Separable Automorphisms on Matrix Algebras over Finite Field Extensions. Applications to Ideal Codes.[1]

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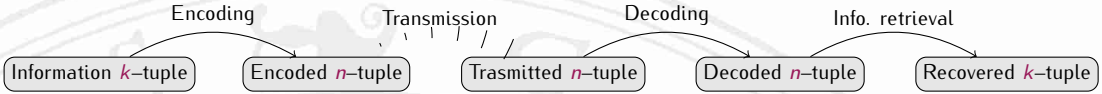
An extremely short overview of coding theory



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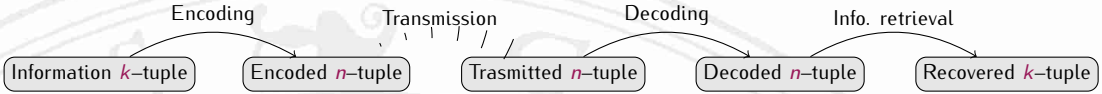

Block (linear) codes

The encoding process is provided by a linear map, i.e. there is a right invertible matrix $G \in \mathcal{M}_{k\times n}(\mathbb{F})$ such that the encoding is $v_t = u_t G \in \mathbb{F}^n$ for each information word $u_t \in \mathbb{F}^k$.

Additional algebraic structure improves encoding and decoding, $\mathbb{F}^n \cong \mathbb{F}[x]/\langle x^n - 1\rangle$, e.g. Reed-Solomon codes.

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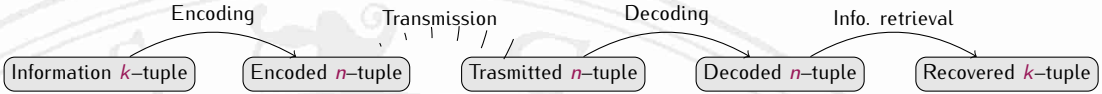

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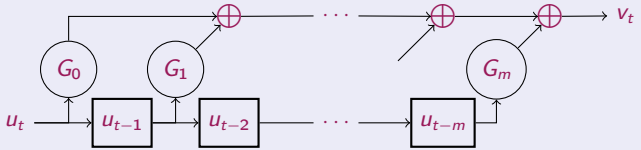
Convolutional codes



Convolutional codes and cyclicity

Cyclicity in convolutional codes $\rightsquigarrow$ non commutative structures on $\mathbb{F}[z]^n$.
[Piret'76, Gluesing and Schmale'04, Lopez-Permouth and Szabo'13, GLN'14b]

For each ring A, the Ore extension $A[z;\sigma,\delta]$ is the free right A–module with basis the powers of z and multiplication defined by the rule $az = z\sigma(a) + \delta(a)$ for all $a \in R$, where σ is a ring endomorphism of A, and δ a σ–derivation.

Let A be a finite semisimple algebra of dimension n over a finite field $\mathbb{F}$. Each $\mathbb{F}$–basis $\mathcal{B}$ of A induces a natural isomorphism of $\mathbb{F}[z]$–modules $\mathfrak{v} : A[z;\sigma,\delta] \to \mathbb{F}[z]^n$. A is the word–ambient of the convolutional code, while $A[z;\sigma,\delta]$ is the sentence–ambient.

Definition 1 ([Lopez-Permouth and Szabo'13])

An ideal code is a left ideal $I \leq A[z;\sigma,\delta]$ such that $\mathfrak{v}(I)$ is a direct summand of $\mathbb{F}[z]^n$.

We focus ourselves in the case $\delta = 0$, i.e. skew polynomials.

Two questions about ideal codes

Are ideal codes direct summands as left ideals?

- A positive answer is shown in [GLN'14b, GLN ISSAC'14 Poster] if $\mathbb{F}[z] \subseteq A[z;\sigma]$ is a separable ring extension.
- This answer generalizes previous works [Gluesing and Schmale'04] when the word–ambient is a commutative semisimple $\mathbb{F}$–algebra, and [Lopez-Permouth and Szabo'13] whenever it is a separable group algebra of a finite group over a $\mathbb{F}$.

