An Introduction to Finite Element Methods

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Numerical methods for finding *approximate solutions* to *partial differential equations* (PDEs) on *non-trivial domains*.

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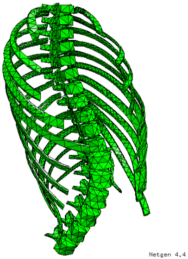

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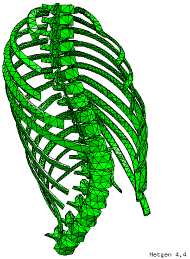
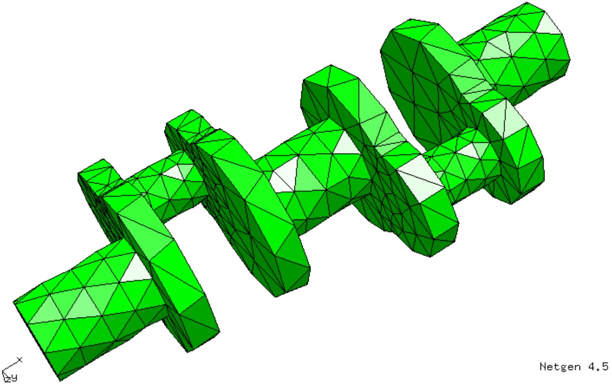


The task of solving the PDE is reduced to the solution of a (large scale) linear system.

$$(\text{Solving the PDE}) \longrightarrow (\text{Solving } A\underline{u} = \underline{f})$$

Another non-trivial domain



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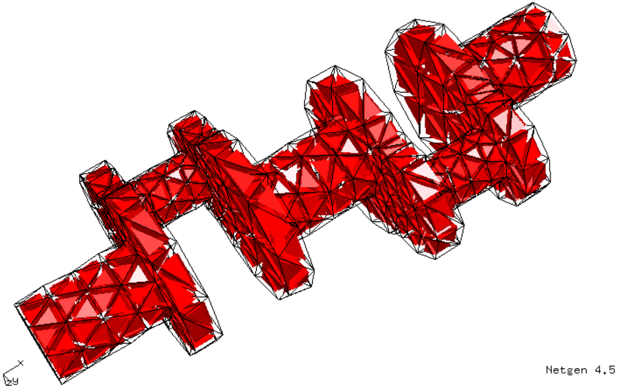

Variational/Weak formulation (1D)

- Consider the two-point boundary value problem: given f, find u such that

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\begin{aligned}
-u''(x) &= f(x), \quad \text{in} \quad \Omega = (0,1) \\
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$$- \qquad u''(x)v(x) \quad = \quad f(x)v(x)$$

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$$-\int_0^1 u''(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx$$
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Galerkin method

- *Problem:* Given a *bilinear form* $a(\cdot,\cdot)$ and a *linear form* $F(\cdot)$,

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- Let $N_h = \dim V_h$ and $\{\phi_1, \ldots, \phi_{N_h}\}$ be a basis for V_h, then we can expand

$$u_h(x) = \sum_{i=1}^{N_h} u_i \phi_i(x)$$

  and it is sufficient to consider $v_h(x) = \phi_j(x)$ for $j = 1, \ldots, N_h$.

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  where $\underline{u} = (u_1, \ldots, u_{N_h})$.
- Let

  $$A = (a(\phi_i, \phi_j))_{i,j=1}^{N_h} \quad \text{and} \quad \underline{f} = (f_1, \ldots, f_{N_h}),$$

  then we arrive at the *linear system*

  $$\text{find } \underline{u} \in \mathbb{R}^{N_h} : \quad A\underline{u} = \underline{f}.$$

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Weak derivative

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$$\int_a^b f(x)v'(x)\,dx = -\int_a^b w(x)v(x)\,dx$$

for all $v \in C_0^\infty(a,b)$. If such a w exists, then it is unique a.e. and we write $f'(x) = w(x)$.

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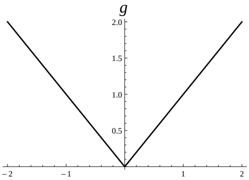
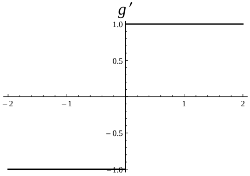
- Functions with jumps are *not* weakly differentiable.

Examples

- $$g(x) = |x|$$





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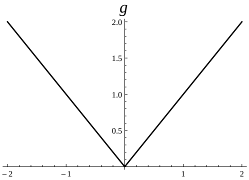
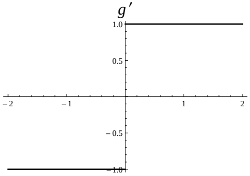
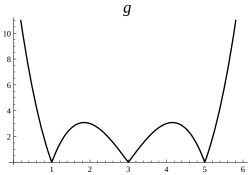
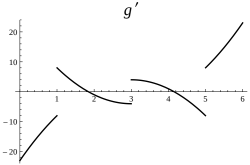




- $$g(x) = |(x-1)(x-3)(x-5)|$$





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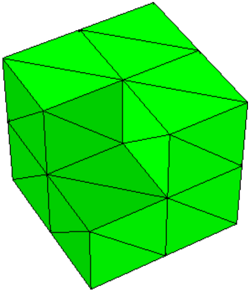

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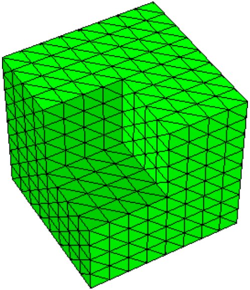

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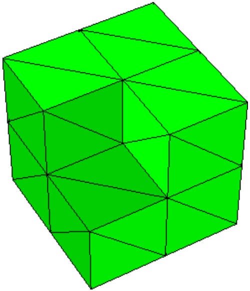

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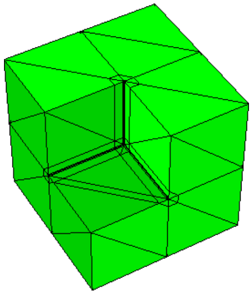

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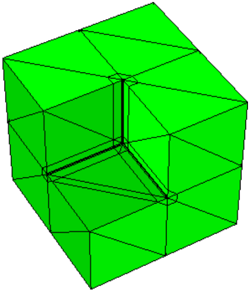

- higher order (i.e., p- and hp-) methods converge faster with respect to the number of unknowns, but require more effort in the implementation.

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- For 1D we consider next examples for two types of conforming finite element bases: a *nodal basis* and a *hierarchic basis*.

Conforming finite element bases

- The given domain is subdivided into *simple geometric objects* (intervals, quadrilaterals, triangles, prisms, tetrahedra, etc.) that are called *elements*.
- A *finite element basis* consists of *locally supported, piecewise polynomial basis functions* $\phi_i(x)$.
- A finite element basis is *conforming* if *for all* i, $\phi_i \in V$, where V is the function space where the variational problem is posed.
- For 1D we consider next examples for two types of conforming finite element bases: a *nodal basis* and a *hierarchic basis*.
- The basis functions are defined on a *reference element* and then mapped to the actual element in the *mesh*.

Setting for the 1D model problem

- Given $f \in L^2(0,1)$, find $u \in V$ such that

$$\underbrace{\int_0^1 u'(x)v'(x)\,dx}_{=a(u,v)} = \underbrace{\int_0^1 f(x)v(x)\,dx}_{=F(v)}, \quad \forall\, v \in V,$$

where $V = \{v \in L^2(0,1) \colon a(v,v) < \infty \text{ and } v(0) = 0\}$.

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High order nodal finite element basis

- reference element $\hat{\mathcal{I}} = [-1, 1]$, maximal polynomial degree p

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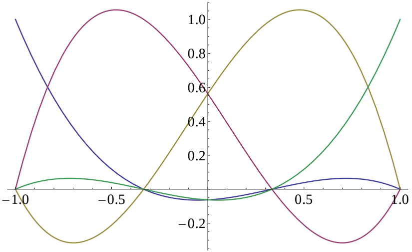
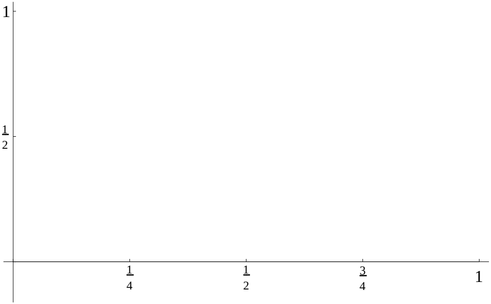
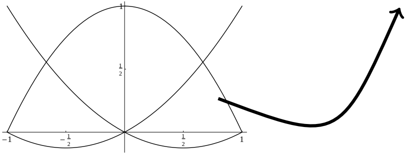
- Example $p = 3$:



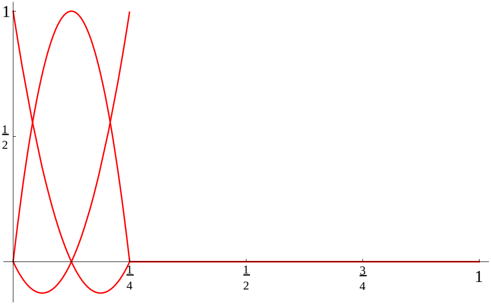
Local and global basis ($p = 2$, 4 elements)



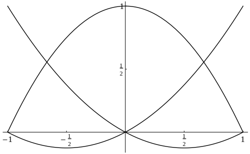


$\ell_1(x), \ell_2(x), \ell_3(x)$

Local and global basis ($p = 2$, 4 elements)



$\ell_1^{(1)}(x), \ell_2^{(1)}(x), \ell_3^{(1)}(x)$



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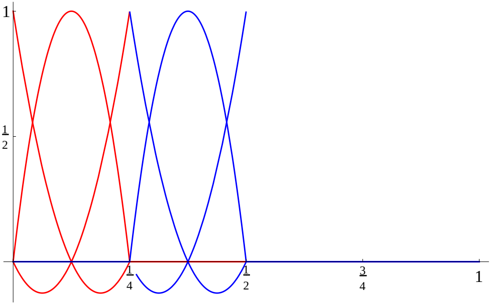
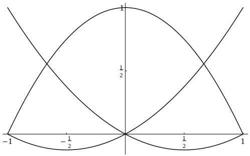


