Buchberger Theory for Effective Associative Rings

T. M., *Seven variations on standard bases*, (1988)

A solution if the ring is a vectorspace over a field

Apel J., *Computational ideal theory in finitely generated extension rings*, T.C.S. 224 (2000), 1–33

Extension to suitable rings which are algebra over a ring

Gateva, Weispfenning and Passau group, Reinert, ...

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Theorem

For an (associative but not necessarily commutative) ring with identity $\mathcal{A}$, there is a (not necessarily finite nor necessarily countable) set $\overline{\mathbf{Z}}$ and a projection $\Pi : \mathcal{Q} := \mathbb{Z}\langle\overline{\mathbf{Z}}\rangle \twoheadrightarrow \mathcal{A}$ so that, denoting $\mathcal{I} \subset \mathcal{Q} = \mathbb{Z}\langle\overline{\mathbf{Z}}\rangle$ the bilateral ideal $\mathcal{I} := \ker(\Pi)$, we have $\mathcal{A} = \mathcal{Q}/\mathcal{I}$.

Effectively given Associative Rings

Let R be an (associative but not necessarily commutative) ring with identity $\mathbf{1}_R$ and $\mathcal{A}$ another (associative but not necessarily commutative) ring with identity $\mathbf{1}_{\mathcal{A}}$ which is a left module on R.

We consider $\mathcal{A}$ to be effectively given when we are given

- sets $\overline{\mathbf{v}} := \{x_1, \ldots, x_j, \ldots\}$, $\overline{\mathbf{V}} := \{X_1, \ldots, X_i, \ldots\}$, which are *countable* and
- $\overline{\mathbf{Z}} := \overline{\mathbf{v}} \sqcup \overline{\mathbf{V}} = \{x_1, \ldots, x_j, \ldots, X_1, \ldots, X_i, \ldots\}$;
- rings $\mathcal{R} := \mathbb{Z}\langle\overline{\mathbf{v}}\rangle \subset \mathcal{Q} := \mathbb{Z}\langle\overline{\mathbf{Z}}\rangle$;
- projections $\pi : \mathcal{R} = \mathbb{Z}\langle x_1, \ldots, x_j, \ldots\rangle \twoheadrightarrow R$ and
- $\Pi : \mathcal{Q} := \mathbb{Z}\langle x_1, \ldots, x_j, \ldots, X_1, \ldots, X_i, \ldots\rangle \twoheadrightarrow \mathcal{A}$ which satisfies

$$\Pi(x_j) = \pi(x_j)\mathbf{1}_{\mathcal{A}}, \text{ for each } x_j \in \overline{\mathbf{v}},$$

so that $\Pi(\mathcal{R}) = \{r\mathbf{1}_{\mathcal{A}} : r \in R\} \subset \mathcal{A}$.

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Thus denoting

- $\mathcal{I} := \ker(\Pi) \subset \mathcal{Q}$ and
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$$X_i x_j \equiv \sum_{l=1}^{i} \pi(a_{lij}) X_l + \pi(a_{0ij}) \bmod \mathcal{I}, a_{lij} \in \mathbb{Z}\langle\overline{\mathbf{v}}\rangle, \forall X_i \in \overline{\mathbf{V}}, x_j \in \overline{\mathbf{v}}.$$

If not $\mathbb{Z}\langle X, Y\rangle$ as left $\mathbb{Z}[X]$-module requires $1 \geq X \geq X^2$

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If we fix a term-ordering $<$ on $\langle \overline{\mathbf{Z}} \rangle$ we can assume $\mathcal{I}$ to be given via its Gröbner basis G w.r.t. $<$ and, if $<$ satisfies

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$$X_i x_j = \mathbf{T}(f_{ij}) \text{ for each } X_i \in \overline{\mathbf{V}}, x_j \in \overline{\mathbf{v}}$$

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Zacharias G., *Generalized Gröbner bases in commutative polynomial rings*, Bachelor's thesis, M.I.T. (1978)

Effective description of the canonical form of the ring

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Spear D.A., *A constructive approach to commutative ring theory*, in *Proc. of the 1977 MACSYMA Users' Conference*, NASA CP-2012 (1977), 369–376

Importing a Buchberger Theory from $\mathbb{Z}\langle\overline{\mathbf{Z}}\rangle$ to $\mathcal{A}$

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Möller H.M., *On the construction of Gröbner bases using syzygies*, J. Symb. Comp. 6 (1988), 345–359
Pritchard F. L., *The ideal membership problem in non-commutative polynomial rings*,
J. Symb. Comp. 22 (1996), 27–48

