Formulas for Continued Fractions: an Automated Guess and Prove Approach

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Rational approximation

The Taylor and Padé approximants at order 4 for the logarithm are:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5) = \frac{x + x^2/2}{1 + x + x^2/6} + O(x^5).$$

Convergence for $x \in \mathbb{R}$,	and for $x \in \mathbb{C}$.
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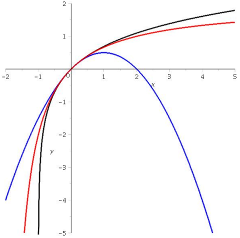	

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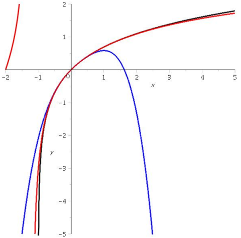	

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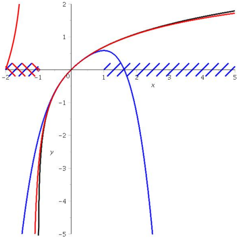	

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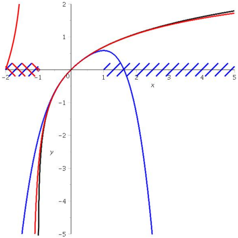	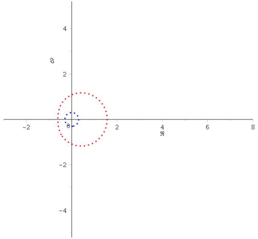

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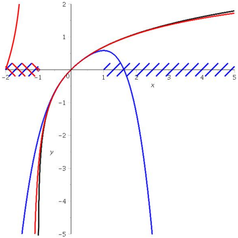	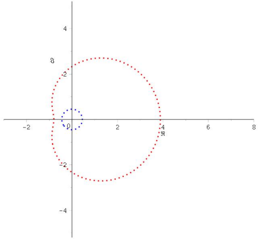

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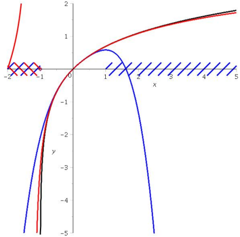	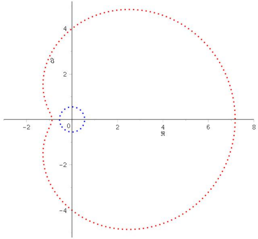

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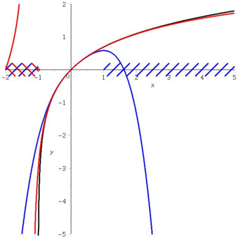	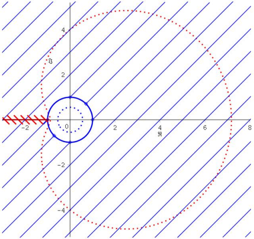

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Explicit formulas

Diagonal Padé approximants for $\ln(1+x)$ are viewed as *corresponding fractions*:

$\frac{x}{1+x/2}$	$\frac{x+x^2/2}{1+x+x^2/6}$	$\frac{x+x^2+11x^3/60}{1+3x/2+3/5x^2+x^3/20}$	$\ldots$
$= \frac{x}{1+x/2}$	$= \cfrac{x}{1+\cfrac{x/2}{1+\cfrac{x/6}{1+x/3}}}$	$\cfrac{x}{1+\cfrac{x/2}{1+\cfrac{x/6}{1+\cfrac{x/3}{1+\cfrac{x/5}{1+3x/10}}}}}$	$\cfrac{a_1(x)}{1+\cfrac{a_2(x)}{1+\cfrac{a_3(x)}{\ddots\,\overline{1+a_{2m}(x)}}}}, \quad \begin{cases} a_1 = x \\ a_{2m} = \frac{m\,x}{4m-2} \\ a_{2m+1} = \frac{m\,x}{4m+2} \end{cases}.$

with a_m monomial in x.

This leads to many formulas:

for $\tan(x)$, for $x\exp(x^2)\operatorname{erf}(x)$, for Bessel ratios $\frac{J_{\nu+1}(x)}{J_\nu(x)}$ ...

$$a_m = \frac{-x^2}{(2m-1)(2m-3)} \qquad \begin{aligned} a_{2m} &= \frac{-2(2m-1)x^2}{(4m-3)(4m-1)} \\ a_{2m+1} &= \frac{4mx^2}{(4m-1)(4m+1)} \end{aligned} \qquad a_m = \frac{-x^2}{4(\nu+m-1)(\nu+m)}.$$

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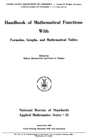
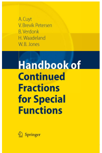
$\rightsquigarrow$ We study *C-fractions* with $(a_{2m})_{m\geq 1}$ and $(a_{2m+1})_{m\geq 0}$ rational in m.