$\ell_1^{(1)}(x), \ell_2^{(1)}(x), \ell_3^{(1)}(x), \ell_1^{(2)}(x), \ell_2^{(2)}(x), \ell_3^{(2)}(x)$



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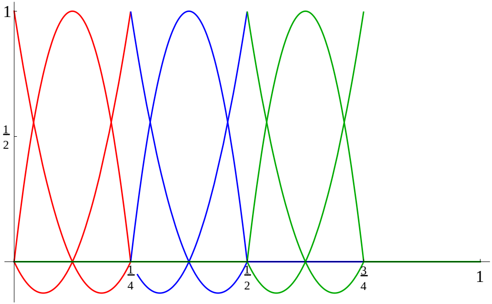

$\ell_1^{(1)}(x), \ell_2^{(1)}(x), \ell_3^{(1)}(x), \ell_1^{(2)}(x), \ell_2^{(2)}(x),$
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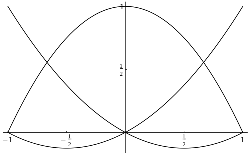

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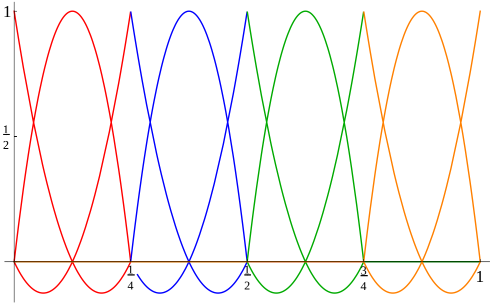

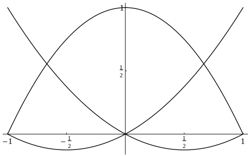

$\ell_1(x), \ell_2(x), \ell_3(x)$

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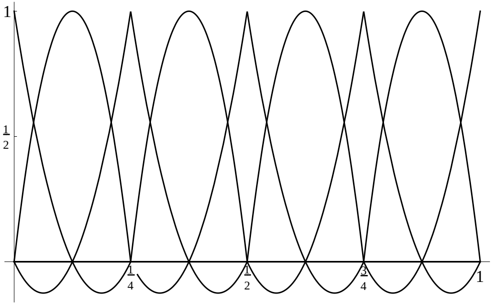

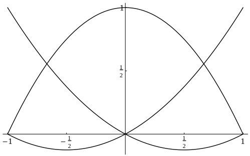

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$\rightarrow$ *local basis functions*

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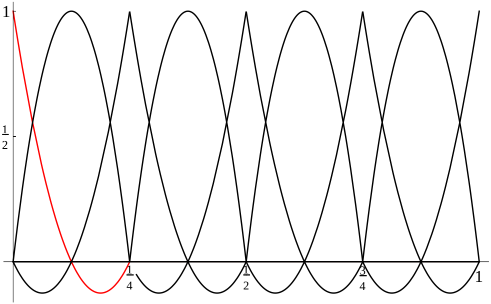

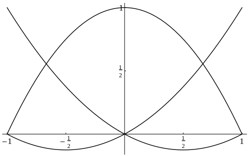

$\ell_1(x), \ell_2(x), \ell_3(x)$

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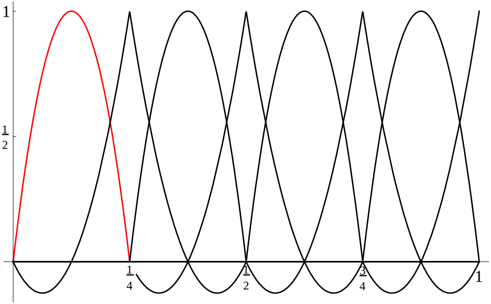

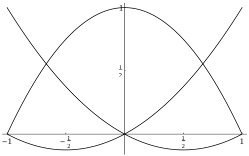

$\ell_1(x), \ell_2(x), \ell_3(x)$

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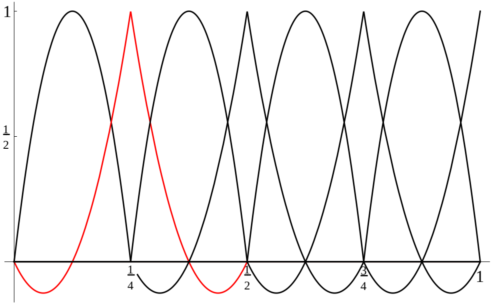

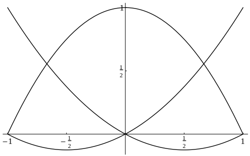

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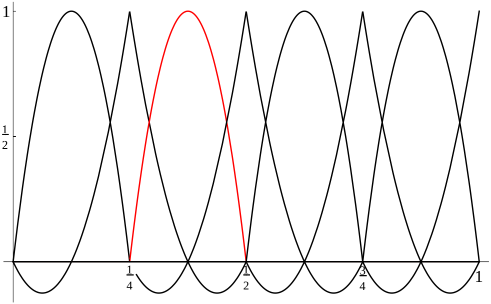

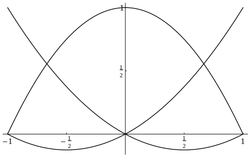

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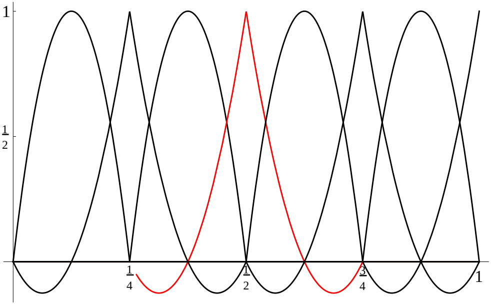

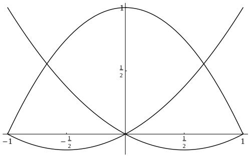

$\ell_1(x), \ell_2(x), \ell_3(x)$

$\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x),$

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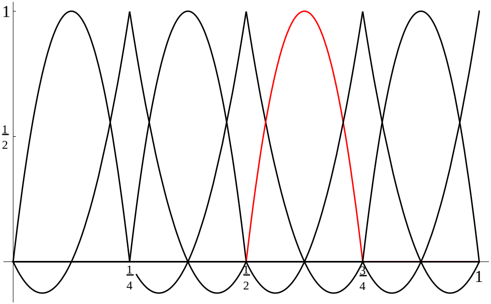

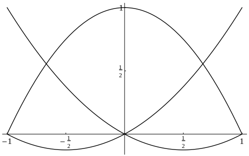

$\ell_1(x), \ell_2(x), \ell_3(x)$

$\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x), \phi_5(x),$

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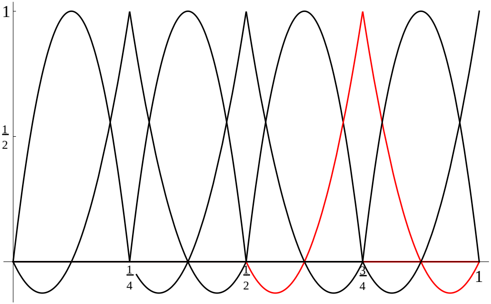

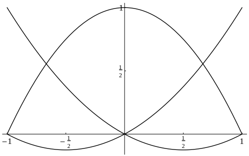

$\ell_1(x), \ell_2(x), \ell_3(x)$

$\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x), \phi_5(x),$
$\phi_6(x),$

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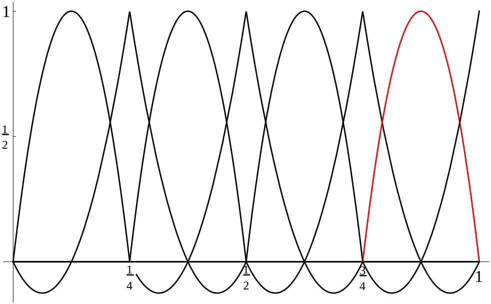
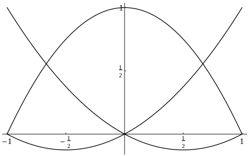
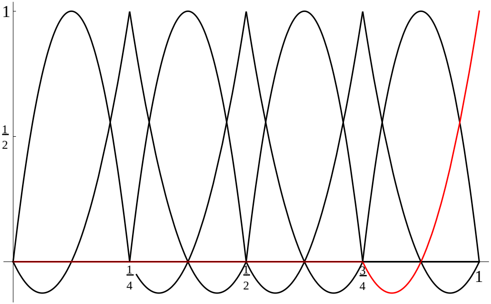


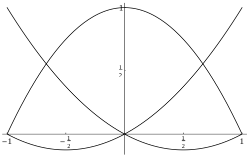


$\ell_1(x), \ell_2(x), \ell_3(x)$

$\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x), \phi_5(x),$
$\phi_6(x), \phi_7(x),$

Local and global basis ($p = 2$, 4 elements)





$\ell_1(x), \ell_2(x), \ell_3(x)$

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$\phi_6(x), \phi_7(x), \phi_8(x)$

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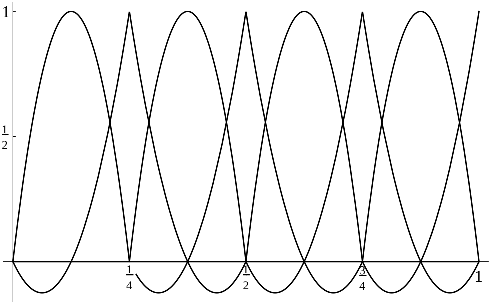
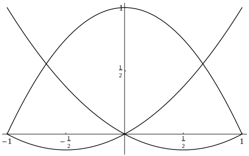


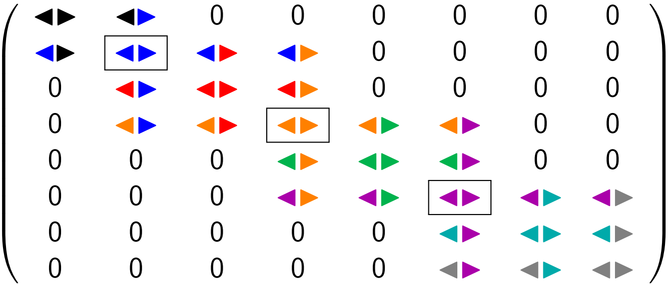


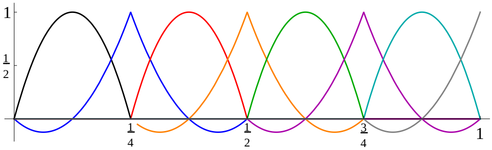
$\ell_1(x), \ell_2(x), \ell_3(x)$

$\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x), \phi_5(x),$
$\phi_6(x), \phi_7(x), \phi_8(x)$
$\rightarrow$ *global basis functions*

The system matrix

- $A = (a(\phi_i, \phi_j))_{i,j=1}^{N_h}$ with $a(u, v) = \int_0^1 u'(x)v'(x)\,dx$.