Lifting Theorem

Zacharias Canonical form

The obvious canonical forms of A_c for $\mathbb{Z}_c$ namely
$A_c := \{r : 0 \leq r < c\}$ or $A_c := \{r : 0 < r \leq c\}$ or
$A_c := \{r : -\frac{c}{2} < r \leq \frac{c}{2}\}$ give naturally a computational canonical form for $\mathbb{Z}\langle\overline{\mathbf{Z}}\rangle^m$
For a module $\mathsf{I} \subset \mathbb{Z}\langle\overline{\mathbf{Z}}\rangle^m$, and each term

$$\tau \in \langle\overline{\mathbf{Z}}\rangle^{(m)} = \left\{ve_i, v \in \langle\overline{\mathbf{Z}}\rangle, 1 \leq i \leq m\right\},$$

considering the principal ideals

$$\mathbb{I}(c_\tau)u := \{\mathrm{lc}(f) : f \in \mathsf{I}, \mathbf{T}(f) = \tau\} \cup \{0\} \subset \mathbb{Z},$$

we have
$\mathbb{Z}\langle\overline{\mathbf{Z}}\rangle^m/\mathsf{I} \cong \mathbf{Zach}(\mathbb{Z}\langle\overline{\mathbf{Z}}\rangle^m/\mathsf{I}) =: \bigoplus_{\tau \in \langle\overline{\mathbf{Z}}\rangle^{(m)}} A_{c_\tau}\tau$

$\mathsf{I} = \mathbb{I}(4X, 2X^2)\ \mathcal{A} = \mathbb{Z} \oplus \mathbb{Z}_4 X \oplus X^2\mathbb{Z}_2[X]$

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$$\mathcal{A} \cong \bigoplus_{w \in \langle\overline{\mathbf{V}}\rangle} \left(\bigoplus_{v \in \langle\overline{\mathbf{v}}\rangle} A_{c_{vw}} v \right) w =: \bigoplus_{w \in \langle\overline{\mathbf{V}}\rangle} R_w w \subset \mathsf{R}\langle\overline{\mathbf{V}}\rangle \subset \mathcal{Q}$$

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$$\mathcal{B} := \{\omega \in \langle\overline{\mathbf{V}}\rangle : \mathcal{I}_\omega \neq R\} \subset \left\{v\omega : v \in \langle\overline{\mathbf{v}}\rangle, \omega \in \langle\overline{\mathbf{V}}\rangle\right\}$$

$$\mathcal{A} \cong \bigoplus_{\omega\in\langle\overline{\mathbf{V}}\rangle} \left(\bigoplus_{v\in\langle\overline{\mathbf{v}}\rangle} A_{c_{v\omega}} v \right) \omega =: \bigoplus_{\omega\in\langle\overline{\mathbf{V}}\rangle} R_\omega \omega \subset \mathsf{R}\langle\overline{\mathbf{V}}\rangle \subset \mathcal{Q}$$

$$R_\omega := \mathcal{R}/\mathcal{I}_\omega \cong \bigoplus_{v\in\langle\overline{\mathbf{v}}\rangle} A_{c_{v\omega}} v \subset \mathsf{R}\langle\overline{\mathbf{V}}\rangle$$

Zacharias Canonical form

$$\Pi : \mathcal{Q} := \mathbb{Z}\langle x_1, \ldots, x_j, \ldots, X_1, \ldots, X_i, \ldots\rangle \twoheadrightarrow \mathcal{A} = \mathcal{Q}/\mathcal{I}$$

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Spear's Theorem

$\mathcal{B} := \{w \in \langle \overline{\mathbf{V}} \rangle : \mathcal{I}_w \neq R\} \subset \left\{ vw : v \in \langle \overline{\mathbf{v}} \rangle, w \in \langle \overline{\mathbf{V}} \rangle \right\}$

Spear's intuition that a Buchberger Theory defined in a ring can be exported to its quotients allow us to impose on $\mathcal{A}$ the "natural" Γ-valuation/filtration

$$\mathbf{T}(\cdot) : \mathcal{A}^m \mapsto \mathcal{B}^{(m)} : f \to \mathbf{T}(f)$$

where $(\Gamma, \circ)$, $\mathcal{B} \subset \Gamma \subset \langle \overline{\mathbf{V}} \rangle$, is a suibale semigroup.