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Can we compute a generator for an ideal code?

- In general it is not known if ideal codes are even principal.
- If the ideal code is a direct summand as left ideal, which we call a *split ideal code*, then it is principal and generated by an idempotent.
- This generator can be effectively computed under the separability conditions, see [GLN'14b, Algorithm 1] and [GLN ISSAC'14 Poster, Algorithm 1]

Separable extensions

Definition 2

A non commutative ring extension $S \subseteq R$ is called separable if the multiplication map

$$\begin{aligned} \mu : R \otimes_S R &\longrightarrow R \\ \textstyle\sum_i a_i \otimes b_i &\longmapsto \textstyle\sum_i a_i b_i \end{aligned}$$

splits, or equivalently if there exists

$$p = \sum_i a_i \otimes b_i \in R \otimes_S R$$

such that

$$\mu(p) = \sum_i a_i b_i = 1 \text{ and } \forall r \in R,\ rp = pr.$$

This element is called a *separability element*.

Proposition 3 ([Hirata and Sugano'66])

In a separable extension, R–submodules which are S–direct summands are also R–direct summands.

Separability and skew polynomials

- A is a separable $\mathbb{F}$–algebra and $\sigma \in \mathrm{Aut}_{\mathbb{F}}(A)$,
- $\sigma^{\otimes} : A \otimes_{\mathbb{F}} A \to A \otimes_{\mathbb{F}} A$ is defined by $\sigma^{\otimes}(a \otimes b) = \sigma(a) \otimes \sigma(b)$.

Corollary 4

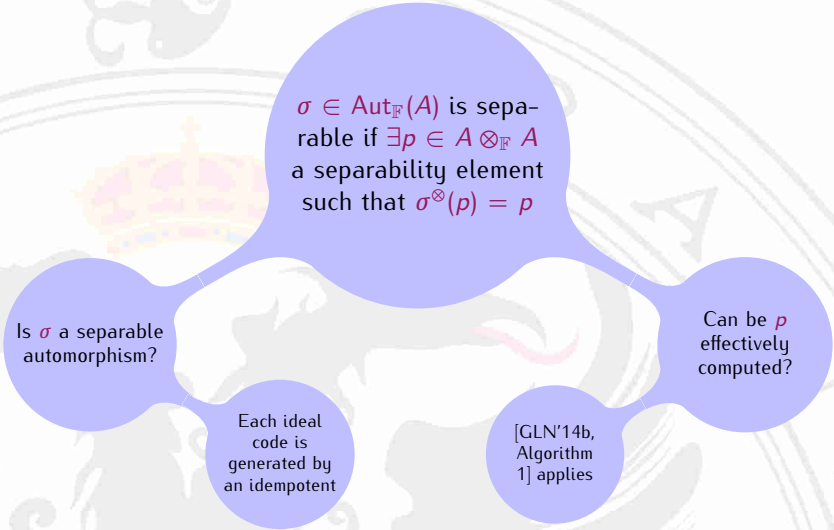
Let $\mathbb{F}$ be a finite field and let A be a separable $\mathbb{F}$–algebra with separability element $p = \sum_i a_i \otimes_{\mathbb{F}} b_i \in A \otimes_{\mathbb{F}} A$. Let $\sigma \in \mathrm{Aut}_{\mathbb{F}}(A)$ such that $\sigma^{\otimes}(p) = p$. Then $\mathbb{F}[z] \subseteq A[z;\sigma]$ is a separable extension and a separability element is given by

$$\overline{p} = \sum_i a_i \otimes_{\mathbb{F}[z]} b_i \in A[z;\sigma] \otimes_{\mathbb{F}[z]} A[z;\sigma].$$

In particular each ideal code is a split ideal code and it is generated by an idempotent.

This corollary follows from [GLN'14b, Theorem 6].

Computational problems



Framework and target

- $\mathbb{F} \subseteq \mathbb{K}$ is a finite fields extension with dual normal bases $\{\alpha^{q^0}, \ldots, \alpha^{q^{t-1}}\}$ and $\{\beta^{q^0}, \ldots, \beta^{q^{t-1}}\}$.
- Word–ambient is $A = \mathcal{M}_n(\mathbb{K})$.
- The center of an A–bimodule M is $M^A = \{m \in M \mid rm = mr \ \forall r \in A\}$.

