Algebraic Diagonals and Walks

Alin Bostan **Louis Dumont** Bruno Salvy

INRIA, France

July 8, 2015

Diagonals

Diagonals of rational functions

Definition

$$F(x_1, \ldots, x_k) = \sum_{i_1, \ldots, i_k \geq 0} a_{i_1, \ldots, i_k} x_1^{i_1} \ldots x_k^{i_k}$$

$$\downarrow$$

$$\mathrm{Diag}(F)(x) = \sum_{n \geq 0} a_{n, \ldots, n} x^n$$

Example:

$$\frac{1}{1 - x - y} = \sum_{n,m \geq 0} \binom{n+m}{n} x^n y^m$$

$$\mathrm{Diag}\left(\frac{1}{1 - x - y}\right) = \sum_{n \geq 0} \binom{2n}{n} x^n$$

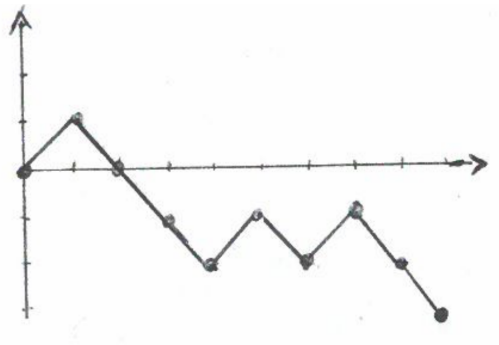
1	1	1	1	1
1	2	3	4	5
1	3	6	10	15
1	4	10	20	35
1	5	15	35	70

