Formal Solutions of Linear Differential Systems with Essential Singularities in their Coefficients

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Introduction / Motivation

General Purpose of the Talk

◇ **Notation**: z complex variable, $\mathbb{C}$ field of complex numbers

◇ We consider systems of linear ordinary differential equations:

$$Y' = A(z)\,Y \qquad\qquad ' := \frac{d}{dz}$$

A square matrix (size n) of analytic functions of z

Y vector of n unknown functions of z

◇ General purpose: local analysis around singularities

$\rightarrow$ more precisely, algorithms for computing formal local solutions

Different Types of Singularities

$$Y' = A(z)\,Y$$

Entries of A are holomorphic in a punctured neighborhood of $z = 0$

◇ Singularity at $z = 0$:

1. Removable (holomorphic function): $\sum_{n=0}^{+\infty} a_n\, z^n$
2. Pole (meromorphic function): $\sum_{n=-k}^{+\infty} a_n\, z^n$, with $k \in \mathbb{N}$
3. Essential: neither removable nor pole: $\sum_{n=-\infty}^{+\infty} a_n\, z^n$

◇ Cases 1. and 2. widely studied, various computer algebra algorithms exist for computing formal solutions

→ This work tackles a class of systems with essential singularities

Example

$\diamond$ $\hbar, m, g, k$ and E (physical) constants

$\diamond$ $X = \exp\left(-\frac{k\,m}{z}\right)$ non-zero solution of the scalar linear differential equations

$$\frac{dX}{dz} = z^{-2}\,k\,m\,X$$

$\diamond$ Schrödinger equation with Yukawa potential (*Hamzavi et al*'12):

$$\frac{dY}{dz} = z^{-2}\underbrace{\begin{pmatrix} 0 & -1 \\ \frac{2\,m\,g^2}{\hbar^2}\,z\,X + \frac{2mE}{\hbar^2} & 0 \end{pmatrix}}_{A(z,X)} Y$$

$$A(z,X) = \begin{pmatrix} 0 & -1 \\ \frac{2mE}{\hbar^2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{2\,m\,g^2}{\hbar^2}\,z & 0 \end{pmatrix} X$$

Applications

◇ Linear differential systems with essential singularities appear in many applications:

- Linearization of a non-linear differential system around a particular solution (*Aparicio'10*)
- Computation of a closed form of some integrals (*BarkatouRaab'12*)
- Equations from physics (Schrödinger, . . .)

Formal Fundamental Matrix of Solutions of $Y' = A\,Y$

1. Removable singularity:

$$Y = \Phi(z)$$

$\Phi(z)$ matrix of formal power series in z

2. Pole: *Turritin'55*, *Wasow'65*, … **Algo**: *Barkatou'97*

$$Y = \Phi(t)\, t^{\Lambda} \exp(Q(1/t))$$

$z = t^r$, $Q(1/t) = \mathrm{diag}(q_1(1/t), \ldots, q_n(1/t))$, $\Lambda \in \mathbb{M}_n(\mathbb{C})$, and $\Phi(t) \in \mathbb{M}_n(\mathbb{C}((t)))$

3. Our class of systems with essential singularities: *Bouffet'03* in the particular case $X = \exp(1/z)$, *BCJ'15*

$$Y = \left(\sum_{k=0}^{+\infty} \Phi_k(t)\, X^k \right) t^{\Lambda} \exp(Q(1/t))$$

same as 2. and $\Phi_k(t) \in \mathbb{M}_n(\mathbb{C}((t)))$

Previous works and contributions

◇ Linear differential systems with hyperexponential coefficients in computer algebra:

- *Fredet'01*: algo. for closed form solutions (polynomial, rational) of scalar equations
- *Bouffet'02*: diff. Galois theory, Hensel lemma
- *BarkatouRaab'12*: direct algo. for closed form solutions of systems, applications to indefinite integration

◇ Contribution: algo. for computing a formal fundamental matrix of solutions of a class of systems with essential singularities

Approach: viewing $Y' = z^{-p} A(z, X)\, Y$ as a perturbation of the meromorphic system by letting $X \to 0$

II

Meromorphic Linear Differential Systems

Definitions

$$Y' = z^{-p}\,A(z)\,Y, \quad p \in \mathbb{N}^*, \quad A(z) \in \mathbb{M}_n(\mathbb{C}[[z]]),\ A(0) \neq 0$$

