A fast algorithm for computing the p-curvature

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joint work with
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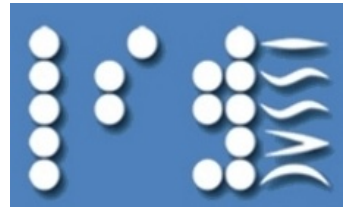
ISSAC'15, Bath, UK  July 7th 2015

Main objects and goal

- $k =$ a field of prime characteristic p, typically $\mathbb{F}_p$
- $k(x)\langle\partial\rangle =$ the non-commutative (right-) Euclidean algebra of linear differential operators $L = a_0 + a_1\partial + \cdots + a_r\partial^r$, for $a_i \in k(x)$

Def: p-curvature $\mathbf{A}_p(L)$ of $L =$ the matrix in $\mathscr{M}_r(k(x))$ whose (i,j) entry is the coefficient of x^i in ∂^{p+j} Rmod L for $0 \leq i,j < r$

Goal: design an efficient algorithm for $\mathbf{A}_p(L)$

- ▷ Efficiency = complexity estimates with a low exponent w.r.t. p
- ▷ Complexity is measured in terms of arithmetic operations in k

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- ▷ Complexity is measured in terms of arithmetic operations in k
- ▷ Caveat: to simplify matters, assume input L has size $\mathcal{O}(1)$

Example

$$L \;=\; (5x^2+4)\partial^2 + (4x^2+6x+5)\partial + 2x + 2 \;\in\; \mathbb{F}_7[x]\langle\partial\rangle$$

Euclidean right division in $\mathbb{F}_7(x)\langle\partial\rangle$:

$$\partial^7 = (\cdots)\,L + \frac{(x+1)(x^2+x-1)}{(x+3)(x-3)^2}\partial + \frac{4x(x-1)}{(x+3)(x-3)^2}$$

$$\partial^8 = (\cdots)\,L + \frac{2(x+1)(x^2+x-1)}{(x+3)(x-3)^2}\partial + \frac{x(x-1)}{(x+3)(x-3)^2}$$

$$\Longrightarrow \qquad \mathbf{A}_7(L) = \begin{bmatrix} \frac{4x(x-1)}{(x+3)(x-3)^2} & \frac{x(x-1)}{(x+3)(x-3)^2} \\ \frac{(x+1)(x^2+x-1)}{(x+3)(x-3)^2} & \frac{2(x+1)(x^2+x-1)}{(x+3)(x-3)^2} \end{bmatrix}$$

Basics on differential equations in characteristic p

Main differences between characteristic zero and p

- (Honda 1981) solutions are simpler in characteristic p

$$\dim_{k(x^p)} \mathcal{S}_L(k[x]) = \dim_{k(x^p)} \mathcal{S}_L(k(x)) = \dim_{k((x^p))} \mathcal{S}_L(k[[x]])$$

- Cauchy's theorem does not hold: the common dimension $\dim \mathcal{S}_L$ of the solution spaces is generally $< r = \mathrm{ord}(L)$

Example: $y' = y$ has no solution in $\mathbb{F}_p[[x]]$

Connection between solutions and p-curvature

- (Katz & Cartier 1970) $\quad \mathrm{rank}(\mathbf{A}_p(L)) = r - \dim(\mathcal{S}_L)$

$\longrightarrow$ p-curvature measures to what extent $\dim(\mathcal{S}_L)$ is close to r

Example

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- 7-curvature of L

$$\mathbf{A}_7(L) = \begin{bmatrix} \frac{4x(x-1)}{(x+3)(x-3)^2} & \frac{x(x-1)}{(x+3)(x-3)^2} \\ \\ \frac{(x+1)(x^2+x-1)}{(x+3)(x-3)^2} & \frac{2(x+1)(x^2+x-1)}{(x+3)(x-3)^2} \end{bmatrix}$$

- Katz-Cartier:

$$1 = \mathrm{rank}(\mathbf{A}_7(L)) = 2 - \dim_{\mathbb{F}_7(x^7)}(\mathcal{S}_L) \quad\Longrightarrow\quad \dim_{\mathbb{F}_7(x^7)}(\mathcal{S}_L) = 1$$

- In fact

$$\mathrm{Basis}_{\mathbb{F}_7(x^7)}(\mathcal{S}_L) = \{1 - 2x^2 - x^3\}$$

A useful tool for theory

Grothendieck's conjecture ('70s) $\Gamma \in \mathbb{Q}[x]\langle\partial\rangle$ has a basis of algebraic solutions over $\mathbb{Q}(x)$ iff $\mathbf{A}_p(\Gamma) = 0$ for almost all primes p.

Def: A power series $\sum_{n\geq 0} \frac{a_n}{b_n} x^n$ in $\mathbb{Q}[[x]]$ is called a *G-series* if it is (a) D-finite; (b) analytic at $x=0$; (c) $\exists C>0$, $\mathrm{lcm}(b_0, \ldots, b_n) \leq C^n$.