Applications of diagonals

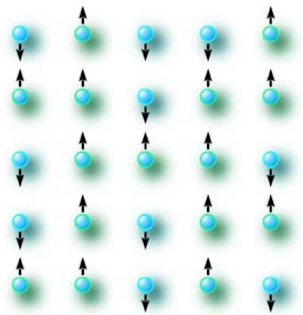
Number theory

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$

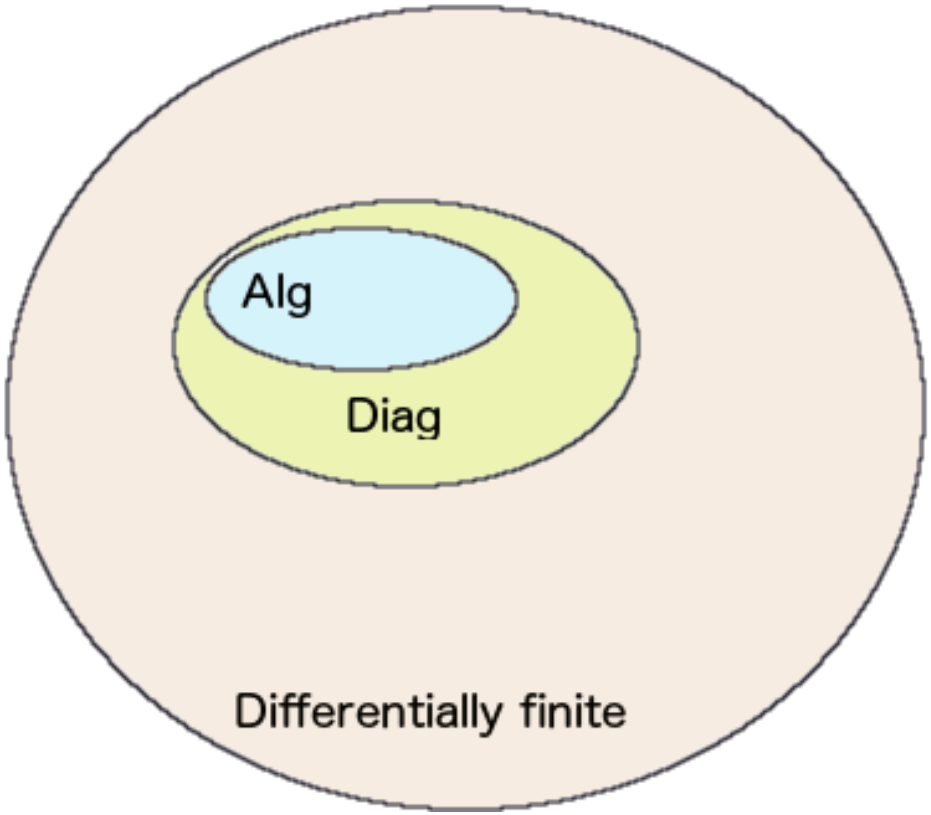
Enumerative combinatorics



Statistical physics



Univariate power series

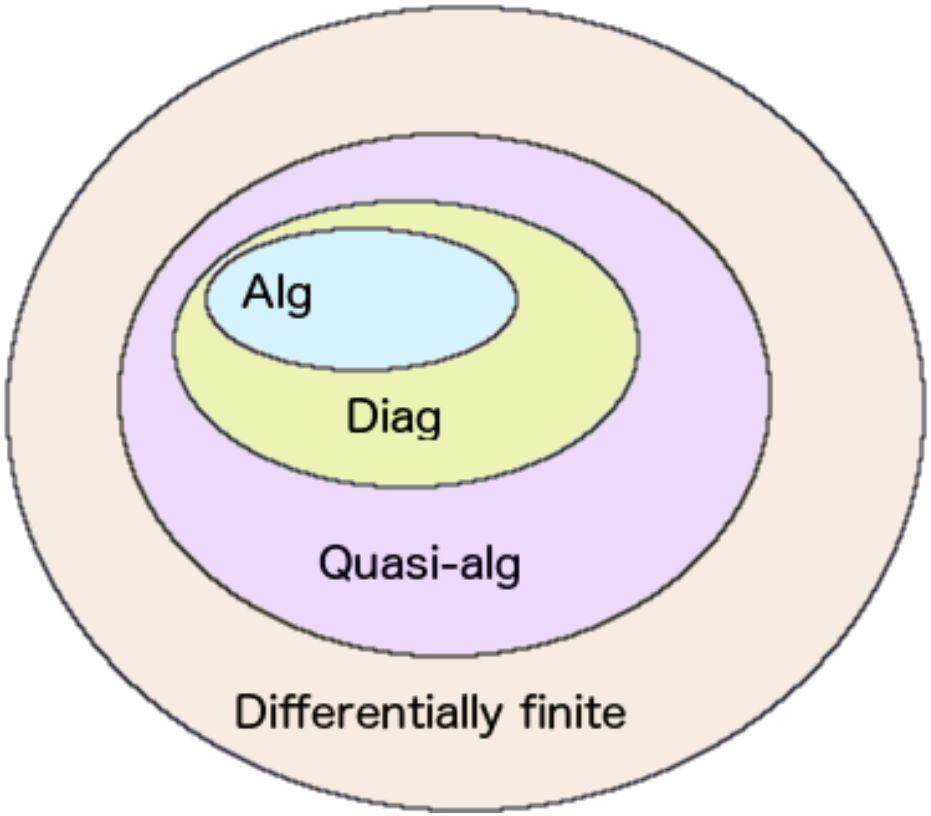


Alg: algebraic series
Diag: diagonals of rational functions

Alg $\subset$ Diag: Furstenberg (1967)
Diag $\subset$ Diff. finite: Christol (1982), Lipshitz (1988)

Algebraic: $P(x, f(x)) = 0,\ P \in \mathbb{Q}[x, y]$
Differentially finite: $L(x, \partial_x) \cdot f = 0,\ L \in \mathbb{Q}(x)\langle\partial_x\rangle$

