Computer Algebra Applied to Solitary Waves

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Co-authors

Co-authors:

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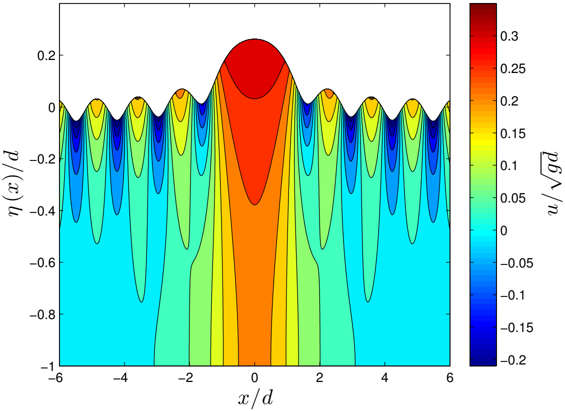
Collaboration

They are specialized in Fluid Mechanics.

Equations

- Surface waves propagation is governed by Euler equations, with nonlinear boundary conditions.
- Simpler sets of equations are derived for specific regimes.
- Here, we consider a **shallow water** of constant depth d, capillary-gravity waves,
- generalizing so called Serre's equations.

Horizontal velocity for Euler equations



Ansatz:

$$\boldsymbol{u}(\boldsymbol{x}, y, t) \approx \bar{u}(\boldsymbol{x}, t),$$

$$v(\boldsymbol{x}, y, t) \approx (y + d)(\eta + d)^{-1} \, \tilde{v}(\boldsymbol{x}, t)$$

Nonlinear Shallow Water Equations:

$$h_t + \boldsymbol{\nabla} \cdot [h\bar{u}] = 0,$$

$$\bar{u}_t + (\bar{u} \cdot \boldsymbol{\nabla})\bar{u} + g\boldsymbol{\nabla} h = 0.$$

1D case: Serre's equations with surface tension

- Governing equations (mass and momentum):

$$h_t + [h\,\bar{u}]_x = 0,$$
$$[h\,\bar{u}]_t + \left[h\,\bar{u}^2 + \tfrac{1}{2} g\,h^2 + \tfrac{1}{3} h^2\,\tilde{\gamma} - \tau\,R\right]_x = 0,$$

- Vertical acceleration:

$$\tilde{\gamma} = h(\bar{u}_x^2 - \bar{u}_{xt} - \bar{u}\bar{u}_{xx}) = 2\,h\,\bar{u}_x^2 - h[\,\bar{u}_t + \bar{u}\,\bar{u}_x\,]_x$$

- Surface tension:

$$R = h\,h_{xx}\left(1 + h_x^2\right)^{-3/2} + \left(1 + h_x^2\right)^{-1/2},$$

Two conservation laws

- After rewriting: 2 Momentum conservations:

$$\left[h\bar{u}-\tfrac{1}{3}(h^3\bar{u}_x)_x\right]_t+\left[h\bar{u}^2+\tfrac{1}{2}gh^2-\tfrac{1}{3}2h^3\bar{u}_x^2-\tfrac{1}{3}h^3\bar{u}\bar{u}_{xx}-h^2h_x\bar{u}\bar{u}_x-\tau\,R\right]_x=0$$

$$\left[\bar{u}-\frac{(h^3\bar{u}_x)_x}{3\,h}\right]_t+\left[\tfrac{1}{2}\bar{u}^2+gh-\tfrac{1}{2}h^2\,\bar{u}_x^2-\frac{\bar{u}\,(h^3\bar{u}_x)_x}{3\,h}-\frac{\tau\,h_{xx}}{(1+h_x^2)^{3/2}}\right]_x=0$$

Permanent waves

$\mathrm{Fr} = c/\sqrt{gd}$, $\mathrm{Bo} = \tau/gd^2$, $\mathrm{We} = \mathrm{Bo}/\mathrm{Fr}^2 = \tau/c^2d$

