Minkowski Decomposition and Geometric Predicates in Sparse Implicitization

I.Z. Emiris, C. Konaxis and Z. Zafeirakopoulos

University of Athens

Talk given by Ilias Kotsireas

Wilfrid Laurier University

ISSAC'15, Bath, July 2015

Outline

1. Implicitization by interpolation.
2. Minkowski decomposition of the predicted polytope.
3. Geometric predicates as matrix operations.

Implicitization by interpolation

Implicitization

Given parameterization

$$x_0 = \alpha_0(t), \ldots, x_n = \alpha_n(t), \qquad t := (t_1, \ldots, t_n),$$

compute the smallest algebraic variety containing the closure of the image of

$$\alpha : \mathbb{R}^n \to \mathbb{R}^{n+1} : t \mapsto \alpha(t), \qquad \alpha := (\alpha_0, \ldots, \alpha_n).$$

This is contained in the variety defined by the ideal

$$\langle p(x_0, \ldots, x_n) \mid p(\alpha_0(t), \ldots, \alpha_n(t)) = 0, \ \forall t \rangle.$$

When this is a *principal* ideal we wish to compute its defining polynomial $p(x)$, given its *Newton polytope* $N(p(x))$ (*implicit polytope*), or, a superset of the exponents of its monomials with nonzero coefficient (*implicit support*).

Approach

Support prediction followed by interpolation.

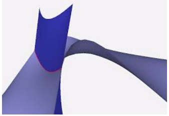
- Superset S of the implicit support: support of specialized sparse resultant.
- Construct $\mu \times |S|$ ($\mu \geq |S|$) matrix M: columns indexed by monomials with exponents in S, rows indexed by values of t at which the monomials are evaluated.
- The (ideally) unique kernel vector of M contains the coefficients of these monomials in the implicit equation.

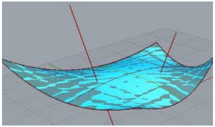
Example (Folium of Descartes)

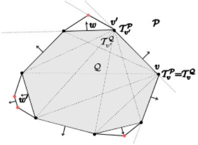
$$x_0 = \frac{3t^2}{t^3+1},\ x_1 = \frac{3t}{t^3+1}, \qquad S = \{(0,3),(3,0),(1,1)\} \quad \Rightarrow$$

$$M = \begin{bmatrix} x_1^3(\tau_1) & x_0^3(\tau_1) & x_0(\tau_1)x_1(\tau_1) \\ x_1^3(\tau_2) & x_0^3(\tau_2) & x_0(\tau_2)x_1(\tau_2) \\ x_1^3(\tau_3) & x_0^3(\tau_3) & x_0(\tau_3)x_1(\tau_3) \\ x_1^3(\tau_4) & x_0^3(\tau_4) & x_0(\tau_4)x_1(\tau_4) \end{bmatrix} \Rightarrow \ p(x_0,x_1) = x_0^3 - 3x_0x_1 + x_1^3.$$

Previous work







16 variables

- Integration of matrix SS^T over each parameter $t_1, \ldots, t_n$. Successively larger supports to capture sparseness [Corless-Giesbrecht-Kotsireas-Watt '00].
- Successively larger supports also used in [Dokken-Thomassen '03] in the setting of approximate implicitization.
- Tropical geometry [Sturmfels-Tevelev-Yu '07] leads to algorithms for the polytope of (specialized) resultants [Jensen-Yu '12].
- Our method developed in [Emiris-Kalinka-Konaxis-Luu Ba '13a, '13b].

Implicitization reduced to elimination

Setup (for omitted details see paper)

Given $x_i = \frac{\alpha_i(t)}{\beta_i(t)},\ i = 0, \ldots, n$, define polynomials in $(\mathbb{R}[x_0, \ldots, x_n])[t]$:

$$f_i := x_i \beta_i(t) - \alpha_i(t),\ i = 0, \ldots, n,$$

with supports $A_i \subset \mathbb{Z}^n$. If the parameterization is generically 1-1, then

$$p(x_0, \ldots, x_n) = Res(f_0, \ldots, f_n),\ \text{provided that}\ Res(f_0, \ldots, f_n) \not\equiv 0.$$

Projection

Let $F := \{F_0, \ldots, F_n\} \in \mathbb{R}(c_{ij})[t]$ be generic polynomials wrt A_i and let ϕ be the specialization of their symbolic coefficients c_{ij} to those of the $f_i's$:

$$\phi:\ c_{ij} \mapsto a_{ij} + b_{ij} x_i.$$

The ϕ-projection of the Newton polytope $N(Res(F))$ of the sparse resultant of polynomials in F, contains (a translate of) the Newton polytope of the implicit polynomial, or *implicit polytope*.

