

# Real root finding for rank defects in linear Hankel matrices

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**I S S A C   2 0 1 5**  
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# Linear matrices and semidefinite programming

A **linear matrix** is a polynomial matrix of degree 1:

$$A(x) = A_0 + x_1 A_1 + \dots + x_n A_n,$$

with  $x = (x_1, \dots, x_n)$ . Suppose:

- ▶  $A_i \in \mathbb{Q}^{m \times m}$
- ▶  $A_i$  are **Hankel**

A linear Hankel matrix

$$\begin{bmatrix} x_1 + x_2 & x_4 + x_2 & x_3 - 1 \\ x_4 + x_2 & x_3 - 1 & x_4 \\ x_3 - 1 & x_4 & x_5 + 4 \end{bmatrix}$$

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Lyapunov Stability is an LMI

$$\boxed{\frac{df}{dt} = M \cdot f} \rightarrow \begin{array}{l} \text{Solve in } P \\ P \succ 0 \\ M^T P + P M \prec 0 \end{array}$$

The set  $\mathcal{S} = \{x \in \mathbb{R}^n : A(x) \succeq 0\}$  is called a **spectrahedron**.

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For  $A_i = A_i^T$ , a **linear matrix inequality** is the formula

$$A_0 + x_1 A_1 + \dots + x_n A_n \succeq 0$$

where  $\succeq 0$  is *positive semidefinite*.

## Semidefinite programming

$$\begin{array}{ll} \min & c_1 x_1 + \dots + c_n x_n \\ \text{s.t.} & x \in \mathcal{S} = \{x \in \mathbb{R}^n \mid A(x) \succeq 0\} \end{array}$$

# A Hankel spectrahedron

## The Carathéodory orbitope

Consider:

- ▶ Linear action of  $SO(2, \mathbb{R}) = \{M \in \mathbb{R}^{2 \times 2} : MM^T = I_2, \det M = 1\}$

$$(X, Y)^T \mapsto M(X, Y)^T$$

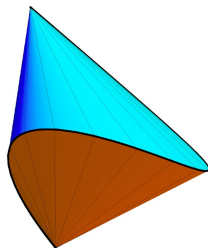
- ▶ over the set of homogeneous binary quartics

$$f(X, Y) = x_0 X^4 + 4x_1 X^3 Y + 6x_2 X^2 Y^2 + 4x_3 X Y^3 + x_4 Y^4$$

The convex hull of the orbit of  $f(X, Y) = X^4$  is the Hankel spectrahedron

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix} \succeq 0$$

$$\text{s.t. } x_0 + 2x_2 + x_4 = 1.$$



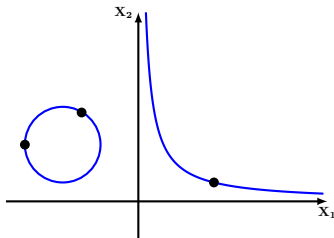
## Problem statement

Given

- ▶  $m, n, r \in \mathbb{N}$ ,  $0 \leq r \leq m - 1$
- ▶  $A_0, A_1 \dots A_n \in \mathbb{Q}^{m \times m}$
- ▶  $A_i$  Hankel matrices

let

- ▶  $A(x) = A_0 + x_1 A_1 + \dots + x_n A_n$
- ▶  $\mathcal{D}_r = \left\{ x \in \mathbb{C}^n \mid \text{rank } A(x) \leq r \right\}$ .



**Pb.** : Compute one point in each connected component of  $\mathcal{D}_r \cap \mathbb{R}^n$ .

Fact

Henrion-N.-Safey

Let  $A(x)$  be symmetric.

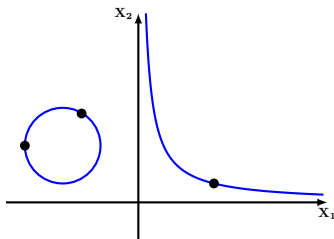
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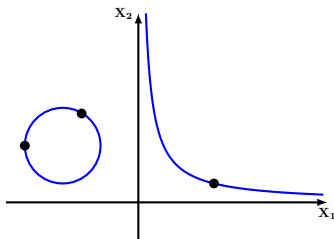
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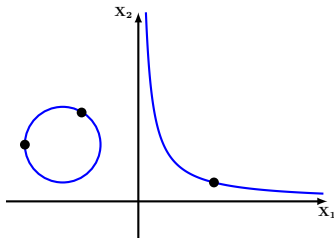
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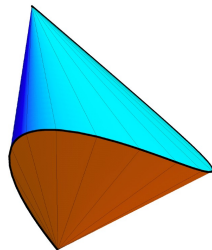
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## State of the art and main contributions

Compute at least one point in each conn. comp. of  $\mathcal{D}_r \cap \mathbb{R}^n$ .