- Mass conservation: $\bar{u} = -cd \;/\; h$
- Momentum conservations lead to:

$$\frac{\mathrm{Fr}^2 d}{h} + \frac{h^2}{2\,d^2} + \frac{\tilde{\gamma} h^2}{3gd^2} - \frac{\mathrm{Bo}\, h h_{xx}}{\left(1 + h_x^2\right)^{\frac{3}{2}}} - \frac{\mathrm{Bo}}{\left(1 + h_x^2\right)^{\frac{1}{2}}} = \mathrm{Fr}^2 + \frac{1}{2} - \mathrm{Bo} + K_1$$

$$\frac{\mathrm{Fr}^2\, d^2}{2\, h^2} + \frac{h}{d} + \frac{\mathrm{Fr}^2\, d^2\, h_{xx}}{3\, h} - \frac{\mathrm{Fr}^2\, d^2\, h_x^2}{6\, h^2} - \frac{\mathrm{Bo}\, d\, h_{xx}}{\left(1 + h_x^2\right)^{\frac{3}{2}}} = \frac{\mathrm{Fr}^2}{2} + 1 + \frac{\mathrm{Fr}^2\, K_2}{2}$$

- K_1 and K_2 are integration constants.

Solitary waves

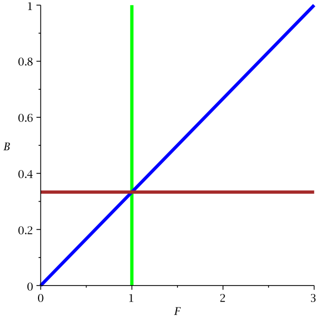
- $h(\infty) = d, \;\; h'(\infty) = 0$ imply $K_1 = K_2 \equiv 0.$
- Combining the two previous equations, h_{xx} is eliminated:

$$F(h, h') \;\equiv\; \frac{\mathrm{Fr}^2\, h'^2}{3} + \frac{2\,\mathrm{Bo}\, h/d}{\left(1 + h'^2\right)^{\frac{1}{2}}} - \mathrm{Fr}^2 + \frac{(2\mathrm{Fr}^2 + 1 - 2\mathrm{Bo})\, h}{d} - \frac{(\mathrm{Fr}^2 + 2)\, h^2}{d^2} + \frac{h^3}{d^3} \;=\; 0$$

- This non linear differential equation depends only on h'^2 and h.

Parametric plane

- The balance between the effects of gravity, inertia and capilarity is expressed by the quantities $\mathrm{Fr}, \mathrm{Bo}, \mathrm{We} = \frac{\mathrm{Bo}}{\mathrm{Fr}}$.
- With $\mathrm{Fr} = 1,\ \mathrm{Bo} = \frac{1}{3},\ \mathrm{We} = \frac{1}{3}$ as critical values.
- Domains in the parametric plane ($\mathscr{F} := \mathrm{Fr}^2, \quad \mathscr{B} := \mathrm{Bo}$), correspond to different behaviors of the solutions.



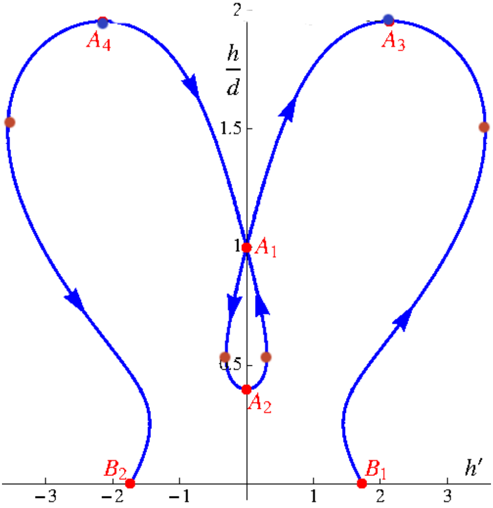
Phase plane analysis for Solitary waves

$$F(h,h') \equiv \frac{\mathscr{F}\,h'^2}{3} + \frac{2\,\mathscr{B}\,h/d}{\left(1+h'^2\right)^{\frac{1}{2}}} - \mathscr{F} + \frac{(2\mathscr{F}+1-2\mathscr{B})\,h}{d} - \frac{(\mathscr{F}+2)\,h^2}{d^2} + \frac{h^3}{d^3} = 0$$