$A =$ 



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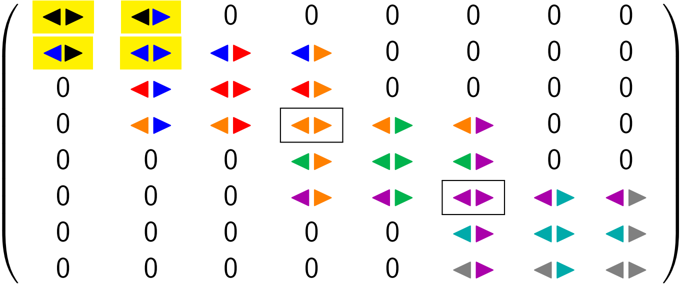
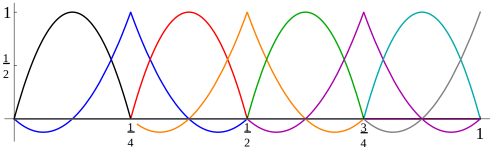
$A =$ 



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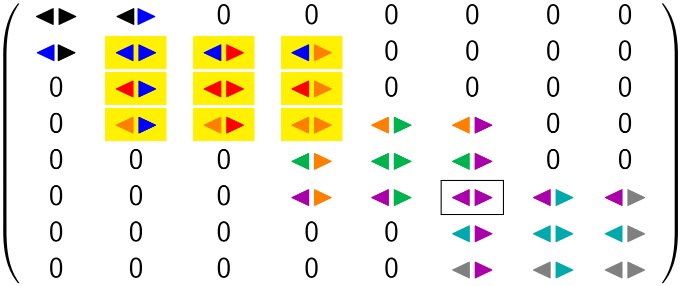
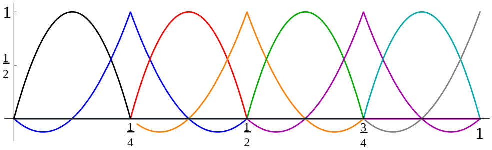
$A =$ 



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High order hierarchical finite element basis

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$$\varphi_{V,0}(x) = \frac{1-x}{2}, \quad \varphi_{V,1}(x) = \frac{1+x}{2}$$

- *cell based basis functions*: vanish at the boundary of the element, polynomial inside:

$$\varphi_{C,i}(x) = L_i(x) := \int_{-1}^{x} P_{i-1}(x)\,dx, \quad i \geq 2,$$

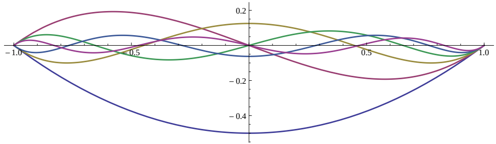
  where $P_n(x)$ is the nth Legendre polynomial. The polynomials $L_n(x)$ are called *integrated Legendre polynomials*.
- Legendre polynomials are orthogonal w.r.t. the $L^2(-1, 1)$ inner product, i.e.,

$$\int_{-1}^{1} P_i(x)P_j(x)\,dx = 0, \quad \text{if} \quad i \neq j.$$

Integrated Legendre polynomials

- For $n \geq 2$ with $L_0(x) = -1$ and $L_1(x) = x$ we have

$$L_n(x) = \frac{2n-3}{n} x L_{n-1}(x) - \frac{n-3}{n} L_{n-2}(x).$$



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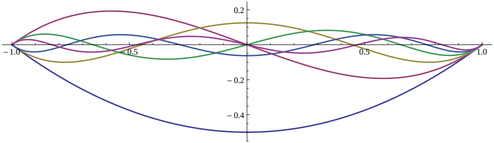
- $L_n(\pm 1) = 0$



Integrated Legendre polynomials

- For $n \geq 2$ with $L_0(x) = -1$ and $L_1(x) = x$ we have

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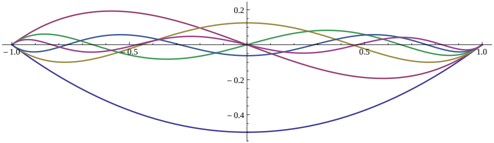
- Orthogonality relation

$$\int_{-1}^{1} L_i'(x) L_j'(x)\, dx = 0, \quad \text{if} \quad i \neq j$$



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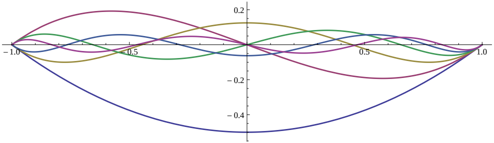
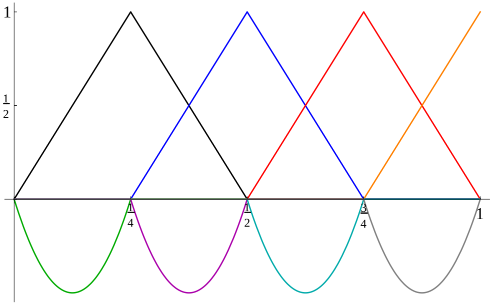
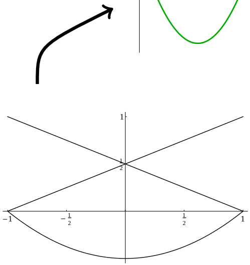
- Sparse w.r.t. the L^2 inner product

$$\int_{-1}^{1} L_i(x) L_j(x)\, dx = 0, \quad \text{if} \quad |i-j| \neq 0, 2$$



Local and global basis ($p = 2$, 4 elements)





$\varphi_{V,0}(x), \varphi_{V,1}(x), \varphi_{C,2}(x)$

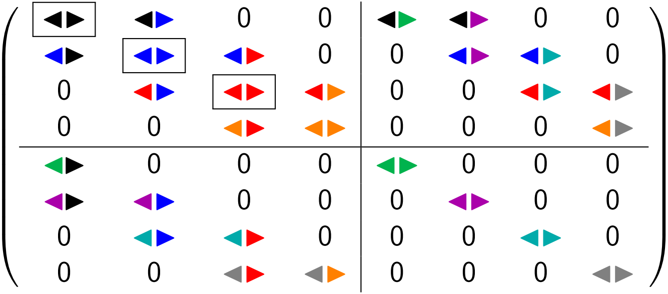
global basis:

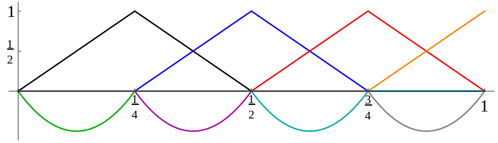
$\underbrace{\phi_{V_2}(x), \phi_{V_3}(x), \phi_{V_4}(x), \phi_{V_5}(x)}_{=\Phi_V},$

$\underbrace{\phi_{C_1,2}(x), \phi_{C_2,2}(x), \phi_{C_3,2}(x), \phi_{C_4,2}(x)}_{=\Phi_C}$

The system matrix

- $A = (a(\phi_i, \phi_j))_{i,j=1}^{N_h} = \left(\begin{array}{c|c} A_{VV} & A_{VC} \\ \hline A_{CV} & A_{CC} \end{array} \right)$ with $a(u,v) = \int_0^1 u'v'dx$.

$A =$ 



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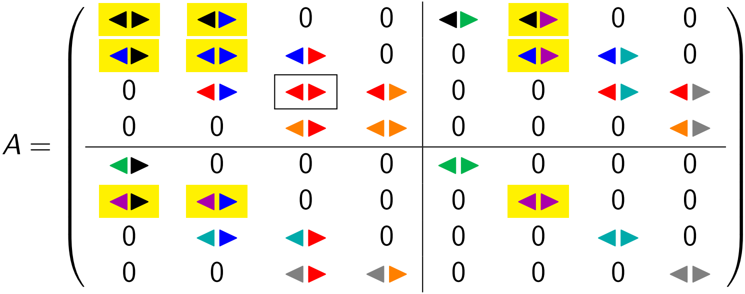

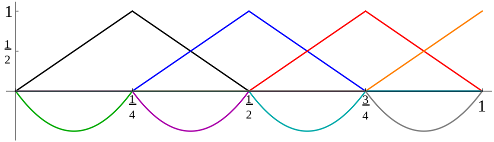

Example

$$\text{Find } u: \quad -u''(x) = \tfrac{1}{8}(2x-1)(4x-3)\sin(\pi x), \quad \text{in} \quad (0,1)$$
$$u(0) = 0, \quad u'(1) = 0.$$

