Computing Hypergeometric Solutions of Second Order Linear Differential Equations using Quotients of Formal Solutions

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Outline of the Talk

1. Introduction and Riemann-Hurwitz Type Formula

2. Algorithm Outline and Examples

1. Introduction and Riemann-Hurwitz Type Formula

${}_2F_1$-type Solutions

Let a second order differential equation $L_{inp}(y) = 0$ be given.

Here $\quad L_{inp} \in \mathbb{C}(x)[\partial], \;\; \mathrm{ord}(L_{inp}) = 2, \quad \partial = \frac{d}{dx}.$

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We will give an algorithm to find ${}_2F_1$-type solutions of L_{inp} in the form of

$$\exp\left(\int r\,dx\right) {}_2F_1(a_1, a_2; b_1; f)$$

where $a_1, a_2, b_1 \in \mathbb{Q}$ and $r, f \in \overline{\mathbb{Q}(x)}$.

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We need to find

1. $a_1, a_2, b_1 \in \mathbb{Q}$,
2. *Pullback function* $f \in \overline{\mathbb{Q}(x)}$,
3. $r \in \overline{\mathbb{Q}(x)}$.

Base Operator L_B

If solutions of a second order differential operator L at $x = 0$ are,

$$y_1 = x^0 + \dots,$$
$$y_2 = x^{\frac{1}{3}} + \dots,$$

then we say that *exponent difference* of L at $x = 0$ is $\Delta(L, 0) := |0 - \frac{1}{3}|$.

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Hypergeometric function ${}_2F_1(a_1, a_2; b_1; x)$ is a solution of a Gauss Hypergeometric Differential Operator (GHDO)

$$L_B = x(1-x)\partial^2 + (b_1 - (a_1 + a_2 + 1)x)\partial - a_1 a_2$$

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If we can find $\alpha_0, \alpha_1, \alpha_\infty$, then we will have a_1, a_2, b_1.

Set of Candidates for $\alpha_0, \alpha_1, \alpha_\infty$

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Problem: This set is finite but **too large**.

Differential Operators on Algebraic Curves

L_B and L_{inp} are in $\mathbb{C}(x)[\partial]$, i.e., they are defined on $\mathbb{P}^1$.

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In general:

Let X be an algebraic curve.

Let $\mathbb{C}(X)$ be its function field.

$D_{\mathbb{C}(X)} := \mathbb{C}(X)[\partial_t]$ where $t \in \mathbb{C}(X)$ with $t' \neq 0$.

$L \in D_{\mathbb{C}(X)}$ is a differential operator.

Riemann-Hurwitz Type Formula for Differential Equations

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f corresponds a homomorhism $\mathbb{C}(Y) \to \mathbb{C}(X)$.

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If $L_1 \in D_{\mathbb{C}(Y)}$ with ord(L_1)=2 and

L_2 is the corresponding element in $D_{\mathbb{C}(X)}$, then

$$2 - 2g_X + \Delta^{\Sigma-1}(L_2) = \deg(f)\left(2 - 2g_Y + \Delta^{\Sigma-1}(L_1)\right)$$

where g_X, g_Y are genus of X, Y resp., and

$\Delta^{\Sigma-1}(L) := \sum_p (\Delta(L, p) - 1)$,

$\Delta(L, p) :=$ exponent difference of L at $x = p$.

Riemann-Hurwitz Type Formula for Our Case

In our case $X = Y = \mathbb{P}^1$, so $g_X = g_Y = 0$.

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where L_B has exponent differences $[\alpha_0, \alpha_1, \alpha_\infty]$ at $\{0, 1, \infty\}$.

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Explicitly:

$$2 + \sum_{p \in \mathbb{P}^1} (\Delta(L_{inp}, p) - 1) = \deg(f)\left(2 + \sum_{i \in \{0,1,\infty\}} (\alpha_i - 1)\right)$$

Computing Candidates for $\alpha_0, \alpha_1, \alpha_\infty$

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Formula

$$2 + \sum_{p \in \mathbb{P}^1} (\Delta(L_{inp}, p) - 1) = \deg(f) \left(2 + \sum_{i \in \{0,1,\infty\}} (\alpha_i - 1) \right)$$

eliminates **vast majority** of the candidates.

2. Algorithm Outline and Examples

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A second order differential operator $L_{inp} \in \mathbb{C}(x)[\partial]$.

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or

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Note: If $f \in \overline{\mathbb{Q}(x)} \setminus \mathbb{Q}(x)$, then additional input is required.

Algorithm Outline

STEP 1: Candidate Exponent Differences

Compute the list of candidate exponent differences

$$[\alpha_0, \alpha_1, \alpha_\infty, d]$$

for candidate base operators L_B.

If there is no candidate, then return [ ].

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STEP 2: Quotient Method

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y_1, y_2 of L_B at s, and Y_1, Y_2 of L_{inp} at $f(s)$ up to order a.

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Let $q = \frac{y_1}{y_2}$ and $Q = \frac{Y_1}{Y_2}$. Then

$$q(f) = c \cdot Q(x) \implies f = q^{-1}(c \cdot Q(x))$$

for an unknown constant c.

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STEP 2 (continued):

There are infinitely many choices for c.

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Idea: Choose a suitable prime p and compute

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by looping $c = 1, \ldots, p-1$.

For each c try Rational Function Reconstruction to find $f \in \mathbb{F}_p[x]$.

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If this succeeds, then try p-adic Hensel lifting

and Rational Number Reconstruction to find $f \in \mathbb{Q}(x)$.