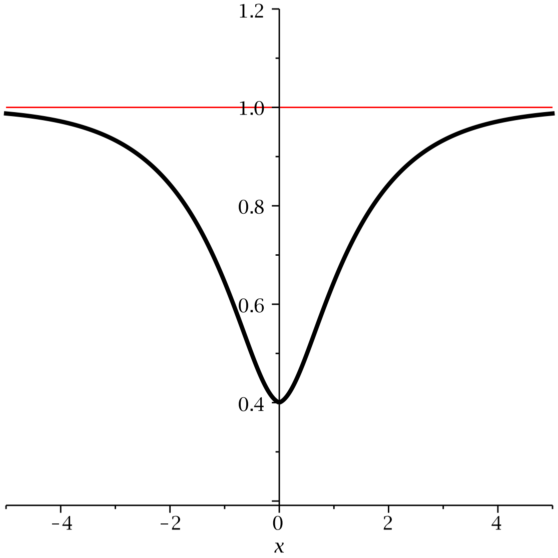
- This is viewed as the implicit equation of a curve $C_{\mathscr{F},\mathscr{B}}$ in the plane (h', h).

Phase-plane analysis

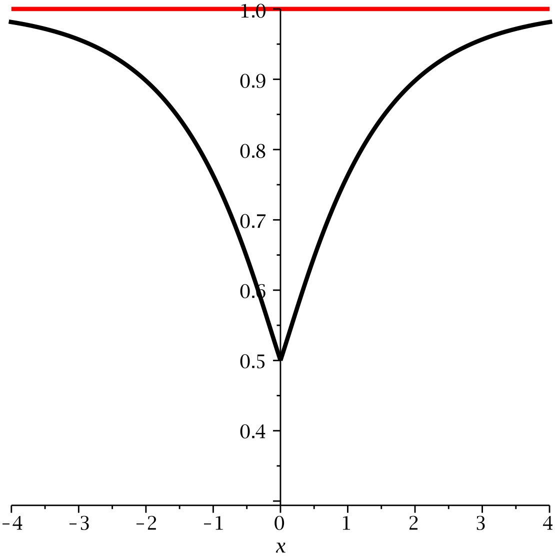
An example for $\mathscr{F} = 0.4,\ \mathscr{B} = 0.9$



Regular Solitary wave



Angular Solitary wave



Local phase-plane analysis: where $h' = 0$

- The solutions of $F_{\mathscr{F},\mathscr{B}}(h,0) = 0$ are $h = d$, or $h = d\mathscr{F}$.
- We compute Taylor expansions at these points.
  - at $A_1 = (0, d)$, we get:

$$d^2\,(\mathscr{F}-3\mathscr{B})\,h'^2 - 3\,(\mathscr{F}-1)\,(h-d)^2 \;=\; 0 + \mathrm{O}\Big((h-d)^3, h'^4\Big)\,.$$

    • If $(\mathscr{F} - 1)(\mathscr{F} - 3\mathscr{B}) < 0$, A_1 is isolated.
  - at $A_2 = (0, d\mathscr{F})$ we get:

$$3\,(\mathscr{F} - 1)^2\,(h - d\mathscr{F}) \;=\; d\,\mathscr{F}\,(3\mathscr{B} - 1)\,h'^2.$$

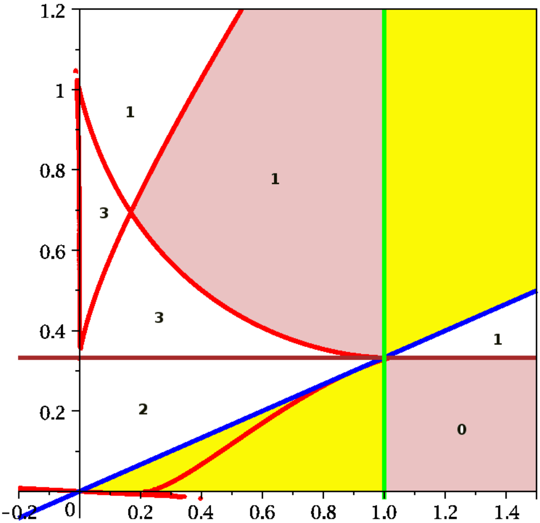
    • If $(\mathscr{F} - 1)(3\mathscr{B} - 1) < 0$ then possibility of regular solitary waves.
    • Else the only possibilities are angular waves.