Univariate power series

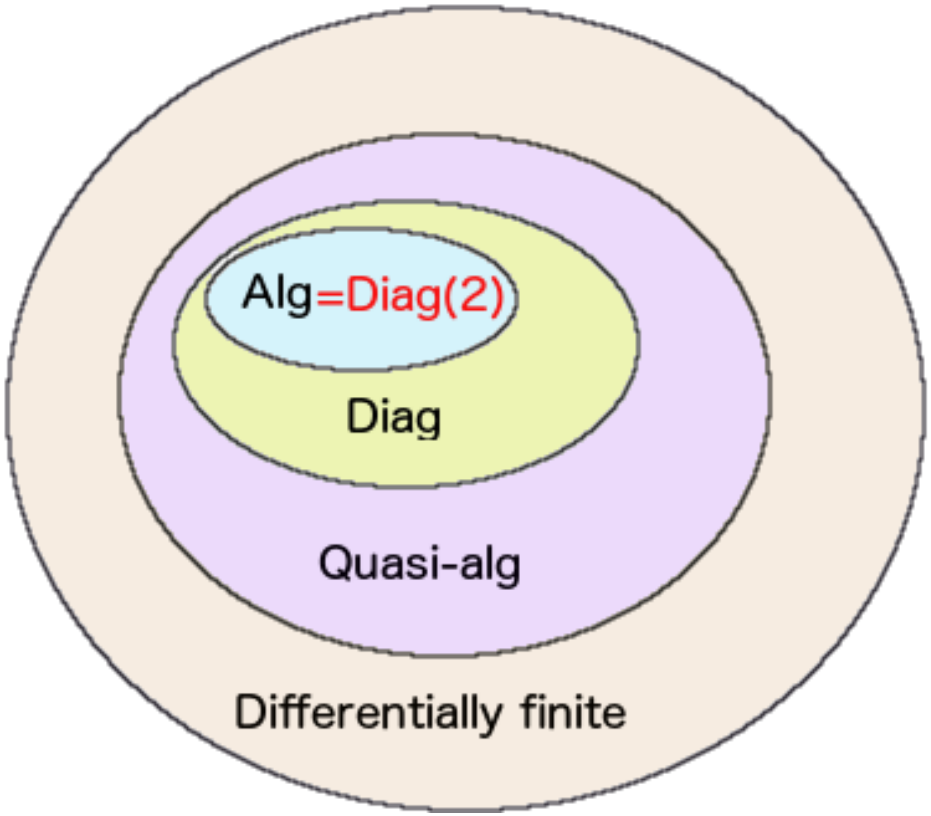


Alg: algebraic series
Diag: diagonals of rational functions
Quasi-alg: quasi-algebraic series

Diag $\subset$ Quasi-alg: Furstenberg (1967)

Quasi-algebraic: algebraic modulo p for all primes p

Univariate power series



Alg: algebraic series
Diag: diagonals of rational functions
Diag(2): diagonals of **bivariate** rational functions
Quasi-alg: quasi-algebraic series

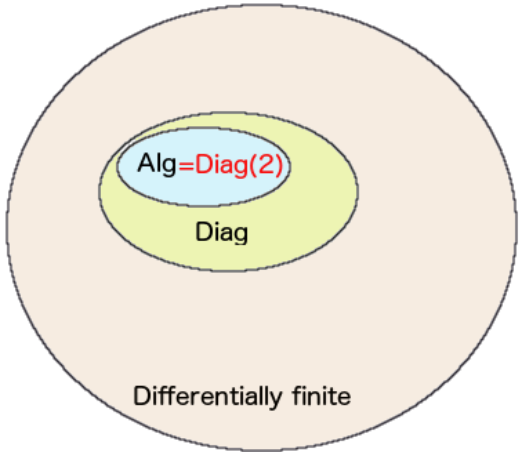
Diag(2) = Alg: Furstenberg (1967)

Quasi-algebraic: algebraic modulo p for all primes p

Diagonals of bivariate rational functions

Theorem (Furstenberg, 1967)

Algebraic univariate power series are exactly the diagonals of bivariate rational functions



Example:

$$\mathrm{Diag}\left(\frac{1}{1-x-y}\right) = \sum_{n\geq 0}\binom{2n}{n}x^n = \frac{1}{\sqrt{1-4x}}$$

$$(1-4x)\Delta^2 - 1 = 0$$

Problem

$F \in \mathbb{Q}(x, y)$

- Compute an annihilating polynomial for $\mathrm{Diag}(F)$
- Study the degree of algebraicity of $\mathrm{Diag}(F)$