Automating the proofs

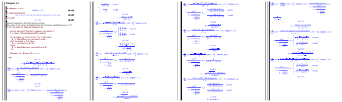
[Abramowitz & Stegun, 1964]
[+the online DLMF]

[Cuyt & others, 2008]





$\rightsquigarrow$



DEMO

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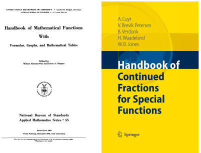

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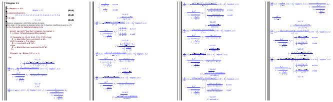

Human proofs for corresponding fractions are:

- clever, e.g. $\exp(x) = \lim_{k\to\infty} \left(1 + \frac{x}{k}\right)^k$,
- hard to automate: generalization/specialization.

They all concern solutions of:

- Riccati equations $y'(x) = py^2 + qy + r$ with rational coefficients,
- and Riccati-like equations (difference and q-difference equations).

These are invariant under the building block: $y \mapsto ax/(1+y)$ for $a \neq 0$.

A generic procedure

$>$ with(gfun:-ContFrac) :
$>$ riccati2cfrac($\{y' = 1 + y^2, y(0) = 0\}, y, x$);

$$y(x) = 0 + \cfrac{x}{1 + \cfrac{-x^2/3}{1 + \cfrac{\cdots}{1 + \cfrac{\frac{-x^2}{(2m-1)(2m-3)}}{\cdots}}}}$$

(formula and proof)

- Input: a Riccati equation $y'(x) = py^2 + qy + r$ with $p, q, r \in \mathbb{C}(x)$,
- Guessing: a finite expansion at a small order gives the conjecture
  $$a_1 = x, \quad a_m = \frac{-x^2}{(2m-1)(2m-3)},\ m > 0.$$
- Proving: do these coefficients lead to $y' = 1 + y^2$?

Todo: prove this.

Miracle: a general recurrence

Elementary property: the *convergents* $f_n := a_1(x)/(1 + \cdots/(1 + a_n(x)))$ satisfy $f_n = P_n/Q_n$ with

$$\begin{aligned} P_n &= P_{n-1} + a_n P_{n-1}, & (P_{-1}, P_0) &= (1, 0), \\ Q_n &= Q_{n-1} + a_n Q_{n-2}, & (Q_{-1}, Q_0) &= (0, 1). \\ \implies P'_n &= P'_{n-1} + a_n P'_{n-2} + a'_n P_{n-2}, & &\text{and same for } Q. \end{aligned}$$

Hence ${f_n}' - 1 - f_n^2 = H_n/Q_n^2$ with $H_n := P'_n Q_n - P_n Q'_n - Q_n^2 - P_n^2$, and $Q_n(0) = 1$.

Lemma: $H_n = O(x^{2n}) \implies \lim_{n\to\infty} f_n = \tan(x)$.

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Hence ${f_n}' - 1 - f_n^2 = H_n/Q_n^2$ with $H_n := P_n' Q_n - P_n Q_n' - Q_n^2 - P_n^2$, and $Q_n(0) = 1$.

Rewriting H_n, H_{n+1}, H_{n+2}, etc. leads to linear combinations of 8 generators with coefficients in $\mathbb{Q}(x, a_{n+2}, a_{n+3}, a_{n+4}, \ldots)$:

$$\begin{aligned} H_n &= P_n' Q_n - Q_n^2 - P_n^2 - P_n Q_n',\\ H_{n+1} &= P_{n+1}' Q_{n+1} - Q_{n+1}^2 - P_{n+1}^2 - P_{n+1} Q_{n+1}',\\ H_{n+2} &= -a_{n+2}^2 P_n Q_n + \cdots P_{n+1} Q_{n+1} + \cdots P_n Q_{n+1}' + \cdots,\\ H_{n+3} &= \cdots. \end{aligned}$$

Using linear algebra leads to a linear recurrence.

Miracle: a general recurrence

- A general recurrence is computed:

$$\frac{1}{a'_{n+1}}\,H_{n+1}+\left(\frac{a_n}{a'_n}-\frac{a_{n+1}+1}{a'_{n+1}}\right)\,H_n-\left(\frac{a_n(a_n+1)}{a'_n}+\frac{a_{n+1}(a_{n+1}+1)}{a'_{n+1}}\right)\,H_{n-1}$$
$$-\left(\frac{a_n+1}{a'_n}-\frac{a_{n+1}}{a'_{n+1}}\right)a_n^2\,H_{n-2}+\frac{a_{n-1}^2a_n^2}{a'_n}\,H_{n-3}=0.$$

- It is independent of the Riccati equation itself!

