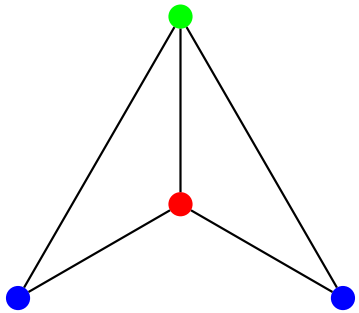

Approach:

- k-coloring $\Leftrightarrow$ system of polynomial equations
- Solve the system or prove unsolvable

The coloring ideal

- Graph $G = (V, E)$
- Variable x_i for each vertex $i \in V$
- Coloring ideal $\mathcal{I}_k(G)$ generated by:

Vertex polynomials $$\nu_i(\mathbf{x}) := x_i^k - 1, \quad \forall i \in V$$

Edge polynomials $$\eta_{ij}(\mathbf{x}) := \frac{x_i^k - x_j^k}{x_i - x_j}, \quad \forall ij \in E$$

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- Solutions $\mathbf{x} \Leftrightarrow$ proper k-colorings [Bayer 1982]
- Need tool for finding solutions to a polynomial system

Gröbner bases

- Polynomial ideal $\mathcal{I}$
- Leading terms of $f \in \mathcal{I}$ form leading term ideal $\mathcal{L}(\mathcal{I})$
- Gröbner basis: $g_1, g_2, \ldots, g_m \in \mathcal{I}$, leading terms generate $\mathcal{L}(\mathcal{I})$
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  - $\mathcal{I} = \langle x + z, x + y \rangle$ with lexicographic monomial order $x > y > z$
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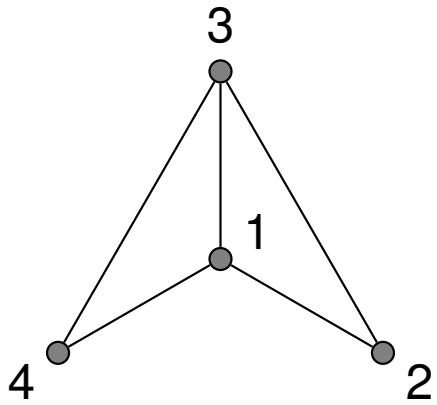
- Gröbner basis $\Rightarrow$ solutions to ideal

Gröbner basis of coloring ideal



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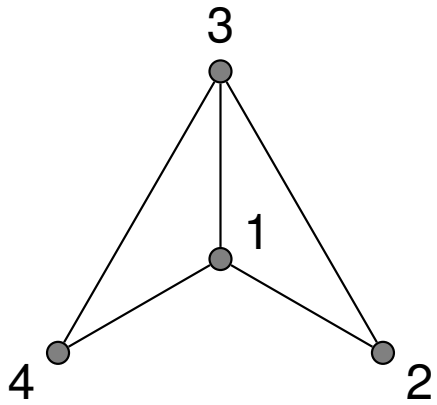

- Coloring ideal for $k = 3$:

$$\langle x_1^3 - 1, x_2^3 - 1, x_3^3 - 1, x_4^3 - 1, x_1^2 + x_1x_2 + x_2^2, x_1^2 + x_1x_3 + x_3^2,$$
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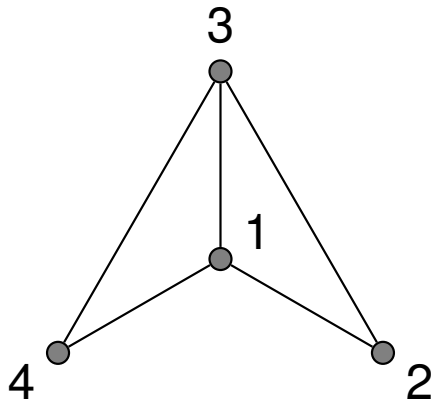

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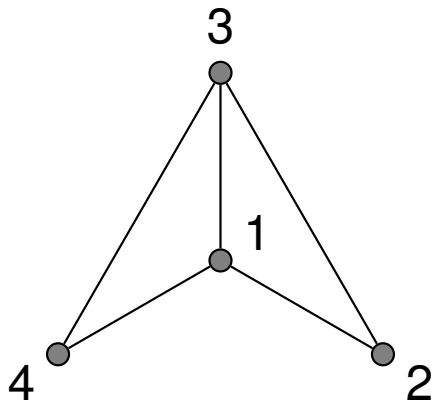