Example

Given parameterization (Buchberger'88):

$$x_0 = t_1 t_2, \quad x_1 = t_1 t_2^2, \quad x_2 = t_1^2,$$

we define:

$$\begin{array}{lll} f_0 := x_0 - t_1 t_2, & f_1 := x_1 - t_1 t_2^2, & f_2 := x_2 - t_1^2, \\ F_0 := c_{00} - c_{01} t_1 t_2, & F_1 := c_{10} - c_{11} t_1 t_2^2, & F_2 := c_{20} - c_{21} t_1^2 \end{array}$$

and $\phi(c_{00}, c_{01}, c_{10}, c_{11}, c_{20}, c_{21}) = (x_0, -1, x_1, -1, x_3, -1)$.

$$\mathrm{Res}(F) = -c_{00}^4 c_{11}^2 c_{21} + c_{01}^4 c_{10}^2 c_{20}, \quad N(\mathrm{Res}(F)) = ((4,0,0,2,0,1),(0,4,2,0,1,0))$$

and projection by ϕ yields implicit polytope $((4,0,0),(0,2,1))$.
We construct M by evaluating (t_1, t_2) at random $\tau_1, \tau_2, \tau_3 \in \mathbb{C}^2$:

$$M = \begin{bmatrix} x_0^4(\tau_1) & x_1^2(\tau_1)x_2(\tau_1) \\ x_0^4(\tau_2) & x_1^2(\tau_2)x_2(\tau_2) \\ x_0^4(\tau_3) & x_1^2(\tau_3)x_2(\tau_3) \end{bmatrix}.$$

Kernel vector $(-1, 1)$ yields implicit equation: $-x_0^4 + x_1^2 x_2$.

Implicit support

- Input: $n+1$ supports $A_i \subset \mathbb{Z}^n$, and projection ϕ.
  Output (predicted implicit polytope): the ϕ-projection of the Newton polytope of the sparse resultant of $A_0, \ldots, A_n$.
- `ResPol` [Emiris-Fisikopoulos-Konaxis-Peñaranda'12]
  computes precise polytope of bicubic surface in 1sec, with 715 terms.

- The lattice points in the implicit polytope correspond to the exponent vectors of the implicit support.

Geometry of predicted support & higher dimensional kernel

Assumption

The interpolation matrix M is built with sufficiently generic evaluation points.

Theorem (Emiris-Kalinka-Konaxis-Luu Ba '13)

For projection $\phi : F_i \mapsto f_i$ s.t. not all leading coefficients of f_i vanish,

$$\phi(Res(F)) = c(x) \cdot p(x).$$

If $P = N(p)$ is the implicit and $Q = \phi(N(Res(F)))$ the predicted polytopes, then

$$Q \supseteq E + P, \qquad \text{for some polytope } E. \ (+ : \text{Minkowski addition}).$$

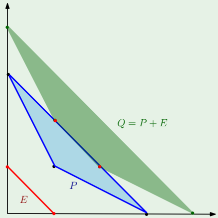
In particular, the kernel of M has dimension equal to the # lattice points in E.

Corollary

If $v_1, \ldots, v_k$ is a basis of the kernel of M, and $g_1, \ldots, g_k$ the corresponding polynomials, i.e. $g_i = v_i^T S$, then gcd$(g_1, \ldots, g_k) = p(x)$; can also use factoring

Geometry of predicted support & higher dimensional kernel

Example (Folium of Descartes: $x_0 = 3t^2/t^3 + 1, \quad x_1 = 3t/t^3 + 1$)



$Q = P + E$

Polynomials from kernel vectors

$$g_1 = x_0(x_0^3 - 3x_0x_1 + x_1^3)$$
$$g_2 = x_1(x_0^3 - 3x_0x_1 + x_1^3)$$

To extract the implicit polytope we employ Minkowski decomposition of Q.

Minkowski decomposition of the predicted polytope

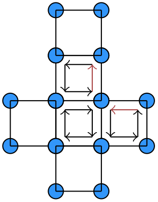
Minkowski Decomposition of the Predicted polytope: Representation

Input

A list of polygons: Facets of the 3-dimensional predicted polytope.