The desired $p \in A \otimes_{\mathbb{F}} A$ must satisfy

(P1) $p \in (A \otimes_{\mathbb{F}} A)^A$

(P2) $p \in E_1 = \left\{ \sum_i a_i \otimes b_i \in (A \otimes_{\mathbb{F}} A)^A \mid \sum_i a_i b_1 = 1 \right\}$

(P3) $p \in \ker(Id_{A \otimes_{\mathbb{F}} A} - \sigma^{\otimes})$

(P1) $(A \otimes_{\mathbb{F}} A)^A$

For each $0 \leq i,j \leq n-1$ and all $0 \leq k \leq t-1$, we denote

$$p_{ijk} = \sum_{l=0}^{n-1} \sum_{h=0}^{t-1} E_{li} \alpha^{q^k} \alpha^{q^h} \otimes \beta^{q^h} E_{jl} \in A \otimes_{\mathbb{F}} A.$$

Lemma 5

The dimension of $(A \otimes_{\mathbb{F}} A)^A$ as an $\mathbb{F}$–vector space is $n^2 t$. An $\mathbb{F}$–basis for $(A \otimes_{\mathbb{F}} A)^A$ is $\{p_{ijk} \mid 0 \leq i,j \leq n-1, 0 \leq k \leq t-1\}$.

(P2) $E_1 = \left\{ \sum_i a_i \otimes b_i \in (A \otimes_{\mathbb{F}} A)^A \mid \sum_i a_i b_1 = 1 \right\}$

Let

$$E_0 = \{p \in (A \otimes_{\mathbb{F}} A)^A \mid \mu(p) = 0\}$$

and

$$E_1 = \{p \in (A \otimes_{\mathbb{F}} A)^A \mid \mu(p) = 1\}.$$

Then E_1 is the set of all separability elements of the extension $\mathbb{F} \subseteq A$. Let $p_1 = \sum_{k=0}^{t-1} \mathrm{Tr}_{\mathbb{K}/\mathbb{F}}(\beta) p_{00k}$.

Proposition 6

E_0 is an $\mathbb{F}$–vector subspace of $(A \otimes_{\mathbb{F}} A)^A$ and E_1 is an affine subspace of $(A \otimes_{\mathbb{F}} A)^A$ both of dimension $(n^2-1)t$. An $\mathbb{F}$–basis of E_0 is

$$\mathcal{E} = \{p_{ijk} \mid 0 \leq i \neq j \leq n-1, 0 \leq k \leq t-1\} \cup \{p_{00k} - p_{iik} \mid 1 \leq i \leq n-1, 0 \leq k \leq t-1\}.$$

Moreover $E_1 = \{p_1 + q \mid q \in E_0\}$.

(P3) $\ker(Id_{A\otimes_{\mathbb{F}} A} - \sigma^{\otimes})$ I

Recall

$$E_0 = \{p \in (A \otimes_{\mathbb{F}} A)^A \mid \mu(p) = 0\}$$

and

$$E_1 = \{p \in (A \otimes_{\mathbb{F}} A)^A \mid \mu(p) = 1\}.$$

Linearizing the problem

$$\exists p \in E_1 \cap \ker(Id_{A\otimes_{\mathbb{F}} A} - \sigma^{\otimes})$$

$$\Updownarrow$$

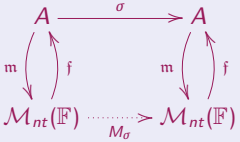
$$\exists q \in E_0 \mid \sigma^{\otimes}(p_1 + q) = p_1 + q$$