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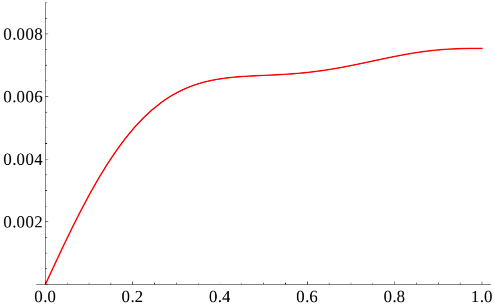

exact solution

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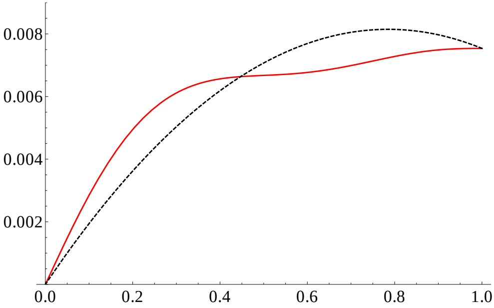

exact solution, approximation for $p = 2$

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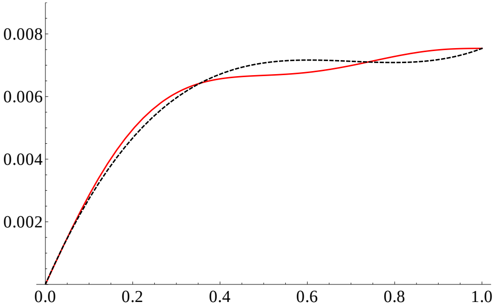

exact solution, approximation for $p = 3$

Example

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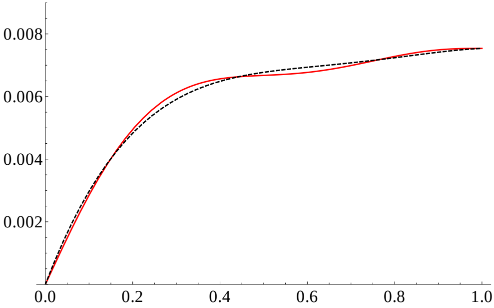

exact solution, approximation for $p = 4$

Example

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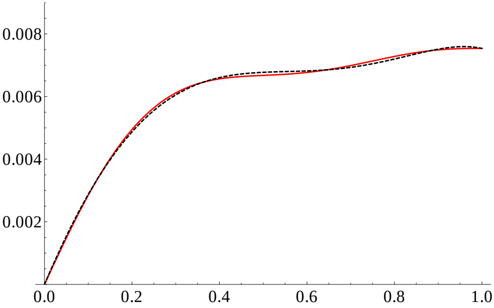

exact solution, approximation for $p = 5$

Example

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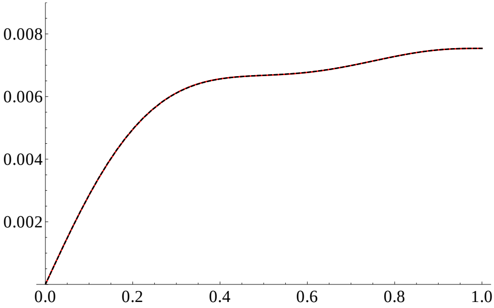

exact solution, approximation for $p = 6$

Example: the system matrix for $p = 6$

in the hierarchic basis: In the nodal basis:

$$\left(\begin{array}{c|ccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{11} \end{array}\right) \left(\begin{array}{cccccc} 40. & -48. & 38. & -24. & 10. & -2.2 \\ -48. & 80. & -76. & 50. & -24. & 5.9 \\ 38. & -76. & 95. & -76. & 38. & -9.8 \\ -24. & 50. & -76. & 80. & -48. & 11. \\ 10. & -24. & 38. & -48. & 40. & -14. \\ -2.2 & 5.9 & -9.8 & 11. & -14. & 8.6 \end{array}\right)$$

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- sparse element matrix

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- full element matrix

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- sparse element matrix
- add new polynomials when increasing the polynomial degree

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- recompute whole basis when increasing the polynomial degree

Example: the system matrix for $p = 6$

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- add new polynomials when increasing the polynomial degree
- recurrence for fast evaluation

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Variational/Weak formulation (2D)

- Model problem: given $f\colon \mathbb{R}^2 \to \mathbb{R}$, find $u\colon \mathbb{R}^2 \to \mathbb{R}$ s.t.

$$\begin{aligned} -\Delta u(x) &= f(x), && \text{in} \quad \Omega \\ u(x) &= 0, && \text{on} \quad \partial\Omega, \end{aligned}$$

where $\Delta g(x) = \sum_{i=1}^{d} \frac{\partial^2 g}{\partial x_i^2}(x)$ denotes the *Laplace operator*.

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$$\begin{aligned} -\int_\Omega \Delta u(x) v(x)\, dx &= \int_\Omega f(x) v(x)\, dx \\ \int_\Omega \nabla u(x) \nabla v(x)\, dx &= \int_\Omega f(x) v(x)\, dx \\ a(u,v) := \int_\Omega \nabla u(x) \nabla v(x)\, dx = \int_\Omega f(x) v(x)\, dx &=: F(v) \end{aligned}$$

Model problem in 2D

- Given $f \in C(\Omega)$, find $u \in C^2(\Omega)$ such that

$$\begin{aligned} -\Delta u(x) &= f(x), \qquad && \text{in} \quad \Omega \\ u(x) &= 0, && \text{on} \quad \partial\Omega, \end{aligned}$$

- Given $f \in L^2(\Omega)$, find $u \in V$ such that

$$\int_\Omega \nabla u(x) \nabla v(x)\, dx = \int_\Omega f(x) v(x)\, dx, \quad \forall\, v \in V,$$

  where $V = \{v \in L^2(\Omega) \colon a(v,v) < \infty \text{ and } v\,|_{\partial\Omega} = 0\}$.

- Find $\underline{u} \in \mathbb{R}^{N_h}$:

$$A\underline{u} = \underline{f},$$

  where $N_h = \dim V_h \subset V$, and the approximate solution is $u_h(x) = \sum_{i=1}^{N_h} u_i \phi_i(x)$ for a basis $\{\phi_1(x), \ldots, \phi_{N_h}(x)\}$.

Hierarchic high order finite element basis in 2D

- *Vertex based basis functions*
- *Edge based basis functions*
- *Cell based basis functions*

Hierarchic high order finite element basis in 2D

- *Vertex based basis functions* 1 at the defining vertex, 0 on all other vertices, linear in between
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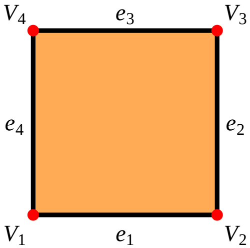
High order finite element basis on quadrilaterals

- reference element $\hat{Q} = [-1, 1]^2$



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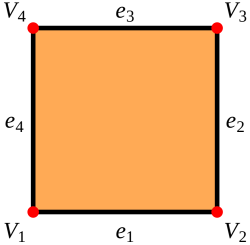
- tensor product of 1D basis functions



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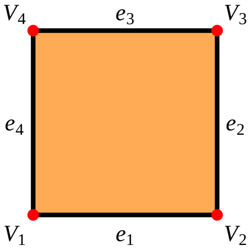
$$\varphi_{V_1}(x, y) = \varphi_{V,0}(x)\varphi_{V,0}(y) = \frac{1}{4}(1 - x)(1 - y)$$



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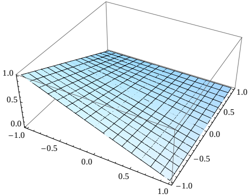

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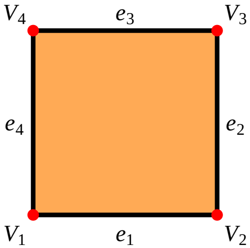

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- *Edge based basis functions:* $\varphi_{e_1,i}(x, y) = L_i(x)\varphi_{V,0}(y)$, for $i = 2, \ldots, p$

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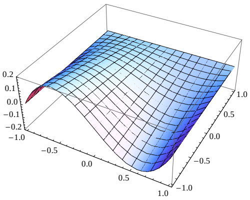

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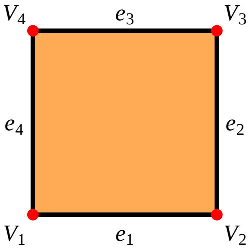

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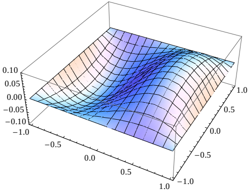

$$\varphi_{V_1}(x,y) = \varphi_{V,0}(x)\varphi_{V,0}(y)$$
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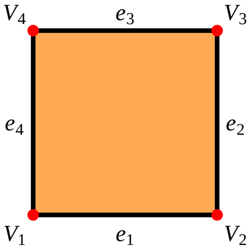

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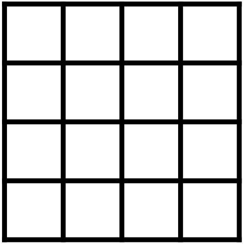
- *Cell based basis functions:* $\varphi_{C,i,j}(x, y) = L_i(x)L_j(y)$ for $i, j = 2, \ldots, p$
- *Local vector of basis functions:*

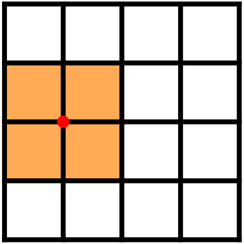
$$(\underbrace{\varphi_{V_1}, \varphi_{V_2}, \varphi_{V_3}, \varphi_{V_4}}_{\underline{\varphi}_V}, \underbrace{\varphi_{e_1,2}, \ldots, \varphi_{e_4,p}}_{\underline{\varphi}_E}, \underbrace{\varphi_{C,2,2}, \ldots, \varphi_{C,p,p}}_{\underline{\varphi}_C})$$

Global vector of basis functions



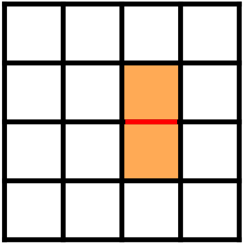
- Collect globally vertex, edge and cell based basis functions (Φ_V, Φ_E, Φ_C)

Global vector of basis functions



- Collect globally vertex, edge and cell based basis functions (Φ_V, Φ_E, Φ_C)
- *Vertex based basis functions* are supported on all elements sharing the defining vertex

Global vector of basis functions



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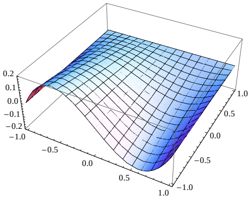

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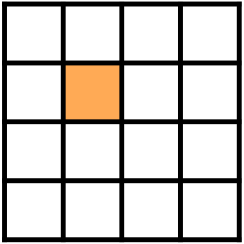

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- *Cell based basis functions* are only supported on the defining element

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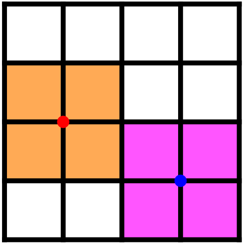

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- *Edge based basis functions* are supported on the two elements sharing the defining edge (Need to care about *orientation* of edges!)
- *Cell based basis functions* are only supported on the defining element
- There can only be nonzero entries in the system matrix, if the *support* of two basis functions *overlaps*.

