Certifying singular isolated points and their multiplicity structure

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A system of equations $\boldsymbol{f} = \{f_1, \ldots, f_s\}$, $f_i \in \mathbb{K}[x_1, \ldots, x_n]$, with an isolated root $\zeta \in \mathbb{K}^n$ of $\boldsymbol{f} = 0$.

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Two types of problems:

- Nearby system with a point of given multiplicity.
- Nearby point of a singular solution of an exact system.

Objectives

- **Numeric:** recover the quadratic convergence.
- **Symbolic:** recover the multiplicity structure (i.e. the differential polynomials which vanish at ζ).

Motivations:

- Numerical improvement of root approximation in homotopy methods (end games), in subdivision methods, . . .
- Certification of approximate roots of (over-determined) polynomial systems.
- Multiplicity structure for topology analysis.

“Desingularisation” strategies

- **Blowup of the singular point:** algebraic tools, need to know the point exactly. *Not applicable for approximate points.*
- **Add new equations to reduce the multiplicity:** Ojika et al. 83; 88, . . . , Lecerf'02, Giusti & Yakoubsohn' 13, Hauenstein & Wampler'13. *Quadratic growth of the system size.*
- **Add new equations and new variables:** Leykin & Verschelde & Zhao'06,'08, *Exponential growth of the number of variables.*
  Li & Zhi'12,'13, *breath one case*.
- **Deform the system of equations:** versal deformations. Exact multiple roots of approximate systems, Mantzaflaris & M' 11.

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Our contributions:

1. An efficient deflation method with no new variable and a linear growth of the system size.
2. A new certification method for the singular point **and** its multiplicity structure.

① Deflation using the first derivatives

For the system $\boldsymbol{f}$ with an isolated root $\boldsymbol{\xi}$ of **multiplicity** δ and **order** o,

- Decompose
$$J_{\boldsymbol{f}}(\boldsymbol{x}) := \begin{bmatrix} A(\boldsymbol{x}) & B(\boldsymbol{x}) \\ C(\boldsymbol{x}) & D(\boldsymbol{x}) \end{bmatrix}$$
  where $A(\boldsymbol{x})$ is an $r \times r$ matrix with $r = \mathrm{rank} J_{\boldsymbol{f}}(\boldsymbol{\xi}) = \mathrm{rank} A(\boldsymbol{\xi})$.
- Take
$$\Delta_{\Lambda}(\partial) = [\partial_1, \ldots, \partial_n] \begin{bmatrix} -A^*(\boldsymbol{x})B(\boldsymbol{x}) \\ \mathrm{Id} \end{bmatrix} \begin{bmatrix} \lambda_{1,1} & \cdots & \lambda_{1,k} \\ \vdots & & \vdots \\ \lambda_{r,1} & \cdots & \lambda_{r,k} \end{bmatrix}$$
  where $A^*(\boldsymbol{x})$ is the co-matrix of A, Λ is a non-zero constant matrix.

Then $\boldsymbol{\xi}$ is an isolated root of the system
$$\boldsymbol{f}^{(1)} = \{\boldsymbol{f}, \Delta_{\Lambda}(\boldsymbol{f})\}$$
of **order** $o' \leq \max(o-1, 0)$ and **multiplicity** $\delta' \leq \max(\delta-1, 1)$.

☞ **Simple point in $\leq o$ steps of deflation.**

① Example

1: $\{x_1^4 - x_2x_3x_4, x_2^4 - x_1x_3x_4, x_3^4 - x_1x_2x_4, x_4^4 - x_1x_2x_3\}$ **at** $(0,0,0,0)$ **with** $\delta = 131$ **and** $o = 10$;

2: $\{x^4, x^2y + y^4, z + z^2 - 7x^3 - 8x^2\}$ **at** $(0,0,-1)$ **with** $\delta = 16$ **and** $o = 7$;

3: $\{14x + 33y - 3\sqrt{5}(x^2 + 4xy + 4y^2 + 2) + \sqrt{7} + x^3 + 6x^2y + 12xy^2 + 8y^3, 41x - 18y - \sqrt{5} + 8x^3 - 12x^2y + 6xy^2 - y^3 + 3\sqrt{7}(4xy - 4x^2 - y^2 - 2)\}$ **at** $Z_3 \sim (1.5055, 0.36528)$ **with** $\delta = 5$ **and** $o = 4$;