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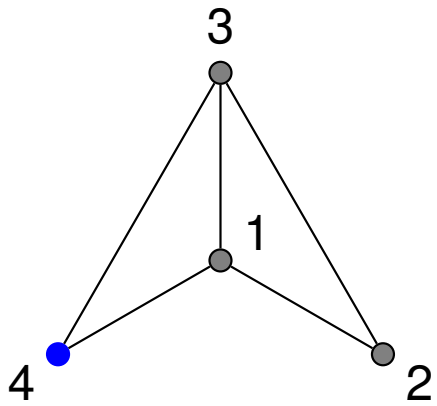

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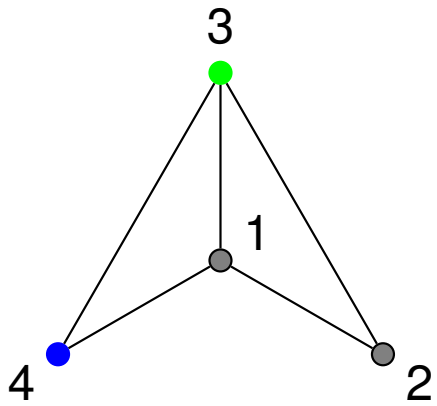

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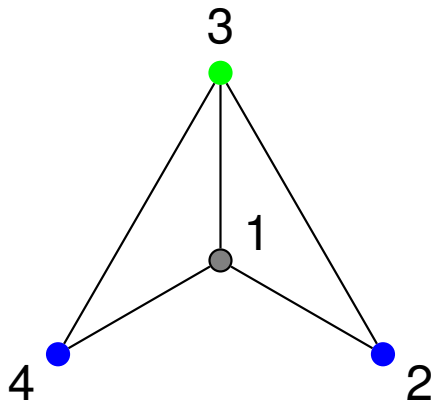

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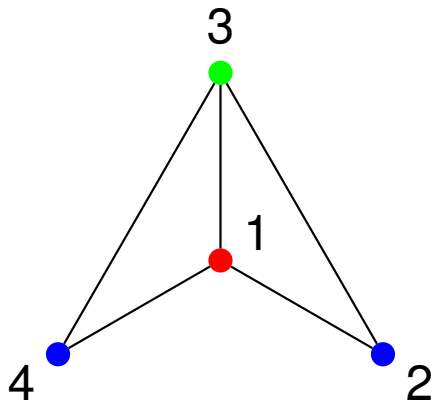

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Computing Gröbner bases

- Buchberger's algorithm works but is slow
- Computation of Gröbner bases is EXPSPACE-complete [Kühnle and Mayr 1996]
- Even hard to write down: maximum degree can be large
- Mayr, Ritscher (2010): upper bound on maximum degree for r-dimensional ideal, n generators of degree d:

$$2\left(\frac{1}{2}d^{n-r}+d\right)^{2^r}$$

- Ritscher (2009): example attaining maximum degree d^n
- In practice, special cases often tractable

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Definition

A graph is chordal if it has no induced cycle of length ≥ 4.

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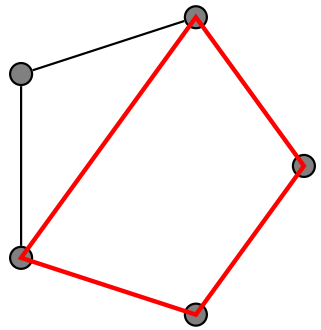

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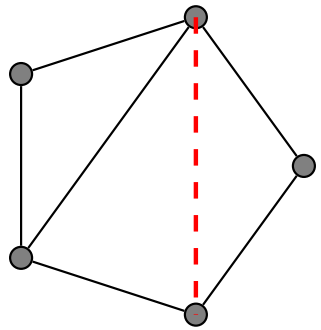

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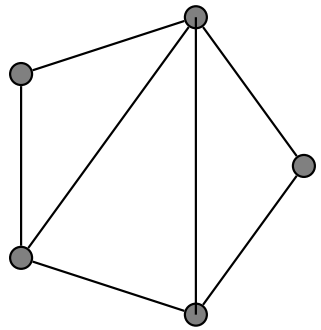