Miracle: a general recurrence

- A general recurrence is computed: (rational fractions of n are omitted)

$$H_{n+1} = (\cdots x^0 + \cdots x^2)\, H_n + (\cdots x^2 + \cdots x^4)\, H_{n-1} + (\cdots x^4 + \cdots x^6)\, H_{n-2} + (\cdots x^8)\, H_{n-3}.$$

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$\not\Rightarrow \; H_n = O(x^{2n}).$

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$\not\Longrightarrow \; H_n = O(x^{2n})$.

- It is independent of the Riccati equation itself!
- It is linear, with rational coefficients in n.

D-finiteness

A generic description of

- power series $y(x)$, as solutions of a LDE with polynomial coefficients in x,
- $(u_n)_{n\geq 0}$, as solution of a linear recurrence, with polynomial coefficients in n.

  ex: $\binom{n}{2}$ can be seen as $\{u_0 = 0, u_1 = 0, u_2 = 1, (n-1)\, u_{n+1} = (n+1)\, u_n\}$.

Algorithmic tools, implemented e.g. in the maple package *gfun*:

- decision of $(a_n)_{n\geq 0} = 0$,
- computation of recurrences for:

  $\left(a_{\lfloor \alpha n+\beta \rfloor}\right)_{n\geq 0}, \left(a_n + b_n\right)_{n\geq 0}, \left(a_n b_n\right)_{n\geq 0}, \left(\sum_{i+j=n} a_i b_j\right)_{n\geq 0} \ldots$
- $+$ reduction of order (new)
- $+$ continued fractions (new)

Reduction of order by *guess and prove*

Problem (2^{-n} and 2^n in the same recurrence)

Show that $\lim_{n\to\infty} a_n = 0$ *with* $\{2a_{n+2} - 5a_{n+1} + 2a_n = 0, (a_0, a_1) = (1, \frac{1}{2})\}$.

```
> with(gfun) : gfun[version](); 3.70
> reducerecorder({2a(n + 2) − 5a(n + 1) + 2a(n) = 0, a(0) = 1, a(1) = 1/2}, a, n); {2a(n + 1) − a(n) = 0, a(0) = 1}
```

- We guess a "small" recurrence on the first values: $\alpha a_{n+1} + \beta a_n = 0$? $(\alpha, \beta \in \mathbb{C})$ $\{2a_{n+1} - a_n = 0, a_0 = 1\}$.
- It defines a *new* sequence $(b_n)_{n\geq 0}$,
- and $b_n = a_n$ is proved by induction on n:
  - $(b_0, b_1) = (a_0, a_1)$;
  - $2b_{n+2} - 5b_{n+1} + 2b_n = (2b_{n+2} - b_{n+1}) - 2(2b_{n+1} - b_n) = 0$.

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  - $\rightsquigarrow$ Euclidean division of recurrence operators (i.e. *Ore polynomials*)

Miracle (2): hypergeometric remainders

Aim: show that $H_n = O(x^{2n})$ with the recurrence $H_1 = -x^2, H_2 = -x^4 \frac{1}{(1-x^2/3)^2}$,
$\cdots H_{n+4} = (\cdots + \cdots x^2)H_{n+3} + (\cdots x^2 + \cdots x^4)H_{n+2} + (\cdots x^4 + \cdots x^6)H_{n+1} + \cdots x^8 H_n\}$.

(polynomials in n are omitted)

Reduction of order concludes all cases:

for $\tan(x)$, for $x\exp(x^2)\,\mathrm{erf}(x)$, for Bessel ratios $\frac{J_{\nu+1}(x)}{J_\nu(x)}$ ...

$\frac{H_{n+1}}{H_n} = \frac{-x^2}{(2n+1)^2}$ $\frac{H_{2n+1}}{H_{2n-1}} = \frac{-n(2n+1)x^4}{(n+1/4)^2(n-1/4)^2}$ $\frac{H_{n+1}}{H_n} = \frac{4x^2}{(\nu+n+2)^2}$

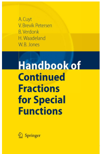
Empirical Fact

All explicit C-fractions formulas for Riccati solutions in the literature have a hypergeometric (two by two) remainder:

$$H_{2n+1}/H_{2n-1} \text{ is monomial in } x, \text{ and rational in } n.$$

This concludes all the proofs, automatically.

Conclusion



Results:

- *One* procedure covers all Riccati solutions we know about,
- it generalizes (by hand) to difference, and q-difference equations.
  $\Longrightarrow$ All explicit C-fractions in [Cuyt et al., 2008] in a Maple worksheet!

(DEMO)

Perspectives:

- Is Khovanskii's formula (1963) the most general, in the Riccati setting?
- (new) Generalize this formula to the other settings.