Algorithm Outline

STEP 2 (continued):

For algebraic pullback functions:

Compute the minimal polynomial of f mod (x^{a-1}, p).

Hensel lift it, and try Rational Number Reconstruction.

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STEP 3:

Compute the operator M such that

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Return a basis $[\mathrm{Sol}_1, \mathrm{Sol}_2]$ of ${}_2\mathrm{F}_1$-type solutions of L_{inp}.

Examples

```
> L := Linp[99];
```

$$L := 400\,(7\,x^2 - 7\,x + 8)\,(x+7)\,(x+8)\,Dx^2 + (7560\,x^3 + 85120\,x^2 + 197560\,x - 98880)\,Dx + 1323\,x^2 + 11270\,x + 12803$$

```
> s := find_2f1(L);
[x, x, x]
[1/5, 1/5, 1/4, 2]
13, 1
[2/15, 2/15, 1/2, 3]
23, 1
[1/15, 1/5, 1/2, 3]
13, 1
[1/5, 1/2, 1/15, 3]
13, 1
[1/5, 1/2, 1/8, 4]
13, 1
[1/5, 1/5, 1/2, 7]
13, 1
13, 2
13, 3
13, 4
13, 5
13, 6
13, 7
13, 8
13, 9
```

$$s := \left[\frac{\mathrm{hypergeom}\left(\left[\frac{1}{20}, \frac{11}{20}\right], \left[\frac{4}{5}\right], \frac{(x+7)^2\,x^5}{(x+8)\,\left((x+7)^3 - 18\,(x+7)^2 + 101\,x + 547\right)^2}\right)}{\left((x+7)^3 - 18\,(x+7)^2 + 101\,x + 547\right)^{1/10}\,(x+8)^{1/20}}, \right.$$
$$\left. \frac{\left(\frac{(x+7)^2\,x^5}{(x+8)\,\left((x+7)^3 - 18\,(x+7)^2 + 101\,x + 547\right)^2}\right)^{1/5} \mathrm{hypergeom}\left(\left[\frac{1}{4}, \frac{3}{4}\right], \left[\frac{6}{5}\right], \frac{(x+7)^2\,x^5}{(x+8)\,\left((x+7)^3 - 18\,(x+7)^2 + 101\,x + 547\right)^2}\right)}{\left((x+7)^3 - 18\,(x+7)^2 + 101\,x + 547\right)^{1/10}\,(x+8)^{1/20}} \right]$$

```
> seq( simplify( eval( diffop2de(L,y(x)), y(x)=i) ), i=s );
```

$$0, 0$$

```
> find_expdiffs( singDiffs(L) );
[x, x, x]
```

$$\left[\left[\frac{1}{5}, \frac{1}{5}, \frac{1}{4}, 2\right], \left[\frac{2}{15}, \frac{2}{15}, \frac{1}{2}, 3\right], \left[\frac{1}{15}, \frac{1}{5}, \frac{1}{2}, 3\right], \left[\frac{1}{8}, \frac{1}{5}, \frac{1}{2}, 4\right], \left[\frac{1}{5}, \frac{1}{5}, \frac{1}{2}, 7\right], \left[\frac{1}{5}, \frac{1}{4}, \frac{1}{2}, 14\right]\right]$$

```
> L := LLinp[92];
```

$$L := Dx^2 - \frac{1}{16}\,\frac{65028096\,x^8 - 140894208\,x^7 + 18047232\,x^6 + 92111232\,x^5 - 14986448\,x^4 - 15758239\,x^3 - 3896340\,x^2 - 744800\,x - 78400}{(x-1)^2\,(21\,x+4)^2\,x^2\,(144\,x^2 - 84\,x - 35)^2}$$

```
> s := find_2f1(L);
[0, 0, x]
[0, x, x]
[0, 1/3, 1/2, 7]
23, 1
23, 2
23, 3
23, 4
23, 5
23, 6
23, 7
```

$$s := \left[\frac{(x-1)^{1/6}\,\sqrt{x}\,\sqrt{144\,x^2-84\,x-35}\,(21\,x+4)^{1/6}\,\mathrm{hypergeom}\left(\left[\frac{1}{12},\frac{7}{12}\right],\left[\frac{2}{3}\right],1-\frac{x^3\,(144\,x^2-84\,x-35)^2}{(x-1)\,(144\,x^3-12\,x^2-23\,x-4)^2}\right)}{(144\,x^3-12\,x^2-23\,x-4)^{1/6}},\right.$$
$$\left.\frac{1}{(144\,x^3-12\,x^2-23\,x-4)^{1/6}}\left((x-1)^{1/6}\,\sqrt{x}\,\sqrt{144\,x^2-84\,x-35}\,(21\,x+4)^{1/6}\,\mathrm{hypergeom}\left(\left[\frac{5}{12},\frac{11}{12}\right],\left[\frac{4}{3}\right],1-\frac{x^3\,(144\,x^2-84\,x-35)^2}{(x-1)\,(144\,x^3-12\,x^2-23\,x-4)^2}\right)\left(\frac{x^3\,(144\,x^2-84\,x-35)^2}{(x-1)\,(144\,x^3-12\,x^2-23\,x-4)^2}-1\right)^{1/3}\right)\right]$$

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$$0,\ 0$$

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> find_expdiffs( singDiffs(L) );
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[0, x, x]
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$$\left[\left[0, \frac{1}{3}, \frac{1}{2}, 7\right]\right]$$

Thank You