The system matrix on the reference element

- $A = (a(\phi_i, \phi_j))_{i,j}$ with $a(u, v) = \int_\Omega \nabla u(x, y) \nabla v(x, y) \, d(x, y)$

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- All basis functions are tensor products: $\phi_i(x, y) = f_i(x) g_i(y)$, hence

$$\nabla \phi_i(x, y) = (f_i'(x) g_i(y), f_i(x) g_i'(y))$$

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and $a(\phi_i(x, y), \phi_j(x, y)) =$

$$= \int_{\hat{Q}} \left(f_i'(x) f_j'(x) g_i(y) g_j(y) + f_i(x) f_j(x) g_i'(y) g_j'(y) \right) d(x, y)$$

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$$= \int_{-1}^{1} f_i'(x) f_j'(x)\, dx \int_{-1}^{1} g_i(y) g_j(y)\, dy$$

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and $a(\phi_i(x,y), \phi_j(x,y)) =$
$$\begin{aligned}
&= \int_{\hat{Q}} \left(f_i'(x) f_j'(x) g_i(y) g_j(y) + f_i(x) f_j(x) g_i'(y) g_j'(y) \right) d(x,y) \\
&= \int_{-1}^{1} f_i'(x) f_j'(x)\, dx \int_{-1}^{1} g_i(y) g_j(y)\, dy \\
&\quad + \int_{-1}^{1} f_i(x) f_j(x)\, dx \int_{-1}^{1} g_i'(y) g_j'(y))\, dy
\end{aligned}$$
- combination of one-dimensional bilinear forms!

The element system matrix for $p = 3$

The matrix can again be subdivided into block according to the different types of basis functions:

$$\begin{pmatrix} A_{VV} & A_{VE} & A_{VC} \\ A_{EV} & A_{EE} & A_{EC} \\ A_{CV} & A_{CE} & A_{CC} \end{pmatrix}$$

and if the basis using integrated Legendre polynomials is used, then the nonzero pattern is as follows...

The element system matrix for $p = 3$

$$
\left(\begin{array}{cccc|cccc|cccccccc}
* & * & * & * & * & * & * & * & * & * & * & * & & & & \\
* & * & * & * & * & * & * & * & * & * & * & * & & & & \\
* & * & * & * & * & * & * & * & * & * & * & * & & & & \\
* & * & * & * & * & * & * & * & * & * & * & * & & & & \\
\hline
* & * & * & * & * & & & & * & & & & * & * & & \\
* & * & * & * & & * & & & & * & & & & & * & * \\
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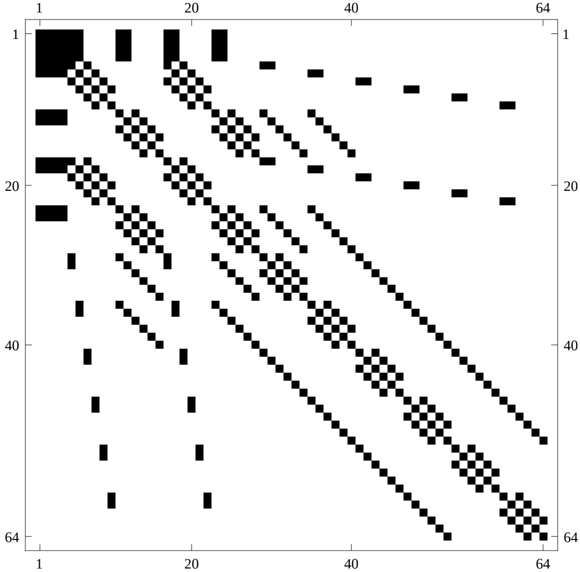
 & & & & & * & * & & & * & * & & & & * & \\
 & & & & & * & & * & & * & & * & & & & *
\end{array}\right)
$$

The element system matrix for $p = 7$



Efficient computations: recurrence relations

- FE-basis on quadrilaterals defined as tensor products of 1D basis functions

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$$\phi_n(x) = (a_n x + b_n)\,\phi_{n-1}(x) + c_n \phi_{n-2}(x), \quad n \geq 1.$$

So do integrated Legendre polynomials!

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  So do integrated Legendre polynomials!
- *Derivatives of orthogonal polynomials* also satisfy three term recurrences.
- *Building blocks* of system matrices are integrals of the form

$$\int_{\hat{\mathcal{I}}} \phi_i(x) \psi_j(x)\, dx,$$

  where ϕ_i and ψ_j are orthogonal polynomials (satisfy three term recurrences).

Efficient computations: recurrence relations

- Let $\phi_n(x), \psi_n(x)$ be polynomials defined by the recurrences

$$\phi_n(x) = (a_n x + b_n)\,\phi_{n-1}(x) + c_n \phi_{n-2}(x),$$

and

$$\psi_n(x) = (\alpha_n x + \beta_n)\,\psi_{n-1}(x) + \gamma_n \psi_{n-2}(x).$$

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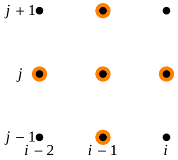
- Let $P_{i,j}(x) = \phi_i(x)\psi_j(x)$, then

$$P_{i,j}(x) = A_{i,j}P_{i-1,j+1}(x) + B_{i,j}P_{i-1,j}(x) + C_{i,j}P_{i-1,j-1}(x) + D_{i,j}P_{i-2,j}(x),$$



with

$$A_{i,j} = a_i/\alpha_{j+1}, \qquad B_{i,j} = (b_i\alpha_{j+1} - a_i\beta_{j+1})/\alpha_{j+1},$$
$$C_{i,j} = -a_i\gamma_{j+1}/\alpha_{j+1}, \qquad D_{i,j} = c_i.$$

Efficient computations: recurrence relations

- We have an x-free recurrence

$$P_{i,j}(x) = A_{i,j}P_{i-1,j+1}(x) + B_{i,j}P_{i-1,j}(x) + C_{i,j}P_{i-1,j-1}(x) + D_{i,j}P_{i-2,j}(x),$$

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  for $P_{i,j}(x) = \phi_i(x)\psi_j(x)$.

- It can be multiplied by any function $f(x)$ and one may integrate over $\hat{\mathcal{I}}$ and the recurrence remains valid, i.e.,

  $$M_{i,j} = A_{i,j}M_{i-1,j+1} + B_{i,j}M_{i-1,j} + C_{i,j}M_{i-1,j-1} + D_{i,j}M_{i-2,j},$$

  for $M_{i,j} = \int_{\hat{\mathcal{I}}} f(x)\phi_i(x)\psi_j(x)\,dx$.

Efficient computations: sum factorization

- Usually the bilinear forms $a(u, v)$ building the system matrix A are evaluated using numerical integration:

$$a(\phi_{i,j}, \phi_{k,l}) = \int\int_{\hat{Q}} C(x, y)\phi_i(x)\psi_j(y)\phi_k(x)\psi_l(y)\, d(x, y)$$
$$\simeq \sum_{\alpha,\beta} w_\alpha w_\beta C(x_\alpha, y_\beta)\phi_i(x_\alpha)\psi_j(y_\beta)\phi_k(x_\alpha)\psi_l(y_\beta),$$

  where
  - $C(\cdot, \cdot)$ is a coefficient function (contains also transformation from the actual element in the mesh to the reference element),
  - w_γ are the quadrature weights and x_α, y_β are the quadrature points.

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  - w_γ are the quadrature weights and x_α, y_β are the quadrature points.
- For the approximate solution of the linear system $A\underline{u} = \underline{f}$ an *iterative scheme* is used

$$\underline{u}^{(k+1)} = \underline{f} - A\underline{u}^{(k)},$$

  hence *not fast assemblance* of the matrix is needed, *but fast application*.

Efficient computation: sum factorization

Let's write the solution vector $\underline{u}$ with two indices, then

$$\left(A\underline{u}\right)_{k,l} \simeq \sum_{\alpha,\beta,i,j} w_\alpha w_\beta C(x_\alpha, y_\beta)\phi_i(x_\alpha)\psi_j(y_\beta)\phi_k(x_\alpha)\psi_l(y_\beta)\underline{u}_{i,j}$$

where each summation is $O(p)$ with maximal polynomial degree p.

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$$M^{(1)}_{\alpha,j} = \sum_i \phi_i(x_\alpha)\underline{u}_{i,j}$$

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$$M^{(3)}_{\beta,k} = \sum_\alpha M^{(2)}_{\alpha,\beta} w_\alpha C(x_\alpha, y_\beta)\phi_k(x_\alpha)$$

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where each summation is $O(p)$ with maximal polynomial degree p.

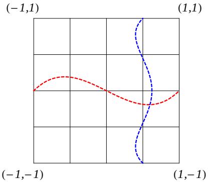
$$\begin{aligned}
M^{(1)}_{\alpha,j} &= \sum_i \phi_i(x_\alpha)\underline{u}_{i,j} \\
M^{(2)}_{\alpha,\beta} &= \sum_j M^{(1)}_{\alpha,j}\psi_j(y_\beta) \\
M^{(3)}_{\beta,k} &= \sum_\alpha M^{(2)}_{\alpha,\beta} w_\alpha C(x_\alpha, y_\beta)\phi_k(x_\alpha) \\
\left(A\underline{u}\right)_{k,l} &\simeq \sum_\beta M^{(3)}_{\beta,k} w_\beta \psi_l(y_\beta)
\end{aligned}$$

Hierarchic high order basis functions on triangles

On quadrilateral elements the basis functions were defined exploiting the tensor product structure:

$$\phi_{i,j}(x,y) = \varphi_i(x)\psi_j(y),$$

with $i,j \geq 0$ and $x,y \in [-1,1]$.