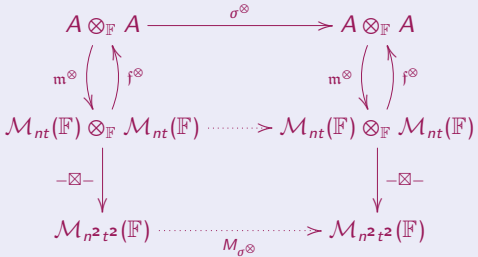
$$\Updownarrow$$

$$(\sigma^{\otimes} - Id_{A\otimes_{\mathbb{F}} A})(p_1) \in (Id_{A\otimes_{\mathbb{F}} A} - \sigma^{\otimes})(E_0)$$

(P3) $\ker(Id_{A\otimes_{\mathbb{F}} A} - \sigma^{\otimes})$ II

Converting to matrices





Example I

- The field extension is $\mathbb{F} = \mathbb{F}_2 \subset \mathbb{F}_4 = \mathbb{K}$.
- The base algebra is $A = \mathcal{M}_2(\mathbb{K})$.
- We fix the normal basis $\mathcal{B} = \{a, a^2\}$, which is also self-dual.
- Two canonical inclusions, $A \to \mathcal{M}_4(\mathbb{F})$ and $A \otimes_{\mathbb{F}} A \to \mathcal{M}_{16}(\mathbb{F})$
- The basis $\{p_{ijk}\}$ of $A \otimes_{\mathbb{F}} A^A$ can now be constructed. For example

$$p_{001} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix},$$

and via the canonical inclusion

$$p_{001} = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Example II

- The basis of E_0 is

$$\mathcal{E} = \{p_{010}, p_{011}, p_{100}, p_{101}, p_{000} + p_{110}, p_{001} + p_{111}\}$$

and $E_1 = p_1 + E_0$ where

$$p_1 = p_{000} + p_{001} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

- The automorphism is $\sigma = \sigma_U \widehat{\tau}$, where $U = \begin{pmatrix} 1 & a \\ a^2 & a \end{pmatrix}$ and τ is the Frobenius automorphism.

Example III

- The matrix M_σ is

$$M_\sigma = \begin{pmatrix}
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1
\end{pmatrix},$$

  and the matrix $M_{\sigma^\otimes}$ has size 256×256.

- In order to check if $(\sigma^\otimes - \mathrm{id})(p_1) \in (\mathrm{id} - \sigma^\otimes)(E_0)$, we had to solve the non homogeneous linear system of size 256×6

$$\mathfrak{v}(p_1) \cdot (M_{\sigma^\otimes} - I_{256}) = \sum_{0 \le i \ne j \le 1} \sum_{0 \le k \le 1} \alpha_{ijk} \left(\mathfrak{v}(p_{ijk}) \cdot (I_{256} - M_{\sigma^\otimes}) \right) + \sum_{0 \le i \le 1} \sum_{0 \le k \le 1} \alpha_{ik} \left(\mathfrak{v}(p_{00k} - p_{iik}) \cdot (I_{256} - M_{\sigma^\otimes}) \right)$$

  whose solution is $(0, 0, 1, 0, 1, 0)$.

Example IV

- The desired separability element is $p = p_1 + p_{100} + p_{000} + p_{110} = p_{100} + p_{110} + p_{001}$. Concretely,

$$p = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
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0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0
\end{pmatrix},$$

Viewed as a tensor product of matrices in $\mathcal{M}_2(\mathbb{K})$,

$$\begin{aligned}
p = &\begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a^2 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a^2 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & a^2 \end{pmatrix} \\
&+ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Summary, conclusions, further work...

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- This paper concerns about convolutional codes with word–ambient matrices over a finite field extension of the symbol–ambient finite field.
- We have presented a complete computational answer to decide if a given automorphism $\sigma \in \mathrm{Aut}_{\mathbb{F}}(A)$ is separable, computing a suitable separability element.
- If the answer in positive, the separability element can be used to compute an idempotent generator of any ideal code.
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Improvements not explained here and further work.

- We have also proved that $\mathbb{F}[z] \subseteq A[z;\sigma]$ is a separable extension if and only if σ is a separable automorphism.
- From the idempotent generator, the dual code can be easily computed if it is also an ideal code over the same word–ambient. This has been also completed.
- Distance profiles of these codes have to be computed.

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