- The integer $p-1 \geq 0$ is called the Poincaré rank of $[z^{-p}\,A]$
- Change of variables $Y = T\,Z$ with $T \in \mathrm{GL}_n(\mathbb{C}((z)))$:

$$\underbrace{Y' = A\,Y}_{[A]} \longrightarrow \underbrace{Z' = T^{-1}\,(A\,T - T')\,Z}_{T[A]}$$

- Equivalence: $[A] \sim_F [B]$ if $\exists T \in \mathrm{GL}_n(F)$ such that $B = T[A]$

$$\left\{ \begin{array}{c} B = T[A] \\ Y_{[B]}\ \text{FFMS of}\ [B] \end{array} \right\} \Longrightarrow Y_{[A]} = T\,Y_{[B]}\ \text{FFMS of}\ [A]$$

Computation of a FFMS of a meromorphic system (1)

◇ *Turritin'55*, *Wasow'65*, ...

$$Y' = z^{-p} A(z)\, Y \longrightarrow \mathit{FFMS} : Y = \Phi(t)\, t^{\Lambda} \exp(Q(1/t))$$

$z = t^r$, $Q(1/t) = \mathrm{diag}(q_1(1/t), \ldots, q_n(1/t))$, $\Lambda \in \mathbb{M}_n(\mathbb{C})$, and $\Phi(t) \in \mathbb{M}_n(\mathbb{C}((t)))$

◇ Sketch of the algorithm of *Barkatou'97*:

1. Case $p \leq 1$: easy, $r = 1$, $Q = 0$ → use *BarkatouPflügel'99*
2. Case $p > 1$: Barkatou-Moser's algo. (*Moser*'60, *Barkatou*'95)

→ equivalent system with minimal Poincaré rank $\tilde{p} - 1$:

- If $\tilde{p} = 1$, then regular ($r = 1$, $Q = 0$) → same as 1.
- If $\tilde{p} > 1$, then irregular ($Q \neq 0$) → see next slide

Computation of a FFMS of a meromorphic system (2)

$$Y' = z^{-p} A(z)\, Y \longrightarrow FFMS : Y = \Phi(t)\, t^{\Lambda} \exp(Q(1/t))$$

◇ Irregular case: Minimal Poincaré rank $> 0 \to$ FFMS with $Q \neq 0$

◇ Method of *Barkatou'97*: reduce to several systems with either Poincaré rank 0 or scalar:

1. 1^{st} gauge transfo. to split the system into smaller systems where $A_0(0)$ has only one eigenvalue
2. 2^{nd} gauge transfo. to get a new system with nilpotent $A_0(0)$ and apply Barkatou-Moser to get minimal Poincaré rank
3. If Poincaré rank still > 0 and $A_0(0)$ nilpotent, then:
   - Compute Katz' invariant κ and perform ramification $z = t^m$
   - Barkatou-Moser to get new system with Poincaré rank $m\,\kappa$ and $A_0(0)$ not nilpotent
4. Apply recursion

III

Linear Differential Systems with Essential Singularities

Class of Systems Considered (1)

- $q \in \mathbb{N}$ such that $q \geq 2$
- $a(z) = \sum_{k=0}^{+\infty} a_k\, z^k \in \mathbb{C}[[z]]$ with $a(0) = a_0 \neq 0$

◇ $X(z)$ non-zero solution of the scalar linear differential equation:

$$X' = z^{-q}\, a(z)\, X$$

$$X(z) = \exp\left(\int z^{-q}\, a(z) dz\right) = \exp\left(\frac{a_0}{(1-q)\, z^{q-1}} + \frac{a_1}{(2-q)\, z^{q-2}} + \cdots\right)$$

◇ Hypotheses on q and $a(z) \Longrightarrow X(z)$ transcendental over $\mathbb{C}((z))$

Class of Systems Considered (2)

- $q \in \mathbb{N}$ such that $q \geq 2$
- $a(z) = \sum_{k=0}^{+\infty} a_k \, z^k \in \mathbb{C}[[z]]$ with $a(0) = a_0 \neq 0$
- $p \in \mathbb{N}^*$
- $A_k(z) \in \mathbb{M}_n(\mathbb{C}[[z]]), \quad k = 0, \ldots, +\infty$ with $A_0(0) \neq 0$

$$\begin{cases} \dfrac{dX}{dz} = z^{-q} \, a(z) \, X \\[2ex] \dfrac{dY}{dz} = z^{-p} \, A(z, X) \, Y, \quad A(z, X) = \sum_{k=0}^{+\infty} A_k(z) \, X^k \end{cases}$$