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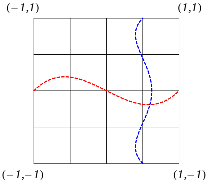

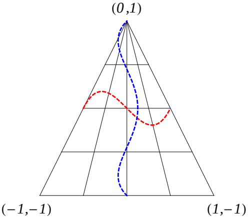

The basis functions on the reference triangle $\hat{T}$ are defined by collapsing the quadrilateral to the triangle:

$$\phi_{i,j}(x,y) = \varphi_i\left(\frac{2x}{1-y}\right)\left(\frac{1-y}{2}\right)^i \psi_j(y).$$

Integral over reference triangle

Let $\phi_{i,j}(x,y) = \varphi_i\left(\frac{2x}{1-y}\right)\left(\frac{1-y}{2}\right)^i \psi_j(y)$, then by means of the substitution $z = \frac{2x}{1-y}$ we *decouple* the integrals:

$$\int_{\hat{T}} \phi_{i,j}(x,y)\phi_{k,l}(x,y)\,d(x,y) = \int_{-1}^{1} \varphi_i(x)\varphi_k(x)\,dx \times \int_{-1}^{1} \left(\tfrac{1-z}{2}\right)^{i+k+1} \psi_j(z)\psi_l(z)\,dz.$$

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$$\int_{\hat{T}} \phi_{i,j}(x,y)\phi_{k,l}(x,y)\,d(x,y) = \int_{-1}^{1} P_i(x)P_k(x)\,dx \\ \times \int_{-1}^{1} \left(\tfrac{1-z}{2}\right)^{i+k+1} \psi_j(z)\psi_l(z)\,dz.$$

- Set $\varphi_i(x) = P_i(x)$ (Legendre polynomials)

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$$\int_{\hat{T}} \phi_{i,j}(x,y)\phi_{k,l}(x,y)\,d(x,y) = g_i\delta_{i,k} \times \int_{-1}^{1} \left(\tfrac{1-z}{2}\right)^{2i+1} \psi_j(z)\psi_l(z)\,dz.$$

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- Set $\varphi_i(x) = P_i(x)$ (Legendre polynomials)
- Set $\psi_j(y) = P_j^{(2i+1,0)}(y)$, the Jacobi polynomials orthogonal w.r.t. the weight function $\left(\frac{1-y}{2}\right)^{2i+1}$, i.e.,

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Dubiner basis

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$$\int_{-1}^{1} \left(\frac{1-y}{2}\right)^{2i+1} P_j^{(2i+1,0)}(y) P_l^{(2i+1,0)}(y)\, dy = h_{ij}\delta_{j,l}$$

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- Proof of sparsity in the latter three cases: exact evaluation of the integrals using symbolic computation

Jacobi and integrated Jacobi polynomials

- For $\alpha > -1$, $-1 \leq x \leq 1$ and $n \geq 0$ we denote by $P_n^{(\alpha,0)}(x)$ the nth *Jacobi polynomial* orthogonal w.r.t. the inner product

$$\langle f, g \rangle_{L^2_\alpha(-1,1)} \int_{-1}^{1} \left(\tfrac{1-x}{2}\right)^\alpha f(x) g(x)\, dx.$$

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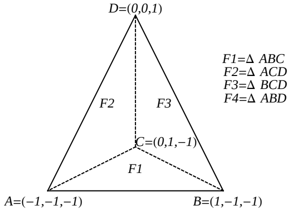
High order basis functions on tetrahedra

- vertex, edge, *face*, and cell based basis functions
- usually defined by collapsing a hexahedron to a tetrahedron



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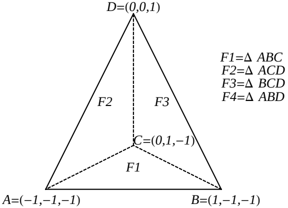

Example: Cell based basis function for H^1:

$$\phi_{ijk}(x,y,z) = \hat{p}_i^0\left(\frac{4x}{1-2y-z}\right)(1-2y-z)^i\,\hat{p}_j^{2i}\left(\frac{2y}{1-z}\right) \times (1-z)^j\,\hat{p}_k^{2i+2j}(z)$$

System matrix

- The FE-matrix that we consider is built from

$$a(u, v) = \int_{\hat{T}} uv \, d(x, y, z) + \int_{\hat{T}} (\nabla u)^T \mathcal{C} \nabla v \, d(x, y, z),$$

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> Let $A = (a(\phi_{ijk}, \phi_{lmn}))_{ijklmn}$, then the entries of the system matrix are zero if $|i - l| > 2$ or $|i + j - l - m| > 3$ or $|i + j + k - l - m - n| > 2$.

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- The results are proven by *explicitly evaluating* the integrals using *rewriting*.

Example for evaluation

After decoupling the integrals for computing

$$\int_{\hat{T}} \phi_{ijk}(x,y,z)\phi_{lmn}(x,y,z)\,d(x,y,z)$$

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- application of 2 rewrite rules and evaluation of 4 integrals

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- application of 6 rewrite rules and evaluation of 40 integrals

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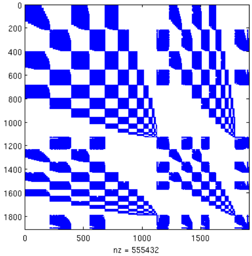
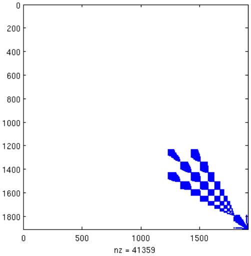
In total 189 nonzero matrix entries

Sparsity pattern for $H(\mathrm{div})$, $p = 15$

- using standard Legendre-type polynomials



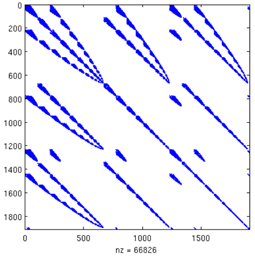
element mass matrix



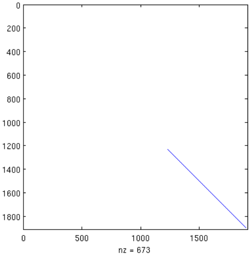
element stiffness matrix

Sparsity pattern for $H(\text{div})$, $p = 15$

- using the Jacobi-type polynomials with optimized parameters



element mass matrix



element stiffness matrix

Fast assembling of the system matrices

- Lehrenfeld+Koutschan+Schöberl: using mixed relations of the form

$$\sum_{r=0}^{R} c_r(i,j)\phi_{i+r_1,j+r_2}(x,y) = \sum_{s=0}^{S} D_x\phi_{i+s_1,j+s_2}(x,y)$$

- Combining recurrences for Jacobi polynomials $\hat{p}_n^{\alpha}(x)$, $p_n^{\alpha}(x)$ and sum factorization techniques [Beuchler et al]

- Computing recurrences for the explicit matrix entries

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- *robust* with respect to parameters such as mesh-size and regularization parameters

Local Fourier Analysis

- Given: Iterative procedure of the form

$$x^{(k+1)} = Ax^{(k)}$$

the *convergence rate* is related to the *matrix norm* of A which can be estimated by the spectral radius, i.e., the *largest eigenvalue in absolute value*

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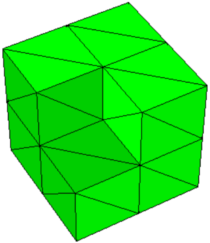
- *symbolic* local Fourier analysis: *exact bounds*

FEM and multigrid

- Multigrid methods operate on two (or more) grids



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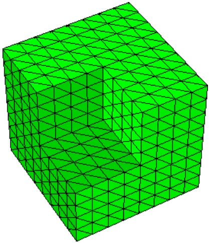


FEM and multigrid

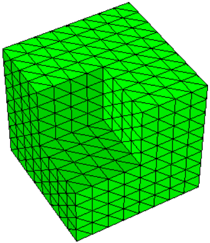
- Multigrid methods operate on two (or more) grids
- One step in a multigrid method consists of
  - (pre)smoothing steps

    $\downarrow$ restriction $\downarrow$

  - coarse grid correction

    $\downarrow$ prolongation $\downarrow$

  - (post)smoothing steps



Bound the convergence rate (2D)

Given: matrix $A(q, c_1, c_2, \eta) \in \mathbb{R}^{8\times 8}$ with $0 < q < 1$ and

$$(c_1, c_2, \eta) \in \Omega = \{(c_1, c_2, \eta) \mid 0 \leq c_1 \leq c_2 < 1 \;\wedge\; \eta > 0\},$$

where $c_i = \cos(\theta_i)$ for some frequencies θ_i and $\eta = h^4/\alpha$ with mesh-size h and regularization parameter α.

Find: bound $B(q)$ for the maximal eigenvalue $\lambda_{max}(q, c_1, c_2, \eta)$ of A over $(0, 1)^2 \times \Omega$.

The Matrix A

$$\begin{pmatrix} \mathbf{A_{1,1}} & 0 & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & A_{1,8} \\ 0 & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & A_{2,8} \\ A_{3,1} & A_{3,2} & A_{3,3} & 0 & A_{3,5} & A_{3,6} & A_{3,7} & A_{3,8} \\ A_{4,1} & A_{4,2} & 0 & A_{4,4} & A_{4,5} & A_{4,6} & A_{4,7} & A_{4,8} \\ A_{5,1} & A_{5,2} & A_{5,3} & A_{5,4} & A_{5,5} & 0 & A_{5,7} & A_{5,8} \\ A_{6,1} & A_{6,2} & A_{6,3} & A_{6,4} & 0 & A_{6,6} & A_{6,7} & A_{6,8} \\ A_{7,1} & A_{7,2} & A_{7,3} & A_{7,4} & A_{7,5} & A_{7,6} & A_{7,7} & 0 \\ A_{8,1} & A_{8,2} & A_{8,3} & A_{8,4} & A_{8,5} & A_{8,6} & 0 & A_{8,8} \end{pmatrix}$$

Common denominator of the matrix entries:

$$\begin{aligned} D =& 256 \Big(16c_2^4c_1^4\eta + 16c_2^2c_1^4\eta + 4c_1^4\eta + 16c_2^4c_1^2\eta + 16c_2^2c_1^2\eta + 4c_1^2\eta \\ &+ 4c_2^4\eta + 4c_2^2\eta + 144c_2^4c_1^4 - 72c_2^2c_1^4 + 9c_1^4 - 72c_2^4c_1^2 - 126c_2^2c_1^2 \\ &+ 36c_1^2 + 9c_2^4 + 36c_2^2 + \eta + 36 \Big) \end{aligned}$$