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The associated Γ-graded ring $\mathcal{G} = G(\mathcal{A})$ coincides as a set with $\mathcal{A}$ and this is sufficient to smoothly export Buchberger test/completion but they don't conside as rings:
the multiplication $\star$ of $\mathcal{A}$ does not coincide with the one, $*$, of $\mathcal{G}$

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For instance, if we consider the Weyl algebra,

$$\mathcal{A} = \mathbb{Q}\langle D, X \rangle / \mathbb{I}(DX - XD - 1)$$

where

$$\mathcal{G} = \mathbb{Q}[D, X], D \star X = XD - 1, D * X = XD.$$

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the multiplication $\star$ of $\mathcal{A}$ does not coincide with the one, $*$, of $\mathcal{G}$
However an old slogan stated that in order to provide a Buchberger Algorithm on $\mathcal{A}$, one just needs to modify, in the algorithm for $\mathcal{G}$, the multiplication procedure!

Gröbner Bases

$$f = \sum_{i=1}^{s} c(f, t_i)t_i : c(f, t_i) \in \mathbb{Z} \setminus \{0\}, t_i \in \langle \overline{\mathbf{Z}} \rangle, t_1 > \cdots > t_s.$$

$$\mathbf{T}(f) := t_1, \mathrm{lc}(f) := c(f, t_1), \mathbf{M}(f) := c(f, t_1)t_1.$$

Pan: $XY = 3X \cdot Y - X \cdot 2Y \in \mathbb{I}(3X, 2Y)$

Let M $\subset \mathcal{A}^m$ be a (left, right, bilateral) $\mathcal{A}$-module. $F \subset$ M will be called

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Let M $\subset \mathcal{A}^m$ be a (left, right, bilateral) $\mathcal{A}$-module. $F \subset$ M will be called

a (left, right, bilateral) *Gröbner basis* of M if F satisfies the following condition:

- for each $f \in$ M, there are $g_i \in F$, $\lambda_i, \rho_i \in \mathcal{B}, a_i \in R_{\lambda_i} \setminus \{0\}, b_i \in R_{\rho_i} \setminus \{0\}$ such that
  - $\mathbf{T}(f) = \lambda_i \circ \mathbf{T}(g_i) \circ \rho_i$ for all i,
  - $\mathbf{M}(f) = \sum_i a_i\lambda_i * \mathbf{M}(g_i) * b_i\rho_i$;

Gröbner Bases

$$f = \sum_{i=1}^{s} c(f, t_i)t_i : c(f, t_i) \in \mathbb{Z} \setminus \{0\}, t_i \in \langle \overline{\mathbf{Z}} \rangle, t_1 > \cdots > t_s.$$

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Let M $\subset \mathcal{A}^m$ be a (left, right, bilateral) $\mathcal{A}$-module. $F \subset$ M will be called

a (left, right, bilateral) *strong Gröbner basis* of I if F satisfies the following equivalent conditions

- for each $f \in$ M there is $g \in F$ such that $\mathbf{M}(g) \mid \mathbf{M}(f)$,
- for each $f \in$ M there are $g \in F$, $\lambda, \rho \in \mathcal{B}, a \in R_\lambda \setminus \{0\}, b \in R_\rho \setminus \{0\}$, such that

$$\mathbf{T}(f) = \lambda \circ \mathbf{T}(g) \circ \rho \text{ and } \mathbf{M}(f) = a\lambda * \mathbf{M}(g) * b\rho.$$

Buchberger Theorem

Theorem

For any set $F \subset \mathcal{A}^m \setminus \{0\}$ *the following conditions are equivalent:*

- $f \in \mathbb{I}(F) \iff$ *it has a representation*

$$f = \sum_{i=1}^{\mu} a_i \lambda_i \star g_i \star b_i \rho_i$$

$\mathbf{T}(f) = \lambda_1 \circ \mathbf{T}(g_1) \circ \rho_1$ *and* $\lambda_i \circ \mathbf{T}(g_i) \circ \rho_i > \lambda_{i+1} \circ \mathbf{T}(g_{i+1}) \circ \rho_{i+1} \forall i$;

- F *is a strong Gröbner basis of* $\mathbb{I}(F)$;

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$\mathbf{T}(f) = \lambda_1 \circ \mathbf{T}(g_1) \circ \rho_1$ *and* $\lambda_i \circ \mathbf{T}(g_i) \circ \rho_i \geq \lambda_{i+1} \circ \mathbf{T}(g_{i+1}) \circ \rho_{i+1} \forall i$;