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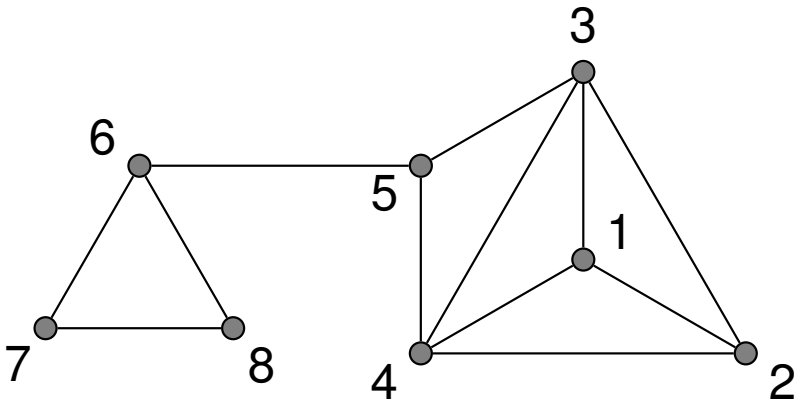

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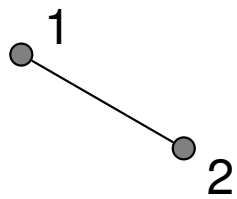

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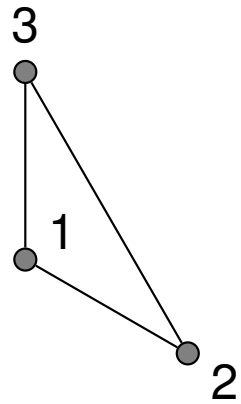

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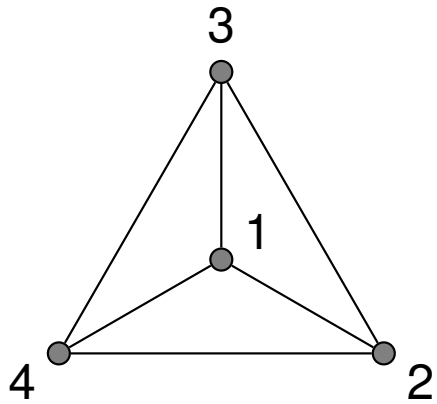

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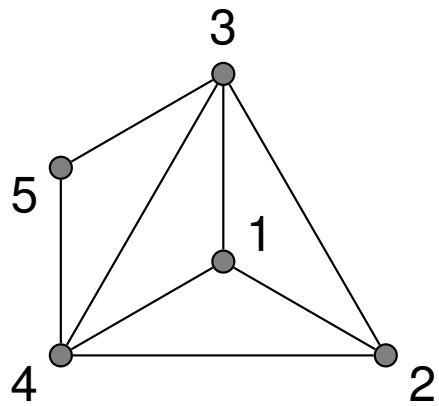

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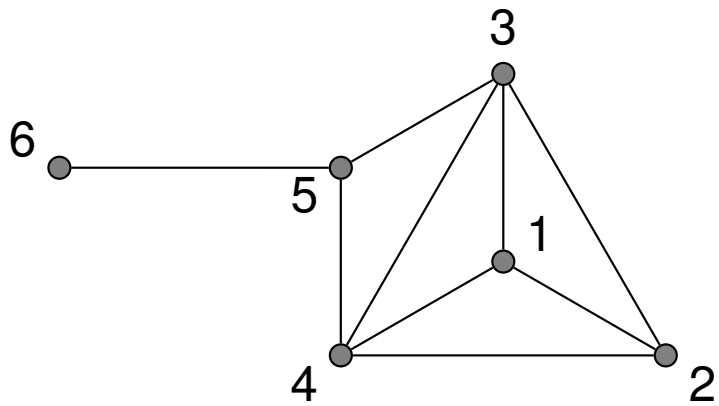