4: $\{2x_1 + 2x_1^2 + 2x_2 + 2x_2^2 + x_3^2 - 1, (x_1 + x_2 - x_3 - 1)^3 - x_1^3, (2x_1^3 + 5x_2^2 + 10x_3 + 5x_3^2 + 5)^3 - 1000x_1^5\}$ **at** $(0,0,-1)$ **with** $\delta = 18$ **and** $o = 7$.

	Method A			Method B			Method C			Method D		
	Poly	Var	It	Poly	Var	It	Poly	Var	It	Poly	Var	It
1	16	4	2	22	4	2	22	4	2	16	4	2
2	24	11	3	11	3	2	12	3	2	12	3	3
3	32	17	4	6	2	4	6	2	4	6	2	4
4	96	41	5	54	3	5	54	3	5	22	3	5

A: intrinsic slicing [Leykin-Verschelde-Zhao'06, Dayton-Zen'05];

B: isosingular deflation [Hauenstein-Wampler'13];

C: "kerneling" method in [Giusti-Yakoubsohn'13];

D: our approach.

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$$\Lambda^{\zeta}[g] = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} \lambda_{\boldsymbol{\alpha}} \frac{1}{\boldsymbol{\alpha}!} \boldsymbol{\partial}_{\zeta}^{\boldsymbol{\alpha}}[g] = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} \lambda_{\boldsymbol{\alpha}} \frac{1}{\alpha_1! \cdots \alpha_n!} \frac{\partial^{|\boldsymbol{\alpha}|} g}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}(\zeta)$$

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Definition (dual local space or inverse system of ζ)

The space $\mathscr{D} \subset R^*$ of differential conditions at $\boldsymbol{\zeta}$, i.e.

$$\mathscr{D}^{\zeta}[g] = 0 \quad \Longleftrightarrow \quad g \in \mathcal{Q}_{\zeta}$$

where $\mathcal{Q}_{\zeta}$ is the primary component of $\boldsymbol{\zeta}$.

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☞ $\mathscr{D} \cong (R/\mathcal{Q}_\zeta)^*$

☞ $\mathscr{D} \subset \mathbb{K}[\partial_1, \ldots, \partial_n]$, stable by $\frac{d}{d\partial_i}$.

- $\delta_\zeta := \dim \mathscr{D} =$ **multiplicity** of ζ,
- $o_\zeta := \max_{\Lambda\in\mathscr{D}} \deg_{\boldsymbol{\partial}}(\Lambda) =$ **order** of ζ.

② Multiplicity structure

The following points are equivalent to $\Lambda \in \mathscr{D}$:

1. $\Lambda^{\zeta}[(\boldsymbol{x}-\zeta)^{\boldsymbol{\alpha}} \cdot f_i] = 0, \forall \boldsymbol{\alpha} \in \mathbb{N}^n, \quad i = 1, \ldots, s.$

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Two methods:

1. Macaulay's dialytic method: [Macaulay'1916], ...
   Kernel of the coeff. matrix of monomial multiples up to valuation o.
   Solve a linear system of size $n\binom{o-1+n}{o-1} \times \binom{o+n}{o}$.

2. Integration method: [M'97], [Mantzaflaris-M'11].
   $\mathscr{D}_k = \{$differentials in $\mathscr{D}$ of degree $\leq k\}$ computed by integration of a basis of $\mathscr{D}_{k-1}$, starting from $\mathscr{D}_0 = \langle 1 \rangle$.
   Solve a linear system of smaller size $(\frac{n(n-1)}{2}\delta + n) \times \delta(n-1) + 1$.

② Example: computing a primal-dual pair

- $f_1 = x_1 - x_2 + x_1^2$, $f_2 = x_1 - x_2 + x_2^2$, $\zeta = (0, 0)$.