Examples: algebraic functions; $\log(1-x)$, ${}_2F_1\left(\begin{matrix}\alpha & \beta \\ \gamma \end{matrix}\middle| x\right)$; diagonals

Chudnovsky's theorem (1985) The minimal-order operator $\Gamma \in \mathbb{Q}[x]\langle\partial\rangle$ annihilating a G-series is globally nilpotent, i.e., the p-curvatures $\mathbf{A}_p(\Gamma)$ are nilpotent for almost all primes p.

Examples: algebraic resolvents; $x(1-x)\partial^2 + (\gamma - (\alpha+\beta+1)x)\partial - \alpha\beta x$

A useful tool for algorithms

p-curvature used in computer algebra:

- [van der Put 1995] for factoring operators in $\mathbb{F}_p(x)\langle\partial\rangle$
- [Cluzeau 2003] for decomposing differential systems over $\mathbb{F}_p(x)$
- [Cluzeau & van Hoeij 2004] as filter in modular algorithms for operators in $\mathbb{Q}(x)\langle\partial\rangle$

Improving the complexity of the p-curvature computation is an interesting problem in its own right

A useful tool for applications

- in enumerative combinatorics (classification of lattice walks)
- in statistical physics (square lattice Ising model)

Typical task: given a power series $S \in \mathbb{Z}[[x]]$, decide if S is D-finite

Differential guessing: from the first $N \gg 0$ terms of S, compute a differential operator $\Gamma \in \mathbb{Q}[x]\langle\partial\rangle$ that annihilates $S \bmod x^N$

▷ One way to empirically certify the correction of Γ is to *look at the p-curvature* $\mathbf{A}_p(\Gamma \bmod p)$ for a random (large) prime p

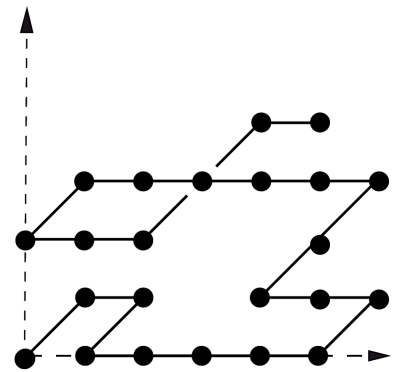
- *if* $\mathbf{A}_p(\Gamma \bmod p)$ *is nilpotent, then* S *is very probably D-finite*
- *if* $\mathbf{A}_p(\Gamma \bmod p)$ *is zero, then* S *is very probably algebraic*

A combinatorial application: Gessel's conjecture

- Gessel walks: walks in $\mathbb{N}^2$ using only steps in $\mathcal{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$
- $g(i,j,n)$ = number of walks from $(0,0)$ to (i,j) with n steps in $\mathcal{S}$

Question: Nature of the generating function

$$G(u,v,x) = \sum_{i,j,n=0}^{\infty} g(i,j,n)\, u^i v^j x^n \in \mathbb{Q}[[u,v,x]]$$



Theorem (B.-Kauers 2010) $G(u,v,x)$ is an algebraic function.*

$\rightarrow$ Effective, computer-driven discovery and proof

$\rightarrow$ Key step in discovery: p-curvature computation of two 11th order (guessed) differential operators for $G(u,0,x)$, and $G(0,v,x)$

*Minimal polynomial $P(u,v,x,G(u,v,x)) = 0$ has $> 10^{11}$ terms; $\approx$30Gb (!)

Previous work

① p-curvature $\mathbf{A}_p(L)$ size $\mathcal{O}(p)$

- [Katz 1982]: algorithm of cost $\mathcal{O}(p^2)$, based on recurrence

$$\mathbf{A}_1 = \text{CompanionMatrix}(L), \quad \mathbf{A}_{k+1} = \mathbf{A}'_k + \mathbf{A}_1 \cdot \mathbf{A}_k$$

- [B. & Schost 2009]: first subquadratic algorithm $\mathcal{O}(p^{1.79})$
- [B. & Schost 2009]: for certain second-order operators $\tilde{\mathcal{O}}(p)$

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② characteristic polynomial of $\mathbf{A}_p(L)$ size $\mathcal{O}(1)$

- [B., Caruso & Schost 2014]: subliniar algorithm $\tilde{\mathcal{O}}(\sqrt{p})$

③ polynomial solutions of L size $O(p)$

- [B. & Schost 2009]: quasi-optimal algorithm $\tilde{\mathcal{O}}(p)$

New result
(B.-Caruso-Schost, 2015)

Computation of $\mathbf{A}_p(L)$

for an arbitrary operator L

in quasi-linear time $\tilde{\mathcal{O}}(p)$.

▷ Precise complexity result for L of bidegree (d, r) in (x, ∂):

$$\tilde{O}(p\, d\, r^{\omega})$$

where ω is the exponent of matrix multiplication.