Our solution

Turn a well-known formula into an algorithm:

$$G(x,y) := \frac{1}{y} F\left(\frac{x}{y}, y\right) = \sum_{i=1}^{d} \frac{\rho_i(x)}{y - y_i(x)} + \frac{\partial}{\partial y}(\dots) \in \mathbb{Q}(x)(y)$$

small branch: $\lim_{x \to 0} y_i(x) = 0$

$$\underbrace{y_1, \dots, y_c}_{\text{small branches}}, \underbrace{y_{c+1}, \dots, y_d}_{\text{large branches}}$$

$$\boxed{\mathrm{Diag(F)} = \sum_{i=1}^{c} \rho_i(x)}$$

Our solution

Turn a well-known formula into an algorithm:

- $\rho_1(x), \ldots, \rho_d(x)$: residues of G at $y_1(x), \ldots, y_d(x)$

$$\mathrm{Diag}(\mathrm{F}) = \sum_{i=1}^{c} \rho_i(x)$$

Subproblem 1: polynomial annihilating the residues

Compute the polynomial $R = \prod_{i=1}^{d}(y - \rho_i(x)) \in \mathbb{Q}(x)[y]$.

Subproblem 2: pure composed sum

Compute the polynomial ($c \leq d$)

$$\Sigma_c R = \prod_{i_1 < \ldots < i_c} (y - (\rho_{i_1}(x) + \ldots + \rho_{i_c}(x)) \in \mathbb{Q}(x)[y]$$

Compute a polynomial cancelling the residues

Subproblem 1: polynomial annihilating the residues

Compute the polynomial $R = \prod_{i=1}^{d}(y - \rho_i(x)) \in \mathbb{Q}(x)[y]$.

- $y_1(x), \ldots, y_d(x)$: distinct poles of $G(x, y) \in \mathbb{Q}(x)(y)$
- $\rho_1(x), \ldots, \rho_d(x)$: residues of G at $y_1(x), \ldots, y_d(x)$

1. If y_i is a **simple** pole, then

$$\rho_i(x) = \frac{P(x, y_i(x))}{\frac{\partial Q}{\partial y}(x, y_i(x))}.$$

   $\rho_i(x)$ is cancelled by the Rothstein-Trager resultant:

$$\mathrm{Res}_z\left(\frac{\partial Q}{\partial y}(x, y)z - P(x, y), Q(x, y)\right)$$

2. if y_i is a **multiple** pole: Bronstein resultants

Compute the pure composed sum of a polynomial

Subproblem 2: pure composed sum

Compute the polynomial ($c \leq d$)

$$\Sigma_c R = \prod_{i_1 < \ldots < i_c} (y - (\rho_{i_1}(x) + \ldots + \rho_{i_c}(x)) \in \mathbb{Q}(x)[y]$$

Main tool: Newton sums.

$$R \longleftrightarrow \mathcal{N}(R) = \sum_{n \geq 0} (\rho_1^n + \ldots + \rho_d^n) \frac{y^n}{n!}$$

Strategy:

$$\begin{array}{ccc} R & \longrightarrow & \mathcal{N}(R) =: S \\ & & \downarrow \\ \Sigma_c R & \longleftarrow & \mathcal{N}(\Sigma_c R) = \Phi(S(y), S(2y), \ldots, S(cy)) \end{array} \qquad \Phi : \text{polynomial}$$

First result: algebraic data structure for diagonals

d_x, d_y: degrees in x and y of the denominator of F

Theorem (B., D., S., 2015)

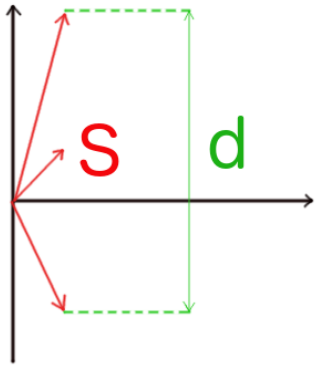
- *$\Sigma_c R$ has degree at most $\binom{d_x+d_y}{d_x}$ in y (tight)*
- *"generically", $\Sigma_c R$ is irreducible over $\mathbb{Q}(x)$*
- *"generically", $\Sigma_c R$ is computed in quasi-optimal time*