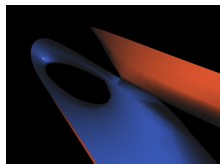
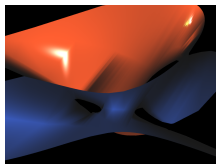
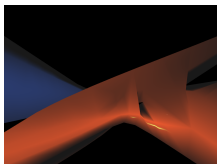
$$\mathcal{D}_r : f_1(x_1, \dots, x_n) = 0, \dots, f_s(x_1, \dots, x_n) = 0$$

- ▶ Cylindrical Algebraic Decomposition: Collins (1975) **doubly exp**
- ▶ Critical point method: Grigoriev, Vorobjov (1988) **singly exp**; Renegar (1992), Heintz, Roy, Solernó (1993) ...  $\rightarrow D^{\mathcal{O}(n)}$
- ▶ Polar varieties: Bank, Giusti, Heintz, Mbakop, Pardo, Safey, Schost
- ▶ Bounds on the exponents: Safey, Schost (2001-present)
  - ▶ smooth equidimensional algebraic sets:  $\mathcal{O}(D^{3n})$ ;
  - ▶ singular algebraic sets  $\mathcal{O}(D^{4n})$ .

### Main contributions:

- ▶ A **dedicated algorithm** for the Real Root Finding for  $\mathcal{D}_r \cap \mathbb{R}^n$
- ▶ Complexity  $\approx$  polynomial in  $\binom{n+2m-r-1}{n}$  (poly in  $n$  when  $m$  fixed)
- ▶ Implementation as MAPLE library: S.P.E.C.T.R.A. (freely available soon)

## Strategy



How to avoid singularities?

Input :  $P_1 = \dots = P_a = 0$   
 $V(P_1, \dots, P_a)$  **possibly singular**



A system  $Q_1 = \dots = Q_b = 0$   
 $V(Q_1, \dots, Q_b)$  **good properties**

How to reduce the dimension?

$\dim V(Q_1, \dots, Q_b) > 0$



A system  $R_1 = \dots = R_c = 0$   
 $V(R_1, \dots, R_c)$  **is finite**

**Main idea:** variant of **critical points method** for **determinantal** varieties with **Hankel** structure

# Exploiting structures

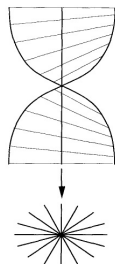
## Determinantal structure

*Room-Kempf* desingularization:

$$\mathbf{Q} = A(\mathbf{x}) \cdot \mathbf{Y} = A(\mathbf{x}) \cdot \begin{bmatrix} y_{1,1} & \cdots & y_{1,m-r} \\ \vdots & & \vdots \\ y_{m,1} & \cdots & y_{m,m-r} \end{bmatrix} = 0.$$

**An incidence relation:**  $\ker A(\mathbf{x}) = \langle \mathbf{Y} \rangle$

$\mathbf{Q}$  = **Bilinear equations** in  $(\mathbf{x}, \mathbf{y})$



# Exploiting structures

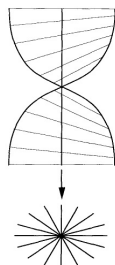
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## Hankel structure

The **kernel** of a Hankel matrix is **structured!**

Heinig, Rost (1984)

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_2 & x_3 & x_4 & x_5 & x_6 \\ x_3 & x_4 & x_5 & x_6 & x_7 \\ x_4 & x_5 & x_6 & x_7 & x_8 \\ x_5 & x_6 & x_7 & x_8 & x_9 \end{bmatrix} \begin{bmatrix} y_1 & 0 & 0 \\ y_2 & y_1 & 0 \\ y_3 & y_2 & y_1 \\ 0 & y_3 & y_2 \\ 0 & 0 & y_3 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 \\ f_2 & f_3 & f_4 \\ f_3 & f_4 & f_5 \\ f_4 & f_5 & f_6 \\ f_5 & f_6 & f_7 \end{bmatrix}$$

**Theorem:**  $Q = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)$  is *radical* and  $V(Q)$  is *smooth and equidimensional*

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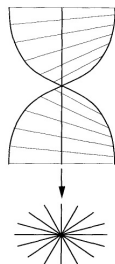
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**Theorem:**  $Q = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)$  is *radical* and  $V(Q)$  is *smooth and equidimensional*