- F *is a weak Gröbner basis of* $\mathbb{I}(F)$;

Buchberger Reduction

In order to perform Buchberger Algorithm we need to solve

Problem

Given $g \in \mathcal{A}^m \setminus \{0\}$ *and* $F \subset \mathcal{A}^m \setminus \{0\}$ *decide whether*

$$\mathbf{M}(g) \in \mathbf{M}\{\mathbb{I}_2(\mathbf{M}\{F\})\} \subset \mathcal{G}^m$$

in which case return $g_i \in F$,
$\lambda_i, \rho_i \in \mathcal{B}, a_i \in R_{\lambda_i} \setminus \{0\}, b_i \in R_{\rho_i} \setminus \{0\}$ *such that*

- $\mathbf{T}(g) = \lambda_i \circ \mathbf{T}(g_i) \circ \rho_i$ *for all* i *and*
- $\mathbf{M}(g) = \sum_i a_i \lambda_i * \mathbf{M}(g_i) * b_i \rho_i$.

which is trivial in $\mathcal{Q}$.
Then in $\mathcal{G}$ performs $g \to g - \sum_i a_i \lambda_i \star g_i \star b_i \rho_i$

Lifting Theorem

Given a finite set

$$F := \{g_1, \ldots, g_u\} \subset \mathcal{A}^m, g_i = \mathbf{M}(g_i) - p_i =: a_i \tau_i \mathbf{e}_{\iota_i} - p_i,$$

we denote $\mathsf{M} := \mathbb{I}(F)$ and the morphisms

$$\mathfrak{s}_L : \mathcal{G}^u \to \mathcal{G}^m \text{ and } \mathfrak{S}_L : \mathcal{A}^u \to \mathcal{A}^m$$

defined as

$$\begin{aligned}
\mathfrak{s}_L \left(\sum_{i=1}^{u} \left(\sum_{\omega \in \mathcal{B}} a_{i\omega} \omega \right) e_i \right) &:= \sum_{i=1}^{u} \sum_{\omega \in \mathcal{B}} a_{i\omega} \omega * \mathbf{M}(g_i), \\
\mathfrak{S}_L \left(\sum_{i=1}^{u} \left(\sum_{\omega \in \mathcal{B}} a_{i\omega} \omega \right) e_i \right) &:= \sum_{i=1}^{u} \sum_{\omega \in \mathcal{B}} a_{i\omega} \omega \star g_i,
\end{aligned}$$

Lifting Theorem

$$\mathfrak{s}_L\left(\sum_{i=1}^{u}\left(\sum_{\omega\in\mathcal{B}} a_{i\omega}\omega\right)e_i\right) := \sum_{i=1}^{u}\sum_{\omega\in\mathcal{B}} a_{i\omega}\omega * \mathbf{M}(g_i),$$

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A reformulation (and a more efficient procedure) of the classical Buchberger test-completion, which states that "a basis F is Gröbner if and only if each S-polynomial reduces to 0".

Lifting Theorem

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A basis F is Gröbner if and only if each element u in a minimal basis of the module $\ker(\mathfrak{s})$ of the syzygies among the leading monomials $\mathbf{M}(g_i)$ lifts, via Buchberger reduction of $\mathfrak{S}(u)$, to a syzygy $U \in \ker(\mathfrak{S})$ among the g_i.

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$$U = \mathfrak{S}(u) - \sum_{i=1}^{\mu} a_i\lambda_i \star e_i \star b_i\rho_i$$

where $\sum_{i=1}^{\mu} a_i\lambda_i \star g_i \star b_i\rho_i$ is the Gröbner repr. of $\mathfrak{S}(u)$ mod. F.

Lifting Theorem

$$\mathfrak{s}_L\left(\sum_{i=1}^{u}\left(\sum_{\omega\in\mathcal{B}} a_{i\omega}\omega\right)e_i\right) := \sum_{i=1}^{u}\sum_{\omega\in\mathcal{B}} a_{i\omega}\omega * \mathbf{M}(g_i),$$

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A basis F is Gröbner if and only if each element u in a minimal basis of the module $\ker(\mathfrak{s})$ of the syzygies among the leading monomials $\mathbf{M}(g_i)$ lifts, via Buchberger reduction of $\mathfrak{S}(u)$, to a syzygy $U \in \ker(\mathfrak{S})$ among the g_i.

As a corollary you get Janet–Schreier Theorem that the lifted elements form a Gröbner basis of $\ker(\mathfrak{S})$