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\end{aligned}$$

The Numerator of $A_{1,1}$

$$
\begin{gathered}
432q^2c_2^6c_1^6 + 3q^2\eta c_2^6c_1^6 + \eta c_2^6c_1^6 + 144c_2^6c_1^6 + 1008q^2c_2^5c_1^6 + 16q^2\eta c_2^5c_1^6 + 8\eta c_2^5c_1^6 + 720c_2^5c_1^6 + 972q^2c_2^4c_1^6 + \\
30q^2\eta c_2^4c_1^6 + 26\eta c_2^4c_1^6 + 1476c_2^4c_1^6 + 1008q^2c_2^3c_1^6 + 28q^2\eta c_2^3c_1^6 + 44\eta c_2^3c_1^6 + 1584c_2^3c_1^6 + 108q^2c_1^6 + 1080q^2c_2^2c_1^6 + \\
27q^2\eta c_2^2c_1^6 + 41\eta c_2^2c_1^6 + 936c_2^2c_1^6 + 12q^2\eta c_1^6 + 4\eta c_1^6 + 576q^2c_2c_1^6 + 28q^2\eta c_2c_1^6 + 20\eta c_2c_1^6 + 288c_2c_1^6 + 36c_1^6 + \\
1008q^2c_2^6c_1^5 + 16q^2\eta c_2^6c_1^5 + 8\eta c_2^6c_1^5 + 720c_2^6c_1^5 - 360q^2c_2^5c_1^5 + 80q^2\eta c_2^5c_1^5 - 64\eta c_2^5c_1^5 - 1656c_2^5c_1^5 - 3600q^2c_2^4c_1^5 + \\
128q^2\eta c_2^4c_1^5 - 304\eta c_2^4c_1^5 - 7056c_2^4c_1^5 - 2736q^2c_2^3c_1^5 + 80q^2\eta c_2^3c_1^5 - 352\eta c_2^3c_1^5 - 5328c_2^3c_1^5 - 720q^2c_1^5 - \\
1872q^2c_2^2c_1^5 + 80q^2\eta c_2^2c_1^5 - 184\eta c_2^2c_1^5 + 720c_2^2c_1^5 + 64q^2\eta c_1^5 - 96\eta c_1^5 - 2088q^2c_2c_1^5 + 128q^2\eta c_2c_1^5 - 160\eta c_2c_1^5 + \\
1800c_2c_1^5 + 432c_1^5 + 972q^2c_2^6c_1^4 + 30q^2\eta c_2^6c_1^4 + 26\eta c_2^6c_1^4 + 1476c_2^6c_1^4 - 3600q^2c_2^5c_1^4 + 128q^2\eta c_2^5c_1^4 - 304\eta c_2^5c_1^4 - \\
7056c_2^5c_1^4 - 5616q^2c_2^4c_1^4 + 108q^2\eta c_2^4c_1^4 + 2724\eta c_2^4c_1^4 + 21168c_2^4c_1^4 + 1584q^2c_2^3c_1^4 - 136q^2\eta c_2^3c_1^4 - 1672\eta c_2^3c_1^4 - \\
3600c_2^3c_1^4 + 648q^2c_1^4 + 1404q^2c_2^2c_1^4 - 114q^2\eta c_2^2c_1^4 + 3114\eta c_2^2c_1^4 - 15660c_2^2c_1^4 + 120q^2\eta c_1^4 + 616\eta c_1^4 - 576q^2c_2c_1^4 + \\
152q^2\eta c_2c_1^4 - 760\eta c_2c_1^4 - 2304c_2c_1^4 + 792c_1^4 + 1008q^2c_2^6c_1^3 + 28q^2\eta c_2^6c_1^3 + 44\eta c_2^6c_1^3 + 1584c_2^6c_1^3 - 2736q^2c_2^5c_1^3 + \\
80q^2\eta c_2^5c_1^3 - 352\eta c_2^5c_1^3 - 5328c_2^5c_1^3 + 1584q^2c_2^4c_1^3 - 136q^2\eta c_2^4c_1^3 - 1672\eta c_2^4c_1^3 - 3600c_2^4c_1^3 + 12384q^2c_2^3c_1^3 - \\
640q^2\eta c_2^3c_1^3 - 1936\eta c_2^3c_1^3 + 17568c_2^3c_1^3 + 1872q^2c_1^3 + 5904q^2c_2^2c_1^3 - 580q^2\eta c_2^2c_1^3 - 1012\eta c_2^2c_1^3 + 14544c_2^2c_1^3 + \\
112q^2\eta c_1^3 - 528\eta c_1^3 + 720q^2c_2c_1^3 - 16q^2\eta c_2c_1^3 - 880\eta c_2c_1^3 - 1872c_2c_1^3 - 2160c_1^3 + 1080q^2c_2^6c_1^2 + 27q^2\eta c_2^6c_1^2 + \\
41\eta c_2^6c_1^2 + 936c_2^6c_1^2 - 1872q^2c_2^5c_1^2 + 80q^2\eta c_2^5c_1^2 - 184\eta c_2^5c_1^2 + 720c_2^5c_1^2 + 1404q^2c_2^4c_1^2 - 114q^2\eta c_2^4c_1^2 + \\
3114\eta c_2^4c_1^2 - 15660c_2^4c_1^2 + 5904q^2c_2^3c_1^2 - 580q^2\eta c_2^3c_1^2 - 1012\eta c_2^3c_1^2 + 14544c_2^3c_1^2 + 108q^2c_1^2 - 5184q^2c_2^2c_1^2 - \\
525q^2\eta c_2^2c_1^2 + 3729\eta c_2^2c_1^2 - 15552c_2^2c_1^2 + 108q^2\eta c_1^2 + 676\eta c_1^2 - 6624q^2c_2c_1^2 - 4q^2\eta c_2c_1^2 - 460\eta c_2c_1^2 + \\
2880c_2c_1^2 + 6948c_1^2 + 576q^2c_2^6c_1 + 28q^2\eta c_2^6c_1 + 20\eta c_2^6c_1 + 288c_2^6c_1 - 2088q^2c_2^5c_1 + 128q^2\eta c_2^5c_1 - 160\eta c_2^5c_1 + \\
1800c_2^5c_1 - 576q^2c_2^4c_1 + 152q^2\eta c_2^4c_1 - 760\eta c_2^4c_1 - 2304c_2^4c_1 + 720q^2c_2^3c_1 - 16q^2\eta c_2^3c_1 - 880\eta c_2^3c_1 - \\
1872c_2^3c_1 + 1440q^2c_1 - 6624q^2c_2^2c_1 - 4q^2\eta c_2^2c_1 - 460\eta c_2^2c_1 + 2880c_2^2c_1 + 112q^2\eta c_1 - 240\eta c_1 - 3816q^2c_2c_1 + \\
176q^2\eta c_2c_1 - 400\eta c_2c_1 - 5112c_2c_1 - 6048c_1 + 108q^2c_2^6 + 12q^2\eta c_2^6 + 4\eta c_2^6 + 36c_2^6 - 720q^2c_2^5 + 64q^2\eta c_2^5 - \\
96\eta c_2^5 + 432c_2^5 + 648q^2c_2^4 + 120q^2\eta c_2^4 + 616\eta c_2^4 + 792c_2^4 + 1872q^2c_2^3 + 112q^2\eta c_2^3 - 528\eta c_2^3 - 2160c_2^3 + 1728q^2 + \\
108q^2c_2^2 + 108q^2\eta c_2^2 + 676\eta c_2^2 + 6948c_2^2 + 48q^2\eta + 144\eta + 1440q^2c_2 + 112q^2\eta c_2 - 240\eta c_2 - 6048c_2 + 5184
\end{gathered}
$$

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2. How to find the bound?
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   - use *CAD*
   - necessary to split into *subtasks* and consider *sufficient conditions*

The Eigenvalues

Using interpolation on the characteristic polynomial of the matrix we find that the eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = q^4, \quad {}_3\lambda_4 = \frac{1}{D}\left(e(q_2) \pm \sqrt{d(q_2)}\right),$$

each of multiplicity 2 with $q_2 = q^2$ and e, d polynomials in c_1, c_2, η and q_2.