- Points with horizontal tangent satisfy:
  - $F(h', h) = 0, \quad \frac{\partial F(h',h)}{\partial h'} = 2/3h'(-\mathscr{F} + 3\mathscr{B}h/(1+h'^2)^{3/2}) = 0.$
  - To get rid of the square and cubic roots we set:
  - $h = (d\mathscr{F}/3\mathscr{B})Y^3$, thence $h'^2 = Y^2 - 1$ with $Y \geqslant 1$.
  - $$f(Y) = \mathscr{F}^2 Y^9 - (3\mathscr{F} - 2)\mathscr{F}\mathscr{B}Y^6 + 9\mathscr{B}^2(1 + 2\mathscr{F} - 2\mathscr{B})Y^3 + 27\mathscr{B}^3 Y^2 - 36\mathscr{B}^3 = 0.$$

$\rightarrow$ Discriminant of $f(Y)$.

$\rightarrow$ Partition of the parametric plane $(\mathscr{F}, \mathscr{B})$ which refines the previous diagram.

11 cells with 0 to 3 real roots with $Y > 1$



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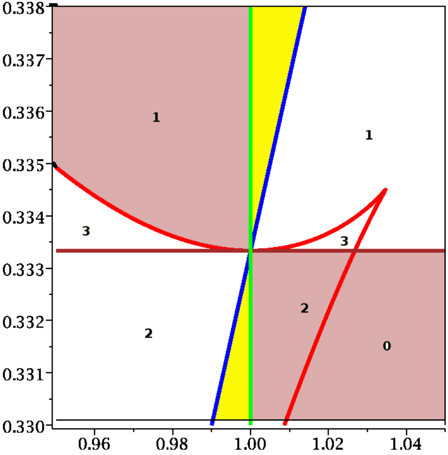
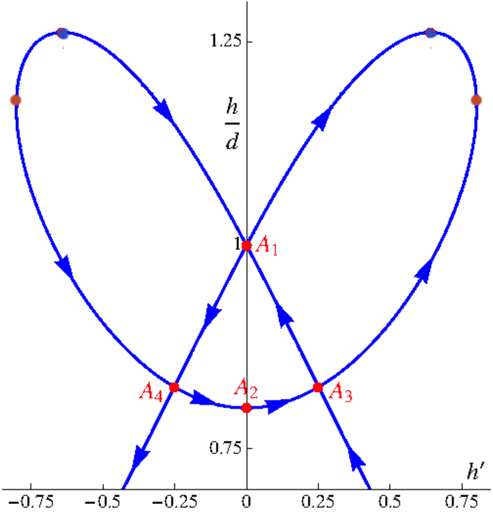


Weakly singular solitary wave

Differentiable but not twice in a special point! $(\mathscr{F}, \mathscr{B}) = (0.8, 0.3538557)$



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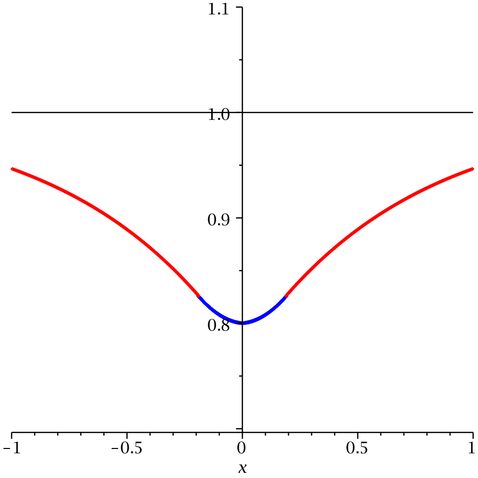
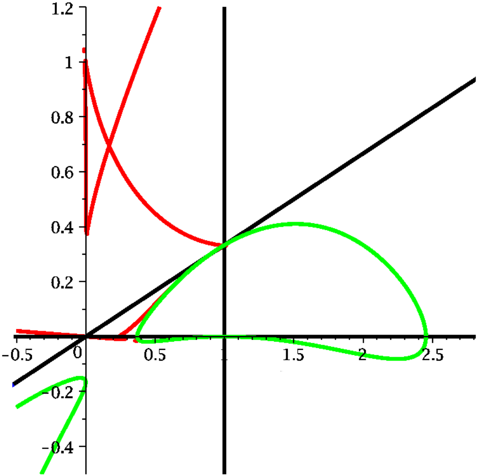


Detection of points with a vertical tangent

- Points with vertical tangent satisfy:
  - $F(h', h) = 0, \quad \frac{\partial F(h',h)}{\partial h} = 0.$
    To get rid of the square roots we set:
  - ${h'}^2 = Z^2 - 1$ hence $\mathscr{F}(Z^2 - 1) = 3(h-1)(2h^2 - h - 1)$,
    $\rightarrow g(Z)$ of degree 6 in Z and degree 3 in $(\mathscr{F}, \mathscr{B})$.
- The two discriminant polynomials for f and g have a common factor.
  $\rightarrow$ A refined partition of the parametric plane $(\mathscr{F}, \mathscr{B})$.