$\rightarrow$ We have an essential singularity at the origin $z = 0$

Computation of a FFMS: different cases

$$\begin{cases} X' = z^{-q}\, a(z)\, X \\ Y' = z^{-p}\, A(z,X)\, Y, \quad A(z,X) = \sum_{k=0}^{+\infty} A_k(z)\, X^k \end{cases}$$

◇ Algorithm for computing a FFMS:

1. Case $p \leq q$: reduction to the meromorphic system $[z^{-p}\, A_0]$
2. Case $p > q$: adapt the process of *Barkatou'97* to reduce to several systems with either $p \leq q$ or scalar

Computation of a FFMS: Case $p \leq q$ (1)

1. Case 1: $p < q$ or $p = q$ and $A_0(0)$ has no eigenvalues that differ by an integer multiple of $a(0)$
2. Case 2: $p = q$ and $A_0(0)$ has eigenvalues that differ by an integer multiple of $a(0)$

Theorem (*BCJ'2015*)

In Case 1., we can compute an invertible matrix transformation

$$T = I_n + T_1(z)\,X + T_2(z)\,X^2 + \cdots\,, \quad T_k(z) \in \mathbb{M}_n(\mathbb{C}[[z]])$$

such that

$$A_0 = T^{-1}(A\,T - z^p\,T') \Longleftrightarrow z^p\,T' = A\,T - T\,A_0$$

Tool: resolution of equations $z^m\,U' = M\,U - U\,N - V$ over $\mathbb{C}[[z]]$

Computation of a FFMS: Case $p \leq q$ (2)

Proposition (*BCJ'2015*)

A system such that $p = q$ and $A_0(0)$ has eigenvalues that differ by an integer multiple of $a(0)$ can be reduced to a system such that $p = q$ and $A_0(0)$ has no eigenvalues that differ by an integer multiple of $a(0)$.

Constructive proof provides the invertible transformation matrix with coeffs. in $\mathbb{C}[[z]][[X]]$

Theorem (*BCJ'2015*)

In the case $p \leq q$, we can compute a FFMS of the form:

$$Y = \left(\sum_{k=0}^{+\infty} T_k(z)\, X^k \right) \Phi(t)\, t^{\Lambda} \exp(Q(1/t)), \quad T_k(z) \in \mathbb{M}_n(\mathbb{C}[[z]])$$

Computation of a FFMS: Case $p > q$ - Scalar Equations

$$\begin{cases} X' = z^{-q}\, a(z)\, X \\ Y' = z^{-p}\, A(z,X)\, Y, \;\; A(z,X) = \sum_{k=0}^{+\infty} A_k(z)\, X^k, \;\; A_k(z) \in \mathbb{C}[[z]] \end{cases}$$

1. $Y_0(z) = \exp(\int z^{-p}\, A_0(z)\, dz)$ solution of $[z^{-p}\, A_0(z)]$
2. $Y = Y_0\, Z \longrightarrow Z' = z^{-p}\, (A_1(z)\, X + A_2(z)\, X^2 + \cdots)\, Z$
3. Normalization $X \to z^c\, X$ with $c \in \mathbb{Z} \longrightarrow$ new system with $p < q$ and $A_0(z) = 0$
4. $\exists\, T \in \mathbb{C}[[z]][[X]]$ such that the equation is reduced to $[0]$

$\to$ formal fundamental solution $Y = Y_0(z)\, T(z, z^c\, X)$

Computation of a FFMS: Case $p > q$ (1)

$\diamond$ Method: adapt the process of *Barkatou'97* to reduce to several systems with either $p \leq q$ or scalar

Proposition (*BCJ'2015*)

We can compute a matrix transformation

$$T = I_n + T_1(z)\,X + T_2(z)\,X^2 + \cdots\,, \quad T_k(z) \in \mathbb{M}_n(\mathbb{C}[[z]]),$$

such that

$$z^p\,T' = A(z,X)\,T - T\,\mathrm{diag}(A^{[1]}(z,X),\ldots,A^{[r]}(z,X)),$$

and each $A^{[i]}(0,0)$ has only one eigenvalue.