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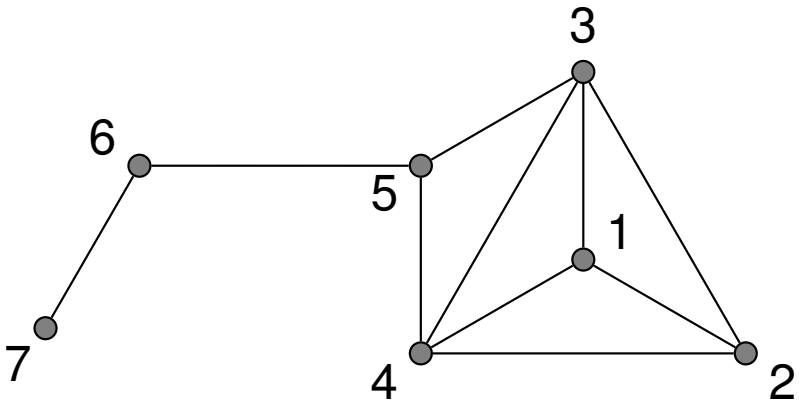

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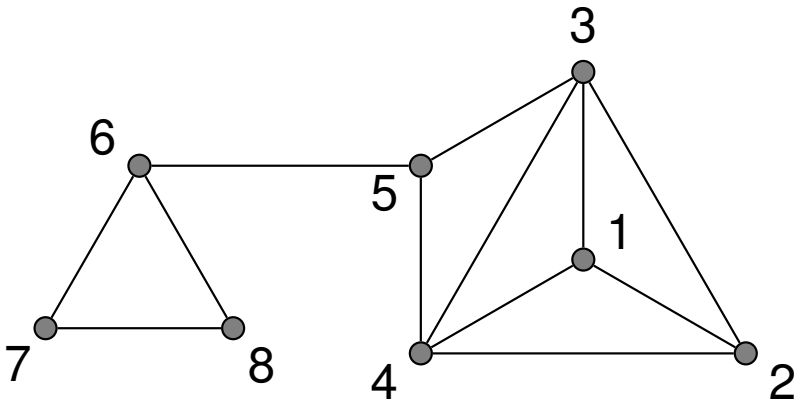

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Theorem (DMPRRSSS)

Let G be a chordal graph on n vertices. Then there exists a Gröbner basis of size n for $\mathcal{I}_k(G)$, and it can be found efficiently.

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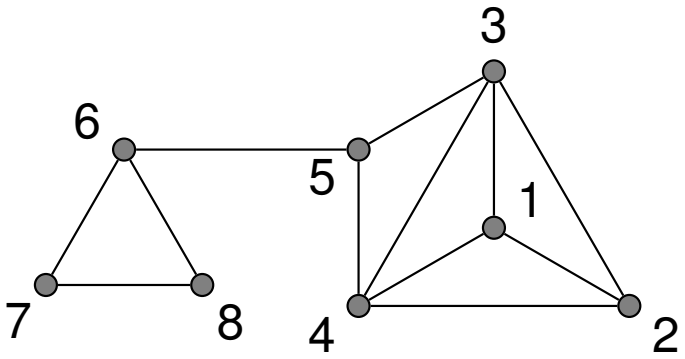

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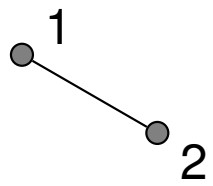

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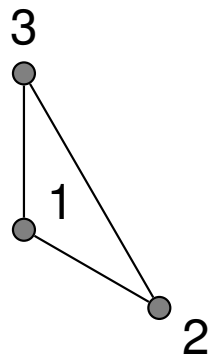

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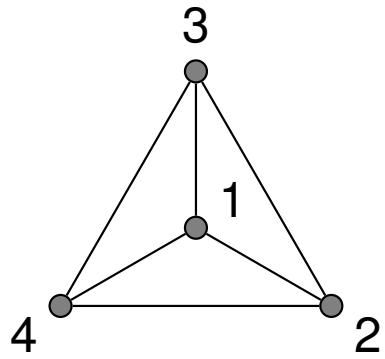

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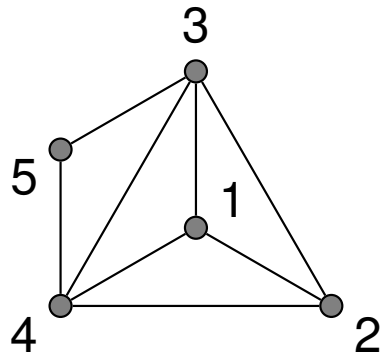