Partition by the number of roots in $Z \geq 1$



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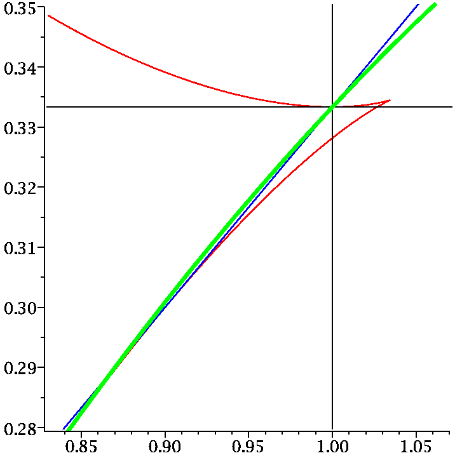
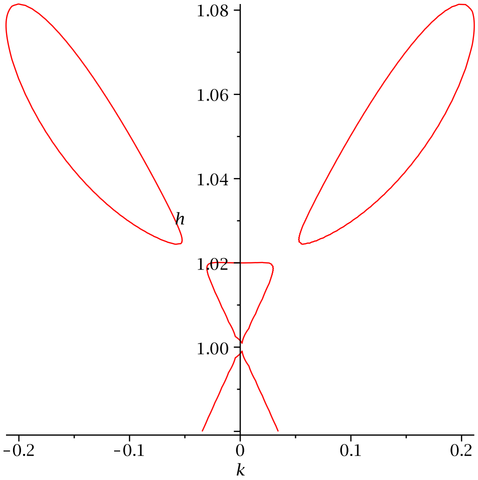


Example of deformation of curves



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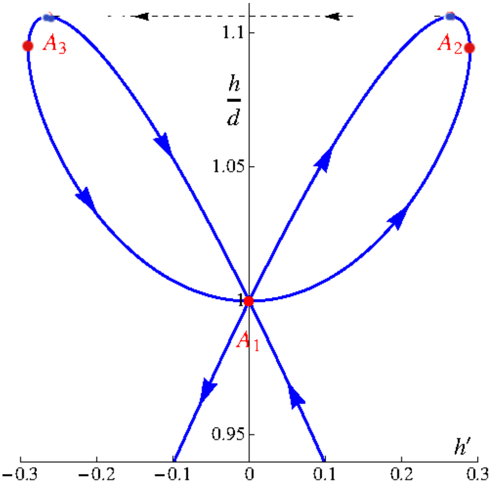

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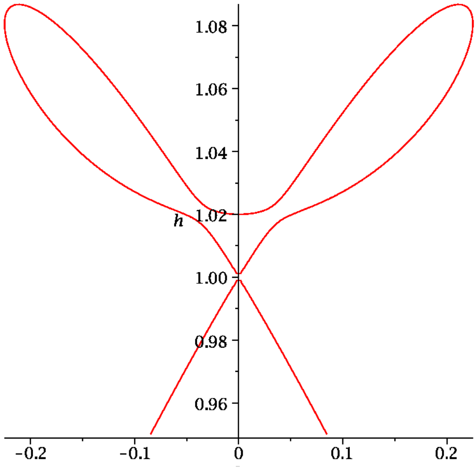

Conclusions & Perspectives

Conclusions:

- A generalization of Serre's equations for Capillary–gravity waves in shallow water regime,
- Weak solitary waves solutions were defined. They depend on two parameters: the square of a Frounde number $\mathscr{F}$, a Bond number $\mathscr{B}$.
- We classified them by exploring the parameter space relying on algebraic techniques; and detected new phenomena.

Perspectives:

- Compute collisions of waves,
- Study permanent waves with $(K_1 K_2 \neq 0)$.