$d_x = d_y = d$	1	2	3	4
$\deg_x \Sigma_c R, \deg_y \Sigma_c R$	(2, 2)	(16, 6)	(108, 20)	(640, 70)

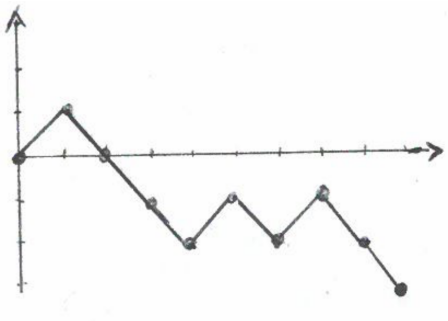
The degree of algebraicity of the diagonal is generically exponential in the size of the input rational function.

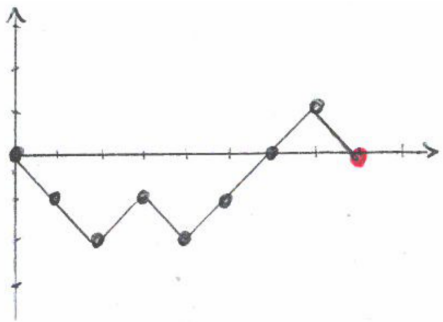
Walks

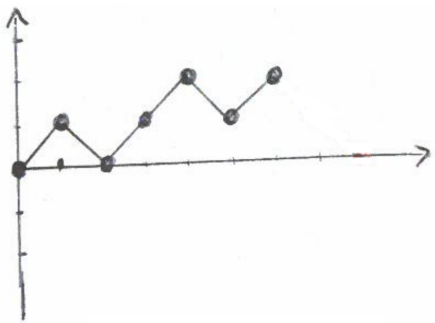
Back to lattice walks

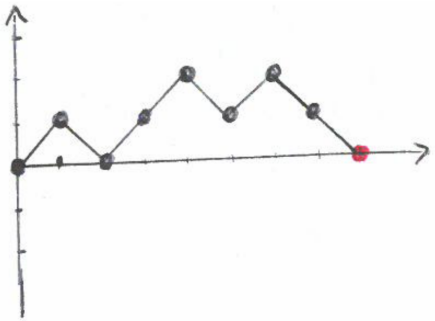


$\mathcal{S}$: set of steps
d: amplitude of $\mathcal{S}$


Walks


Bridges


Meanders


Excursions

Problem

Compute the generating series of the bridges, excursions and meanders at precision N

Naive algorithms: quadratic complexity $\mathcal{O}(d^2N^2)$

A previously known linear time algorithm

Theorem (Banderier, Flajolet, 2002)

The generating series of the bridges, excursions and meanders are algebraic.

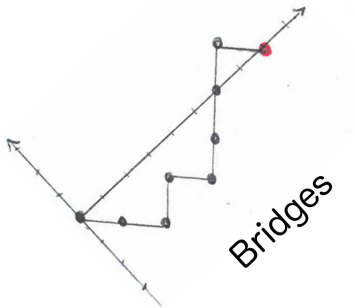
Strategy:

1. Compute an algebraic equation for B, E, or M;
2. Deduce a differential equation; $\mathcal{O}(C^d)$
3. Deduce a recurrence;
4. Compute initial conditions using a naive method;
5. Unroll the recurrence.

- Complexity of the expansion: $\mathcal{O}(d^2 N)$ (linear in N)
- Complexity of the pre-processing: $\mathcal{O}(C^d)$ (exponential in d)

A new quasi-linear time algorithm with small pre-processing

Fundamental fact: the generating series of the **bridges** is a **diagonal**



Theorem (B., Chen, Chyzak, Li (2010))

Diagonals of bivariate rational functions satisfy polynomial-size differential equations.

Strategy:

1. Directly compute a differential equation for the bridges;
2. Deduce the excursions from the formula $B(x) = 1 + xE'(x)/E(x)$

- Complexity of the expansion: $\tilde{\mathcal{O}}(d^2N)$ (quasi-linear in N)
- Complexity of the pre-processing: $\tilde{\mathcal{O}}(d^5)$ (polynomial in d)

The meanders can be computed similarly.

Summary picture