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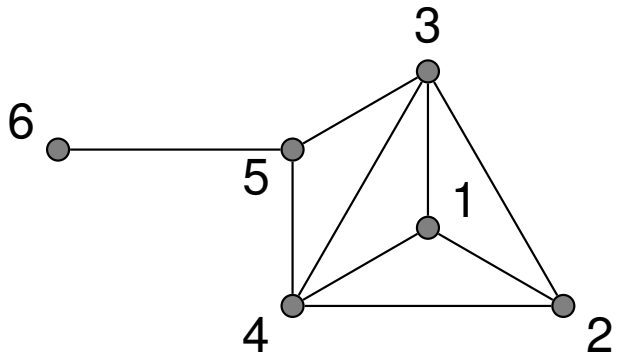

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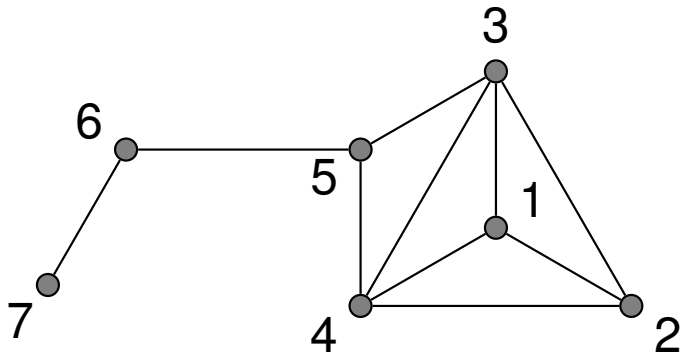

Gröbner basis $(k = 4)$: $\{\nu_1(x_1), S_3(x_1, x_2), S_2(x_1, x_2, x_3),$
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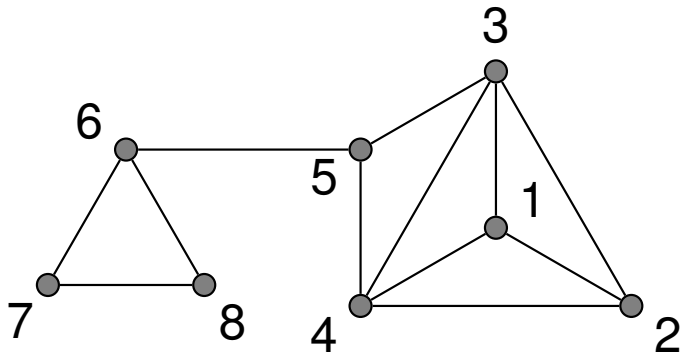

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$$S_m(y_1, \ldots, y_t) := \sum_{1 \le j_1 \le \cdots \le j_m \le t} y_{j_1} \cdots y_{j_m} \quad .$$

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Complete homogeneous symmetric polynomials:

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Lemma

For a positive integer k, let $\zeta_1, \zeta_2, \ldots, \zeta_k$ be the kth roots of unity in some order. Then, for every $k > r$,
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Chordal graph algorithm

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Proof of algorithm

- Perfect elimination order $\Rightarrow$ polynomial $S_m(\mathbf{x})$ for each vertex
- These polynomials generate graph coloring ideal by induction
- Form Gröbner basis, by considering *S-pairs*

Hilbert's Nullstellensatz

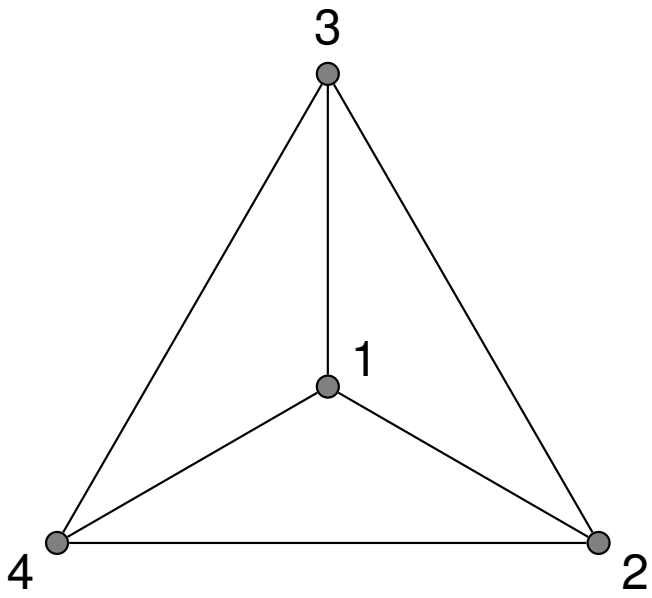
Theorem (Hilbert)