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$$\mathcal{B} = \{1, x_1\}$$

$$\Lambda = \lambda_1 \partial_1 + \lambda_2 \partial_2$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 0 \quad \Rightarrow \quad \Lambda_1 = \partial_1 + \partial_2$$

$$\text{Row constaints: } \begin{cases} \text{Vanishing: } * \ \Lambda[f_1] = 0 \ \ , \ \ \Lambda[f_2] = 0 \\ \text{Stability: } \ \text{(none)} \end{cases}$$

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$$\Lambda = \lambda_1\partial_1 + \lambda_2\partial_2 + \lambda_3\tfrac{1}{2}\partial_1^2 + \lambda_4(\partial_1\partial_2 + \tfrac{1}{2}\partial_2^2)$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = 0 \quad \Rightarrow \quad \Lambda_2 = -\partial_1 + \tfrac{1}{2}\partial_1^2 + \partial_1\partial_2 + \tfrac{1}{2}\partial_2^2$$

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☞ **Primal-Dual pair:**

$$\mathcal{B} = \{1, x_1, x_2\}, \quad \mathscr{D} = \langle 1, \partial_1 + \partial_2, \partial_2 + \tfrac{1}{2}\partial_1^2 + \partial_1\partial_2 + \tfrac{1}{2}\partial_2^2 \rangle.$$

Primal-Dual pair:

$$\mathcal{B} = \{(\boldsymbol{x}-\zeta)^{\beta_1}, \ldots, (\boldsymbol{x}-\zeta)^{\beta_\delta}\} \qquad \begin{array}{c} \\ \Lambda_1 \\ \vdots \\ \Lambda_\delta \end{array} \begin{array}{c} \begin{array}{cccccc} \boldsymbol{\beta}_1 & \cdots & \boldsymbol{\beta}_\delta & \boldsymbol{\gamma}_1 & \cdots & \boldsymbol{\gamma}_l \end{array} \\ \left[\begin{array}{cccccc} 1 & & * & \nu_{1,1} & \cdots & \nu_{1,l} \\ & \ddots & & \vdots & & \vdots \\ \mathbf{0} & & 1 & \nu_{\delta,1} & \cdots & \nu_{\delta,l} \end{array}\right] \end{array}$$

with $\mathscr{D}_t$ spanned by $\Lambda_1, \ldots, \Lambda_{s_t}$ for $t = 0, \ldots, o_\zeta$.

Multiplication operators: $M_k^t : \Lambda \in \mathscr{D}_t \mapsto d_{\partial_k}\Lambda \in \mathscr{D}_{t-1}$.

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Parametric multiplication matrices:

$$M_{k,\boldsymbol{\mu}}^t := \begin{bmatrix} 0 & \boldsymbol{\mu}_{\boldsymbol{\beta}_2,e_k} & \boldsymbol{\mu}_{\boldsymbol{\beta}_3,e_k} & \cdots & \boldsymbol{\mu}_{\boldsymbol{\beta}_\delta,e_k} \\ 0 & 0 & \boldsymbol{\mu}_{\boldsymbol{\beta}_3,\boldsymbol{\beta}_2+e_k} & \cdots & \boldsymbol{\mu}_{\boldsymbol{\beta}_\delta,\boldsymbol{\beta}_2+e_k} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \boldsymbol{\mu}_{\boldsymbol{\beta}_\delta,\boldsymbol{\beta}_{\delta-1}+e_k} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{with} \quad \boldsymbol{\mu}_{\beta_i,\beta_j+e_k} = \begin{cases} 1 & \text{if } \beta_i = \beta_j + e_k \\ 0 & \text{if } \beta_j + e_k \in \mathcal{B}, \\ & \text{and } \beta_i \neq \beta_j + e_k \end{cases}$$

Parametric normal form:

$$\begin{aligned} \mathcal{N}_{\mathbf{z},\boldsymbol{\mu}} : \mathbb{K}[\mathbf{x}] &\rightarrow \mathbb{K}[\mathbf{z},\boldsymbol{\mu}]^\delta \\ p &\mapsto \mathcal{N}_{\mathbf{z},\boldsymbol{\mu}}(p) := \sum_{\gamma \in \mathbb{N}^n} \frac{1}{\gamma!} \partial_{\mathbf{z}}^{\gamma}(p)\, \mathbf{M}_{\boldsymbol{\mu}}^{\gamma}(1). \end{aligned}$$