Primitive edges

- $V = [v_1, v_2, \ldots, v_m]$ ordered list of vertices.
- For every edge (v_{i_1}, v_{i_2}), define vector $v_{i_2} - v_{i_1}$.
- Let ℓ_i be its integer length and e_i the corresponding primitive vector.



A cube flattened. Orientation shown in some of its facets. Common edges can have different orientation (red) in each facet.

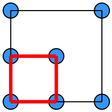
The Integer Linear Program (ILP)

Conditions for Decomposition

Given a facet F, it satisfies

$$\sum_{i \text{ s.t. } e_i \in F} \sigma_{i,F} \ell_i e_i = 0,$$

where $\sigma_{i,F}$ is the sign of e_i and depends on the orientation of the edge $\ell_i e_i$ in the facet F.



ILP

$$a_i \in \mathbb{N}, \quad 0 \leq a_i \leq \ell_i,$$

$$\sum_{i \text{ s.t. } e_i \in F} \sigma_{i,F} a_i e_i = 0, \qquad \text{for every facet } F.$$

Solutions that give summands homothetic to the input are excluded by adding two more appropriate equations.

A set of sub-edges $a_i e_i$ (red) of the square forming a polygon.

Recursion and Balance

Balancing

We can choose the objective function of the ILP in order to decompose to polytopes with specific characteristics.
Due to the nature of the motivating problem, we prefer the decomposition to two summands of almost equal edge length sum, i.e., "balanced".
This is achieved by setting as the objective function the sum of the edge lengths.

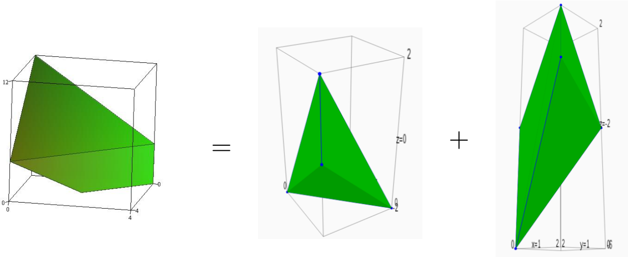
Recursion

We recurse over the obtained two summand polytopes until we reach an indecomposable polytope.

Example (Eight-surface)

$$x_0 = \frac{4s(-1+t_2^2)(-1+t_1^2)}{(1+t_2^2)(1+t_1^2)^2}, \quad x_1 = \frac{-8t_1t_2(-1+t_1^2)}{(1+t_2^2)(1+t_1^2)^2}, \quad x_2 = \frac{2t_1}{(1+t_1^2)}$$

Predicted polytope leads to a 67×67 interpolation matrix. Minkowksi decomposition gives true implicit polytope (2nd summand) and a 10×10 matrix.



Geometric predicates as matrix operations

Geometric predicates using interpolation matrices

Setup

- S: superset of implicit support,
  $m(x)$: vector of monomials (in the variables x_i) with exponents in S.
- Fix *generic* distinct values $\tau_k,\ k = 1, \ldots, |S| - 1$.
- Construct $(|S| - 1) \times |S|$ *numeric* matrix M' whose kth row, $k = 1, \ldots, |S| - 1$, is vector $m(x)$ evaluated at τ_k.
- Construct

$$M(x) = \begin{bmatrix} M' \\ m(x) \end{bmatrix}.$$

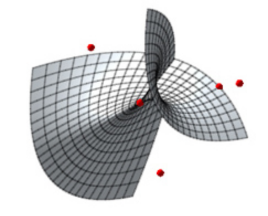
Lemma

Assuming M' is of full rank (equiv., the predicted polytope Q contains only one translate of P), the determinant of matrix $M(x)$ equals the implicit polynomial $p(x)$ up to a constant.

Membership predicate

Membership

Given $x_i = f_i(t)/g_i(t),\ i = 0, \ldots, n$, and query point $q \in \mathbb{R}^{n+1}$, decide if $p(q) = 0$, where $p(x)$ is the implicit polynomial, by using the interpolation matrix.



Lemma

Given $M(x)$ and a query point q in $(\mathbb{R}^*)^{n+1}$, let $M(q)$ denote matrix $M(x)$ where its last row is evaluated by q. Then q lies on the hypersurface defined by $p(x) = 0$ if and only if

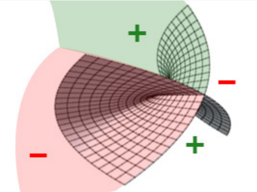
$$corank(M(q)) = corank(M').$$

Numeric matrix M' need not be of full rank.