## Bilinearity of critical points systems

Let  $\mathbf{a} = [a_1, \dots, a_n]^T \in \mathbb{R}^n$  and consider

$$\Pi_{\mathbf{a}}(\mathbf{x}, \mathbf{y}) = a_1 x_1 + \dots + a_n x_n.$$

*Critical points*  $\rightarrow$  solutions of the **bi-linear Lagrange system**  $\mathbf{R}$  :

$$\mathbf{Q}(\mathbf{x}, \mathbf{y}) = 0 \quad \mathbf{z}' \begin{pmatrix} \text{jac}_X \mathbf{Q}(\mathbf{x}, \mathbf{y}) & \text{jac}_Y \mathbf{Q}(\mathbf{x}, \mathbf{y}) \\ a_1, \dots, a_n & 0 \dots 0 \end{pmatrix} = 0.$$

**Theorem:** The set of solutions is *finite*

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By **multilinear Bézout bounds**:

degree of  $V(\mathbf{R}) =$  number of complex solutions  $\leq \binom{n+2m-r-1}{n}^3$

**Remark!** when  $\mathbf{m}$  is fixed, degree is polynomial in  $\mathbf{n}$

Multilinear bounds  $\ll$  Classical Bézout bound

## Complexity and output degree estimates

### Output representation: rational parametrization

$$x_1 = \frac{q_1(t)}{q_0(t)}, \dots, x_n = \frac{q_n(t)}{q_0(t)}, \quad q(t) = 0.$$

Algorithms for computing rational parametrizations:

- ▶ [Giusti, Lecerf, Salvy]  $\rightarrow$  geometric resolution
- ▶ [Jeronimo, Matera, Solernó]  $\rightarrow$  symbolic homotopy

### Output degree

$$\delta(m, n, r) \leq \binom{n + 2m - r - 1}{n}^3$$

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### Output degree

$$\delta(m, n, r) \leq \binom{n + 2m - r - 1}{n}^3$$

### Complexity

$$\mathcal{O} \left( rn(2m - r)(rn(2m - r)(n + 2m)^2 + (n + 2m)^4)\delta(m, n, r)^2 \right)$$

**Remark!** when  $\mathbf{m}$  is fixed, complexity is polynomial in  $\mathbf{n}$   
(this is not the case for general algorithms)

## Experiments

**Input:** randomly generated Hankel matrices  $A_0, A_1, \dots, A_n \in \mathbb{Q}^{m \times m}$

deg = degree of the **output** rational parametrization

$(m, r, n)$	RAGlib	New	deg	$(m, r, n)$	RAGlib	New	deg
(3, 2, 7)	20	21	39	(5, 3, 4)	202	18	110
(3, 2, 8)	53	21	39	(5, 3, 5)	$\infty$	583	338
(4, 2, 4)	43	6.5	40	(5, 4, 4)	8713	885	325
(4, 2, 5)	56575	18	88	(5, 4, 5)	$\infty$	15537	755
(4, 2, 6)	$\infty$	35	128	(6, 3, 5)	$\infty$	10	56
(4, 3, 7)	528	324	264	(6, 3, 6)	$\infty$	809	336
(4, 3, 8)	2638	375	264	(6, 3, 7)	$\infty$	49684	1032
(5, 2, 5)	25	4	21	(6, 4, 5)	$\infty$	30660	973
(5, 2, 6)	31176	21	91	(6, 5, 3)	915	356	186
(5, 2, 7)	$\infty$	135	199	(6, 5, 4)	$\infty$	20310	726

Table: Dense Hankel linear matrices

**Characteristics:** Intel(R) Xeon(R) CPU E7540@2.00GHz 256 Gb of RAM

**For Gröbner bases and rational parametrizations:** FGb (Faugère)

**Comparison with:** RAGlib (Safey El Din)

## Perspectives

The nice property  $\exists \mathcal{C} \subset \mathcal{D}_{\mathcal{R}(A)} : \mathcal{C} \subset \mathcal{S}$  of spectrahedra  
+ this algorithm for generic symmetric matrices  
= **Solve the emptiness problem for spectrahedra!**

Complexity is monotone on  $\mathcal{R}(A) \rightarrow$  low rank is better!

A. Exact solving for SDP

Compute a minimizer

$$\begin{array}{ll} \min & c_1 x_1 + \dots + c_n x_n \\ \text{s.t.} & x \in \mathcal{S} = \left\{ x \in \mathbb{R}^n \mid A(x) \succeq 0 \right\} \end{array}$$

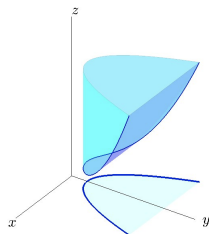
C. Sums of squares

Low rank decompositions

$$f = v^T X v \text{ with } X \succeq 0$$

B. Projected spectrahedra

Emptiness problem



D. Gb for multi-linear systems with additional structures?

Thank you!

Questions?