Computation of a FFMS: Case $p > q$ (2)

◇ Follow the algorithm of *Barkatou'97* applied to $[z^{-p} A_0(z)]$ and at each step perform the transformations needed (e.g., splitting, shift, Moser's reduction, ramification, ...) to the whole system

◇ Ramification $z = t^r$ $\rightarrow$ new differential system in t with

$$\tilde{p} = r\,(p-1)+1,\ \tilde{q} = r\,(q-1)+1,\ \tilde{A}(t,X) = r\,A(t^r,X),\ \tilde{a}(t) = r\,a(t^r)$$

Theorem (*BCJ'2015*)

In the case $p > q$, we can compute a FFMS of the form:

$$Y = \left(\sum_{k=0}^{+\infty} \Phi_k(t)\,X^k\right)\, t^{\Lambda}\,\exp(Q(1/t)), \quad \Phi_k(t) \in \mathbb{M}_n(\mathbb{C}((t)))$$

IV

Extensions and Future Works

Ring of Coefficients

- X given by $X' = z^{-q}\, a(z)\, X$ with same hypotheses as before

- $\mathcal{R} = \mathbb{C}((z))[[X]]$: the ring of constants of $(\mathcal{R}, d/dz)$ is $\mathbb{C}$

$$\mathcal{A} = \left\{ f = \sum_{k=0}^{+\infty} f_k(z)\, X^k \in \mathcal{R} \mid \inf_{k \in \mathbb{N}} v_z(f_k(z)) > -\infty \right\} \subsetneq \mathcal{R}$$

- This talk: systems with coeffs in $\mathcal{A}$ and $A_0(0) \neq 0$

$\to$ This can be generalized to tackle systems with coeffs in

$$\mathcal{B} = \left\{ f = \sum_{k=0}^{+\infty} f_k(z)\, X^k \in \mathcal{R} \mid \exists \alpha, \beta \in \mathbb{Q}\,;\, \forall k,\ v_z(f_k(z)) \geq \alpha\, k + \beta \right\} \supsetneq \mathcal{A}$$

Tool: normalizations $X \to z^c\, X$

Future Work

- ◇ Implementation in progress $\rightarrow$ handle many examples
- ◇ This work: 1^{st} order systems with coeffs in $\mathbb{C}((z))[[X]]$

$\rightarrow$ Future Work: more general systems

1. Systems with coeffs in $\mathbb{C}((z))((X))$:

$$\begin{cases} X' = z^{-q}\, a(z)\, X \\ \\ Y' = z^{-p}\, A(z,X)\, Y, \quad A(z,X) = \sum_{k=-N}^{+\infty} A_k(z)\, X^k \end{cases}$$

$\rightsquigarrow$ Classification of singularities, rank reduction, ...

2. Systems involving several transcendental functions
3. Systems of arbitrary order

Thank you!

Example (1)

◇ $X(z)$ non-zero solution of $X' = z^{-q}\, a(z)\, X$

◇ Consider the scalar equation:

$$Y' = -z^{-(q+1)} \left(\sum_{k=0}^{+\infty} A_k\, X^k \right) Y$$

$$A_0 = z^q, \quad \forall\, k \geq 1,\ A_k = z^{-(k-1)} \left(z^{q-1} - a(z) \right)$$

◇ With the previous notation, we have $p = q + 1 > q$

◇ $\inf_{k \in \mathbb{N}} v_z(A_k) = -\infty \rightarrow$ coeffs not in $\mathcal{A}$

$\rightarrow$ They are in $\mathcal{B}$ since $v_z(A_k) \geq -k + 1$

Example (2)

◇ Normalization $X \to z^{-q} X$

$$\to Y' = -z^{-1} \left(\sum_{k=0}^{+\infty} \tilde{A}_k X^k \right) Y, \quad \tilde{A}_0(z) = 1,\ \tilde{A}_k \in \mathbb{M}_n(\mathbb{C}[[z]])$$

◇ Case $p < q \to$ Using as transformation the formal power series

$$T = 1 + T_1 X + T_2 X^2 + \cdots, \qquad T_k = z^{(q-1)k}$$

we are reduced to the leading equation $y' = -z^{-1} y$

◇ Trivial solution $y = z^{-1} \to$ formal fundamental solution

$$Y = \left(\sum_{k=0}^{+\infty} z^{(q-1)k} (z^{-q} X)^k \right) z^{-1} = z^{-1} \sum_{k=0}^{+\infty} z^{-k} X^k$$