Sidedness predicate

Given a hypersurface in $\mathbb{R}^{n+1}$ defined by $p(x)$, and point $q \in \mathbb{R}^{n+1}$ such that $p(q) \neq 0$, we define

$$\text{side}(q) = \text{sign}(p(q)) \in \{-1, 1\}.$$



Lemma

Assuming Q contains only one translate of P, $M(x)$ is the interpolation matrix and $q \in (\mathbb{R}^*)^{n+1}$ such that $p(q) \neq 0$, then $\det M(q) \neq 0$.

Lemma

Let q_1, q_2 be two query points in $(\mathbb{R}^*)^{n+1}$ not lying on the hypersurface $p(x) = 0$. Assuming Q contains only one translate of P, then

$$side(q_1) = side(q_2) \;\; iff \;\; sign(\det M(q_1)) = sign(\det M(q_2)).$$

Implementation

- Minkowski decomposition implemented in `Sage`.
- Sparse implicitization and geometric predicates implemented in `Maple`.
- Exact and approximate computations available.
- Option to use normal vectors to the curve/surface to build the interpolation matrix (Hermite interpolation).
- Code available at

  http://ergawiki.di.uoa.gr/index.php/Implicitization

Experimental comparison in Maple

Surface	Deg.	*M*size	SI	GB	DR	Memb.	Side
Handkerchief surface	3	10	0.008	0.012	0.004	0.008	0.004
Moebius strip	3	398	16.784	0.268	0.204	15.012	83.0*
Bohemian dome	4	55	0.108	0.092	0.060	0.092	0.260
Eight surface	4	67	0.224	0.372	0.540	1.036	0.620
Swallowtail surface	5	25	0.032	0.032	0.008	0.048	0.044
Sine surface	6	125	0.556	0.160	0.048	0.532	7.916
Enneper's surface	9	103	0.268	0.204	0.032	0.476	3.468
Bicubic	18	715	16.804	$>$3000	8.028	68.176	1676.8*
HyperSurface	Deg.	*M*size	SI	GB	DR	Memb.	Side
Bourgain hypersurface	3	6	0.004	0.0016	0.004	0.004	0.008
Hypercone	2	165	89.86	0.028	0.392	0.892	10.932*
Hypesurface	12	169	18.464	0.792	1.164	1.620	0.280

- Exact computations (Intel i5-2500, 3.30 GHz, 8GB Linux machine): runtimes (s) of: our method (SI), Gröbner bases (`EliminationIdeal()`) (GB), and Dixon resultant (Minimair) (DR).
- Last 2 columns show the runtime to decide Membership and Sidedness using previous Lemmas.
- Nullspace and determinants use Maple's native routines except *: modular det computation.

- Experiments show our method is competitive and less dependent on the degree and the dimension of the (sparse) input. Matrices often ill-conditioned.

Future work

- Numerical computation: stability problems, non-rational coefficients, accuracy (Hausdorff distance).
- Ray shoot? More operations?
- Use Bernstein basis (no conversion to monomial basis).
- Exploit the generalized Vandermonde structure of M: implies $O^*((\dim M)^2)$. Relies on multivariate polynomial interpolation at arbitrary evaluation points.
- Higher codimension: space curves?

Future work

- Numerical computation: stability problems, non-rational coefficients, accuracy (Hausdorff distance).
- Ray shoot? More operations?
- Use Bernstein basis (no conversion to monomial basis).
- Exploit the generalized Vandermonde structure of M: implies $O^*((\dim M)^2)$. Relies on multivariate polynomial interpolation at arbitrary evaluation points.
- Higher codimension: space curves?

These and other similar problems are going to be studied in the framework of the Marie Sklodowska-Curie European Training Network ARCADES, 2016 – 2020.

Future work

- Numerical computation: stability problems, non-rational coefficients, accuracy (Hausdorff distance).
- Ray shoot? More operations?
- Use Bernstein basis (no conversion to monomial basis).
- Exploit the generalized Vandermonde structure of M: implies $O^*((\dim M)^2)$. Relies on multivariate polynomial interpolation at arbitrary evaluation points.
- Higher codimension: space curves?

These and other similar problems are going to be studied in the framework of the Marie Sklodowska-Curie European Training Network ARCADES, 2016 – 2020.

Thank you for your attention!