Given a field $\mathbb{K}$ *and* $f_1, \ldots, f_s \in \mathbb{K}[x_1, \ldots, x_n]$*, the system* $f_1 = f_2 = \cdots = f_s = 0$ *has no solutions in the algebraic closure of* $\mathbb{K}$ *iff there exist polynomials* $\beta_1, \ldots, \beta_s \in \mathbb{K}[x_1, \ldots, x_n]$ *such that*

$$1 = \sum_{i=1}^{s} \beta_i f_i.$$

The set $\{\beta_i\}$ *is a Nullstellensatz certificate.*

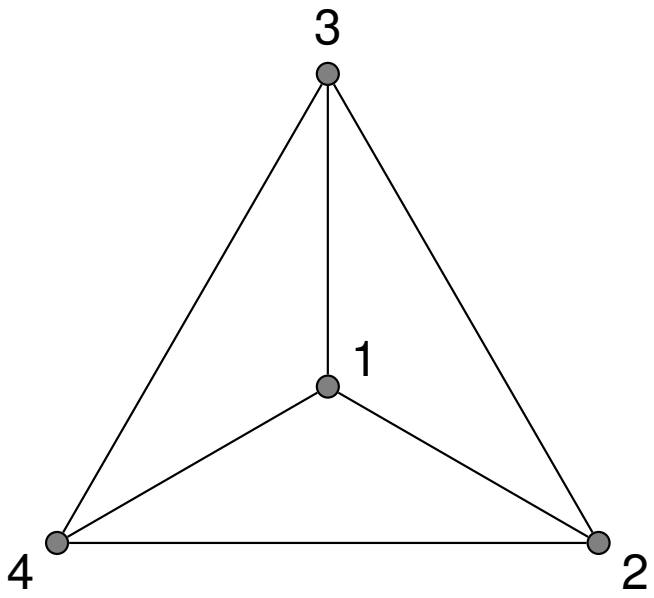
Hilbert's Nullstellensatz



Certificate of infeasibility for 3-coloring ideal:

$$1 = \nu_4(\mathbf{x}) + (-x_1) \cdot \eta_{12}(\mathbf{x}) + (-x_2 - x_4) \cdot \eta_{13}(\mathbf{x}) + (-x_1) \cdot \eta_{14}(\mathbf{x})$$
$$+ (-x_1 - x_4) \cdot \eta_{23}(\mathbf{x}) + (-x_2) \cdot \eta_{24}(\mathbf{x}) + (-x_1 - x_2) \cdot \eta_{34}(\mathbf{x})$$

Hilbert's Nullstellensatz



Gröbner basis for coloring ideal:

$$\{1\}$$

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The set $\{\beta_i\}$ is a Nullstellensatz certificate.

- Degree of the certificate is minimum degree of the β_i
- If degree small, find certificate by brute force over finite field $\mathbb{K}$

Certificates for $\mathcal{I}_k(G)$

Theorem (DMPRRSSS)

Given a non-k-colorable graph G, let d be the minimum degree of a Nullstellensatz certificate.

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- $d \equiv 1 \mod k$
- $d \geq k+1$ *if* $k > 3$.

- Brute force is inefficient for **every** G

Certificates for $\mathcal{I}_k(G)$

Example: degree-4 certificate over $\mathbb{F}_2$:

$$
\begin{aligned}
1 = {} & (1 + x_0x_2x_4 + x_0x_2x_6 + x_0x_3x_4 + x_0x_3x_5 + x_0x_4x_5 + x_0x_4x_6 + x_1^2x_4 + x_1^2x_6 \\
& + x_1x_3x_4 + x_1x_3x_5 + x_1x_5x_6 + x_2x_3x_4 + x_2x_3x_6 + x_3x_5x_6 + x_4x_5x_6)(\mathbf{x_0^3 + 1}) \\
& + (x_1 + x_3 + x_4 + x_0^2x_1x_4 + x_0^2x_1x_6 + x_0^2x_2x_4 + x_0^2x_2x_6 + x_0^2x_3x_4 + x_0^2x_3x_5 \\
& + x_0^2x_5x_6 + x_0x_1x_3x_4 + x_0x_1x_3x_6 + x_0x_2x_3x_4 + x_0x_2x_3x_6 + x_0x_2x_4x_5 + x_0x_2x_4x_6 \\
& + x_0x_2x_5x_6 + x_0x_3x_4x_5 + x_0x_3x_4x_6 + x_0x_3x_5x_6 + x_0x_4x_5x_6 + \\
& + x_1x_3x_4x_5 + x_1x_3x_4x_6 + x_1x_4x_5x_6 + x_2x_3x_4x_5 + x_2x_3x_4x_6)(\mathbf{x_0^2 + x_0x_1 + x_1^2}) \\
& + (x_1 + x_3 + x_4 + x_6 + x_0^2x_1x_4 + x_0^2x_1x_6 + x_0^2x_4x_5 + x_0^2x_4x_6 + x_0^2x_5x_6 \\
& + x_0x_1x_3x_4 + x_0x_1x_3x_6 + x_0x_3x_4x_5 + x_0x_3x_4x_6 + x_1x_3x_4x_5 + x_1x_3x_4x_6 \\
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& + x_0x_2x_5x_6 + x_0x_3x_4x_5 + x_0x_3x_4x_6 + x_0x_3x_5x_6 + x_0x_4x_5x_6 + \\
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& + (x_1 + x_3 + x_4 + x_6 + x_0^2x_1x_4 + x_0^2x_1x_6 + x_0^2x_4x_5 + x_0^2x_4x_6 + x_0^2x_5x_6 \\
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Certificates for $\mathcal{I}_k(G)$

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Given a non-k-colorable graph G, let d be the minimum degree of a certificate.

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Graph	k	Possible degrees	$\mathbb{F}_2$	$\mathbb{F}_3$	$\mathbb{F}_5$	$\mathbb{F}_7$
K_4	3	$1, 4, 7, 10, \ldots$	1	–	4	4
K_5	4	$5, 9, 13, \ldots$	–	5	5	5
K_6	5	$6, 11, 16, \ldots$	6	6	–	11
K_7	6	$7, 13, 19, \ldots$	–	–	13	13
K_8	7	$8, 15, 22, \ldots$	8	≥ 15	≥ 15	–
K_9	8	$9, 17, 25, \ldots$	–	≥ 17	≥ 17	≥ 17
K_{10}	9	$10, 19, 28, \ldots$	≥ 19	–	≥ 19	≥ 19
K_{11}	10	$11, 21, 31, \ldots$	–	≥ 21	–	≥ 21

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- Computation over $\mathbb{F}_p$ possible only if p, k relatively prime

Certificates for $\mathcal{I}_k(G)$

Conjecture

For every field $\mathbb{K}$, the minimum degree of a k-coloring certificate grows superlinearly in k.

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K_8	7	$8, 15, 22, \ldots$	8	≥ 15	≥ 15	–
K_9	8	$9, 17, 25, \ldots$	–	≥ 17	≥ 17	≥ 17
K_{10}	9	$10, 19, 28, \ldots$	≥ 19	–	≥ 19	≥ 19
K_{11}	10	$11, 21, 31, \ldots$	–	≥ 21	–	≥ 21

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An independent set of variables do not pairwise co-occur in any f_i.

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Theorem (Khanna, Linial, Safra 1993)

It is NP-hard to color a k-chromatic graph with at most $k + 2\left\lfloor \frac{k}{3} \right\rfloor - 1$ colors.

Technical details

- Certain monomial orders are elimination orders
- Every lexicographic order $x_1 > \cdots > x_n$ is an elimination order
- For an elimination order, the Gröbner basis allows back-substitution, e.g.

$$x_1^3 + x_2 x_3 - x_3^2 - 1,$$
$$x_2^3 - x_2 + x_3^2 + 1,$$
$$x_2 x_3^2 - 2x_3^3 + x_3,$$
$$x_3^3 + 1.$$

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- Problem: What if one cannot solve for roots of unity, e.g. over a field other than $\mathbb{R}$?
- Solution: Do not solve numerically, merely symbolically

Summary of results

- Polytime algorithm finding Gröbner basis of graph-coloring ideal in chordal graphs
- Complexity of Nullstellensatz certificate for general graphs
- Hardness of approximate Gröbner basis computation, $\Leftarrow$ from hardness of approximate k-coloring

Questions?

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