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The Polynomial $e(q_2)$

$$\eta c_2^6 c_1^6 + 144c_2^6 c_1^6 + 8\eta c_2^5 c_1^6 + 720c_2^5 c_1^6 + 26\eta c_2^4 c_1^6 + 1476c_2^4 c_1^6 + 44\eta c_2^3 c_1^6 + 1584c_2^3 c_1^6 + 41\eta c_2^2 c_1^6 + 936c_2^2 c_1^6 +$$
$$9\eta c_2^6 q_2^2 c_1^6 + 1296c_2^6 q_2^2 c_1^6 - 24\eta c_2^5 q_2^2 c_1^6 - 2160c_2^5 q_2^2 c_1^6 + 138\eta c_2^4 q_2^2 c_1^6 + 6372c_2^4 q_2^2 c_1^6 - 132\eta c_2^3 q_2^2 c_1^6 - 4752c_2^3 q_2^2 c_1^6 +$$
$$177\eta c_2^2 q_2^2 c_1^6 + 4968c_2^2 q_2^2 c_1^6 + 36\eta q_2^2 c_1^6 - 60\eta c_2 q_2^2 c_1^6 - 864c_2 q_2^2 c_1^6 + 324q_2^2 c_1^6 + 4\eta c_1^6 + 20\eta c_2 c_1^6 + 288c_2 c_1^6 +$$
$$6\eta c_2^6 q_2 c_1^6 + 864c_2^6 q_2 c_1^6 + 16\eta c_2^5 q_2 c_1^6 + 1440c_2^5 q_2 c_1^6 + 60\eta c_2^4 q_2 c_1^6 + 1944c_2^4 q_2 c_1^6 + 88\eta c_2^3 q_2 c_1^6 + 3168c_2^3 q_2 c_1^6 +$$
$$54\eta c_2^2 q_2 c_1^6 + 2160c_2^2 q_2 c_1^6 + 24\eta q_2 c_1^6 + 40\eta c_2 q_2 c_1^6 + 576c_2 q_2 c_1^6 + 216q_2 c_1^6 + 36c_1^6 + 8\eta c_2^6 c_1^5 + 720c_2^6 c_1^5 - 64\eta c_2^5 c_1^5 -$$
$$1656c_2^5 c_1^5 - 304\eta c_2^4 c_1^5 - 7056c_2^4 c_1^5 - 352\eta c_2^3 c_1^5 - 5328c_2^3 c_1^5 - 184\eta c_2^2 c_1^5 + 720c_2^2 c_1^5 - 24\eta c_2^6 q_2^2 c_1^5 - 2160c_2^6 q_2^2 c_1^5 +$$
$$32\eta c_2^5 q_2^2 c_1^5 + 2664c_2^5 q_2^2 c_1^5 + 272\eta c_2^4 q_2^2 c_1^5 + 10224c_2^4 q_2^2 c_1^5 + 416\eta c_2^3 q_2^2 c_1^5 + 3312c_2^3 q_2^2 c_1^5 + 296\eta c_2^2 q_2^2 c_1^5 - 2736c_2^2 q_2^2 c_1^5 +$$
$$32\eta q_2^2 c_1^5 + 128\eta c_2 q_2^2 c_1^5 - 792c_2 q_2^2 c_1^5 - 144q_2^2 c_1^5 - 96\eta c_1^5 - 160\eta c_2 c_1^5 + 1800c_2 c_1^5 + 16\eta c_2^6 q_2 c_1^5 + 1440c_2^6 q_2 c_1^5 +$$
$$32\eta c_2^5 q_2 c_1^5 - 1008c_2^5 q_2 c_1^5 + 32\eta c_2^4 q_2 c_1^5 - 3168c_2^4 q_2 c_1^5 - 64\eta c_2^3 q_2 c_1^5 + 2016c_2^3 q_2 c_1^5 - 112\eta c_2^2 q_2 c_1^5 + 2016c_2^2 q_2 c_1^5 +$$
$$64\eta q_2 c_1^5 + 32\eta c_2 q_2 c_1^5 - 1008c_2 q_2 c_1^5 - 288q_2 c_1^5 + 432c_1^5 + 26\eta c_2^6 c_1^4 + 1476c_2^6 c_1^4 - 304\eta c_2^5 c_1^4 - 7056c_2^5 c_1^4 +$$
$$2724\eta c_2^4 c_1^4 + 21168c_2^4 c_1^4 - 1672\eta c_2^3 c_1^4 - 3600c_2^3 c_1^4 + 3114\eta c_2^2 c_1^4 - 15660c_2^2 c_1^4 + 138\eta c_2^6 q_2^2 c_1^4 + 6372c_2^6 q_2^2 c_1^4 +$$
$$272\eta c_2^5 q_2^2 c_1^4 + 10224c_2^5 q_2^2 c_1^4 + 4292\eta c_2^4 q_2^2 c_1^4 + 15408c_2^4 q_2^2 c_1^4 + 1496\eta c_2^3 q_2^2 c_1^4 + 8496c_2^3 q_2^2 c_1^4 + 5018\eta c_2^2 q_2^2 c_1^4 + \dots$$

The Polynomial $e(q_2)$

$$\cdots - 25740c_2^2q_2^2c_1^4 + 1064\eta q_2^2c_1^4 + 680\eta c_2q_2^2c_1^4 - 576c_2q_2^2c_1^4 + 1368q_2^2c_1^4 + 616\eta c_1^4 - 760\eta c_2c_1^4 - 2304c_2c_1^4 +$$
$$60\eta c_2^6q_2c_1^4 + 1944c_2^6q_2c_1^4 + 32\eta c_2^5q_2c_1^4 - 3168c_2^5q_2c_1^4 + 216\eta c_2^4q_2c_1^4 - 11232c_2^4q_2c_1^4 + 176\eta c_2^3q_2c_1^4 -$$
$$4896c_2^3q_2c_1^4 - 228\eta c_2^2q_2c_1^4 + 2808c_2^2q_2c_1^4 + 240\eta q_2c_1^4 + 80\eta c_2q_2c_1^4 + 2880c_2q_2c_1^4 + 1296q_2c_1^4 + 792c_1^4 +$$
$$44\eta c_2^6c_1^3 + 1584c_2^6c_1^3 - 352\eta c_2^5c_1^3 - 5328c_2^5c_1^3 - 1672\eta c_2^4c_1^3 - 3600c_2^4c_1^3 - 1936\eta c_2^3c_1^3 + 17568c_2^3c_1^3 -$$
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$$8496c_2^4q_2^2c_1^3 + 1808\eta c_2^3q_2^2c_1^3 - 13536c_2^3q_2^2c_1^3 + 1628\eta c_2^2q_2^2c_1^3 - 14832c_2^2q_2^2c_1^3 + 176\eta q_2^2c_1^3 + 944\eta c_2q_2^2c_1^3 -$$
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The Polynomial $e(q_2)$

$$\cdots + 740\eta c_2 q_2^2 c_1^2 + 4608 c_2 q_2^2 c_1^2 + 11844 q_2^2 c_1^2 + 676\eta c_1^2 - 460\eta c_2 c_1^2 + 2880 c_2 c_1^2 + 54\eta c_2^6 q_2 c_1^2 + 2160 c_2^6 q_2 c_1^2 -$$
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$$10368 c_2^2 q_2 c_1^2 + 216\eta q_2 c_1^2 - 280\eta c_2 q_2 c_1^2 - 7488 c_2 q_2 c_1^2 + 216 q_2 c_1^2 + 6948 c_1^2 + 20\eta c_2^6 c_1 + 288 c_2^6 c_1 -$$
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$$864 c_2^6 q_2^2 c_1 + 128\eta c_2^5 q_2^2 c_1 - 792 c_2^5 q_2^2 c_1 + 680\eta c_2^4 q_2^2 c_1 - 576 c_2^4 q_2^2 c_1 + 944\eta c_2^3 q_2^2 c_1 - 144 c_2^3 q_2^2 c_1 + 740\eta c_2^2 q_2^2 c_1 +$$
$$4608 c_2^2 q_2^2 c_1 + 80\eta q_2^2 c_1 + 368\eta c_2 q_2^2 c_1 + 6120 c_2 q_2^2 c_1 + 2016 q_2^2 c_1 - 240\eta c_1 - 400\eta c_2 c_1 - 5112 c_2 c_1 +$$
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$$96\eta c_2^5 + 432 c_2^5 + 616\eta c_2^4 + 792 c_2^4 - 528\eta c_2^3 - 2160 c_2^3 + 676\eta c_2^2 + 6948 c_2^2 + 36\eta c_2^6 q_2^2 + 324 c_2^6 q_2^2 + 32\eta c_2^5 q_2^2 -$$
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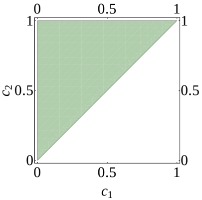
$$2016 c_2 q_2^2 + 9792 q_2^2 + 144\eta - 240\eta c_2 - 6048 c_2 + 24\eta c_2^6 q_2 + 216 c_2^6 q_2 + 64\eta c_2^5 q_2 - 288 c_2^5 q_2 + 240\eta c_2^4 q_2 +$$
$$1296 c_2^4 q_2 + 352\eta c_2^3 q_2 + 1440 c_2^3 q_2 + 216\eta c_2^2 q_2 + 216 c_2^2 q_2 + 96\eta q_2 + 160\eta c_2 q_2 + 4032 c_2 q_2 + 3456 q_2 + 5184$$

The Bound

- Consider extreme cases for (c_1, c_2)
- Consider the limits $\eta \to 0$ and $\eta \to \infty$



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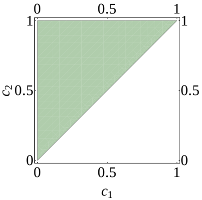
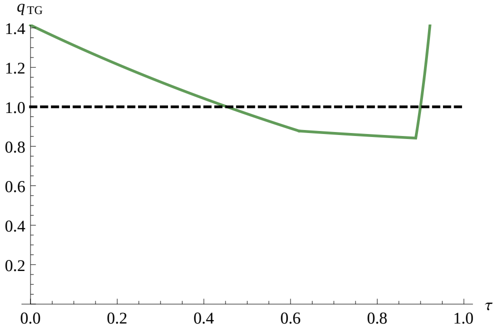


This yields the following *guess*:

$$B(q_2) = \begin{cases} \left(\frac{q_2+3}{4}\right)^2, & 0 < q_2 < Q_2, \\ q_2(q_2+1), & Q_2 \leq q_2 < 1. \end{cases},$$

with $Q_2 = \frac{1}{15}(4\sqrt{10} - 5)$.

Bound for the Convergence Rate



Two-grid convergence factor depending on τ for
$\nu = \nu_{pre} + \nu_{post} = 2 + 2$ smoothing steps

Experimental all-at-once analysis for 2D

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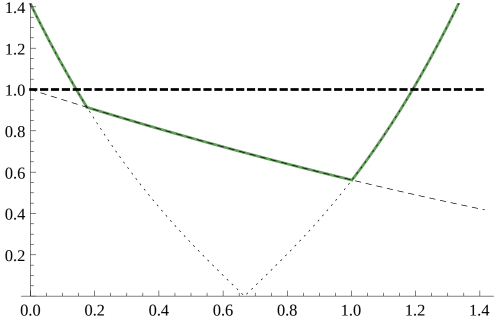

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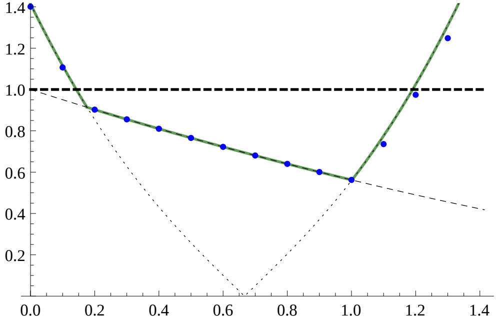

http://www.risc.jku.at/people/vpillwei/sLFA/

Chapter 0, Basic Concepts, Brenner+Scott, 2002

The finite element method provides a formalism for generating discrete (finite) algorithms for approximating the solutions of partial differential equations. It should be thought of as a black box into which one puts the differential equation (boundary value problem) and out of which pops an algorithm for approximating corresponding solutions. *Such a task could conceivably be done automatically by a computer*, but it necessitates an amount of mathematical skill that today still requires human involvement. The purpose of this book is to help people become adept at working the magic of this black box. The book does *not* focus on how to *turn the resulting algorithms into computer codes*, although this is at present also a complicated task. The latter is, however, a more well-defined task than the former and thus *potentially more amenable to automation*.