② Certifying the multiplicity structure

Theorem

Let $\mathbb{K} \subset \mathbb{C}$, $\boldsymbol{f} = \{f_1, \ldots, f_s\} \in \mathbb{K}[\boldsymbol{x}]^s$ *and let* $\boldsymbol{\zeta} \in \mathbb{C}^n$ *be an isolated solution of* $\boldsymbol{f}$*. Then* $(\boldsymbol{z}, \boldsymbol{\mu}) = (\zeta, \nu)$ *is an isolated root with multiplicity one of the polynomial system in* $\mathbb{K}[\boldsymbol{z}, \boldsymbol{\mu}]$*:*

$$\begin{cases} \mathcal{N}_{\boldsymbol{z},\boldsymbol{\mu}}(f_k) = 0 \quad \text{for } k = 1, \ldots, s, \\ M_{i,\boldsymbol{\mu}} \cdot M_{j,\boldsymbol{\mu}} - M_{j,\boldsymbol{\mu}} \cdot M_{i,\boldsymbol{\mu}} = 0 \quad \text{for } i, j = 1, \ldots, n \end{cases} \tag{1}$$

☞ **Over determined system with exact coefficients defining the multiple point ζ and the inverse system ν.**

☞ **Quadratic convergence to the multiple root and its inverse system from a nearby solution**.

② Examples

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Extended system:

$$\begin{array}{ll} M_1 M_2 - M_2 M_1 = 0 & \mu_1\mu_2 - \mu_3, \\ \mathcal{N}(f_1) = 0 & x_1 - x_2 + {x_1}^2,\quad 1 + 2\,x_1 - \mu_2,\quad -1 + \mu_1, \\ \mathcal{N}(f_2) = 0 & x_1 - x_2 + {x_2}^2,\quad 1 + (-1 + 2\,x_2)\,\mu_2,\quad -1 + 2\,x_2 + \mu_2\mu_3 \end{array}$$

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Numerical improvements:

Iter	$[x_1, x_2, \mu_1, \mu_2, \mu_3]$
0	$[0.1, 0.12, 1.1, 1.25, 1.72]$
1	$[0.0297431315, 0.0351989647, 0.9975178694, 1.0480778978, 1.0227973199]$
2	$[0.0005578682, 0.0008806394, 0.9999134370, 0.9997438194, 0.9996904740]$
3	$[0.0000001981, -0.0000001864, 0.9999999998, 1.0000002375, 1.0000002150]$
4	$[2.084095775\,10^{-14}, -1.9808984139\,10^{-14}, 1.0, 1.0000000000, 1.0000000000]$

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Quadratic convergence: $(0,0), (1, \partial_1 + \mu_2\partial_2, \partial_2 + \frac{1}{2}\mu_1\partial_1^2 + \mu_3\partial_1\partial_2 + \frac{1}{2}\mu_2\mu_3\partial_2^2)$

$\mu_1 = 1, \mu_2 = 1, \mu_3 = 1$

② Examples

New family:

$$x_1^3 + x_1^2 - x_2^2,\ x_2^3 + x_2^2 - x_3, \ldots, x_{n-1}^3 + x_{n-1}^2 - x_n,\ x_n^2.$$

$\zeta = \mathbf{0}$: $\delta := 2^n$, $o = 2^{n-1}$, breadth = corank of Jacobian = 2.

		New approach			Null space		
n	mult	vars	poly	time	vars	poly	time
2	4	5	9	1.476	8	17	2.157
3	8	17	31	5.596	192	241	208
4	16	49	100	19.698	7189	19804	> 76000
5	32	129	296	73.168	N/A	N/A	N/A
6	64	321	819	659.59	N/A	N/A	N/A

Experiment in matlab on iMac, 3.4 GHz Intel Core i7.

Perspectives:

- Parameters in the parametric normal form.
- Approximate structure of sum of local algebras.
- Applications to geometric problems
- ...

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Thanks for your attention