▷ Optimality: for $r > 1$, generic size of $\mathbf{A}_p(L)$ is $\Theta(p\, d\, r^2)$

▷ Extension to systems: same results for p-curvature of $Y' = AY$

The starting point

- Question: Given L in $k(x)\langle\partial\rangle$, compute R in $k(x)\langle\partial\rangle$ such that

$$\partial^p = QL + R, \quad \operatorname{ord}(R) < \operatorname{ord}(L) = r$$

- Idea: *evaluation-interpolation*; on "points" = solutions of L

▷ Fruitful strategy in related contexts: product, lclm, gcrd (char. 0)
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 If L had a full basis of power series solutions $\{y_1, \dots, y_r\}$, then $R = \sum_{j=0}^{r-1} a_j(x)\partial^j$ could be determined by solving a linear system

$$(a_0, \dots, a_{r-1}) \cdot \text{Wronskian}(y_1, \dots, y_r) = (\partial^p(y_1), \dots \partial^p(y_r))$$

with coefficients in $k[[x]]$ truncated modulo $x^{\mathcal{O}(p)}$

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 Obstruction: Cauchy's theorem *does not hold* in char. $p > 0$

The key: series with divided powers

- $\ell =$ a ring in which p vanishes
- $\ell[[t]]^{\mathrm{dp}} =$ series with divided powers (Hurwitz series)

$$f = a_0\gamma_0(t) + a_1\gamma_1(t) + a_2\gamma_2(t) + \cdots + a_i\gamma_i(t) + \cdots$$

where $a_i \in \ell$ and $\gamma_i(t) \cdot \gamma_j(t) = \binom{i+j}{i} \gamma_{i+j}(t)$.

Theorem (Cauchy's theorem for Hurwitz series)
For any $r \times r$ matrix A with coefficients in $\ell[[t]]^{\mathrm{dp}}$, and for any initial data $V \in \ell^r$, the Cauchy problem

$$\begin{cases} Y' = A \cdot Y \\ Y(0) = V \end{cases}$$

has a unique solution in $\ell[[t]]^{\mathrm{dp}}$.

Efficient computation with divided powers

Theorem For $N = np^s$, with $s \geq 0$ and $n \in \{1, \ldots, p\}$, there is a canonical isomorphism of ℓ-algebras:

$$\ell[[t]]^{\mathrm{dp}} / \ell[[t]]^{\mathrm{dp}}_{\geq N} \simeq \ell[t_0, \ldots, t_s] / (t_0^p, \ldots, t_{s-1}^p, t_s^n).$$

Proof: Send $\gamma_{p^i}(t)$ to t_i and use Lucas' theorem. If $n = \sum_{i=0}^{s} n_i p^i$,

$$\gamma_n(t) = \gamma_{n_0}(t) \cdot \gamma_{n_1 p}(t) \cdots \gamma_{n_s p^s}(t) \quad \mapsto \quad \frac{t_0^{n_0}}{n_0!} \cdot \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_s^{n_s}}{n_s!}.$$

Theorem The product in $\ell[[t]]^{\mathrm{dp}}$ at precision $N = p^{\mathcal{O}(1)}$ can be performed with $\tilde{O}(N)$ operations in k.

Proof: Use Kronecker's substitution + univariate FFT.

Fast differential system solving in divided powers

- Newton iteration [B.-Chyzak-Ollivier-Salvy-Schost-Sedoglavic'07]

Input: a differential system $Y' = AY$, an integer N
Output: the fundamental system of solutions in $\ell[[t]]^{\mathrm{dp}}_{\geq N}$

1. $Y = I_r + t\,A(0)$; $Z = I_r$; $m = 2$
2. **while** $m \leq N/2$:
3. $\quad Z = Z + \lceil Z(I_r - YZ) \rceil^m$
4. $\quad Y = Y - \left\lceil Y\left(\int Z \cdot (Y' - \lceil A \rceil^{2m-1} Y) \right) \right\rceil^{2m}$
5. $\quad m = 2m$
6. **return** Y

Theorem Solving $Y' = AY$ in $\ell[[t]]^{\mathrm{dp}}$ at precision $N = p^{\mathcal{O}(1)}$ can be performed with $\tilde{O}(N\,r^\omega)$ operations in ℓ.

Example (continued)

$$L = (5x^2+4)\partial^2 + (4x^2+6x+5)\partial + 2x + 2 \in \mathbb{F}_7[x]\langle\partial\rangle$$

- basis of divided power solutions of L in $\mathbb{F}_7[[x]]^{\mathrm{dp}}$:

$$y_1 = \gamma_0 + 3\gamma_2 + \gamma_3$$

$$y_2 = \gamma_0 + 4\gamma_2 + \gamma_4 + 2\gamma_5 + 4\gamma_6 + \gamma_7 + 2\gamma_8 + 4\gamma_9 + \gamma_{10} + 2\gamma_{11} + 4\gamma_{12} + \cdots$$

- $\partial^7 = QL + R$, with $R = a_0 + a_1\partial$, implies

$$\begin{bmatrix} a_0 & a_1 \end{bmatrix} \cdot \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} y_1^{(7)} & y_2^{(7)} \end{bmatrix}$$

with solution

$$a_0 = 4x + 2x^2 + 5x^3 + 2x^4 + 4x^5 + 5x^6 + \cdots = \frac{4x(x-1)}{(x+3)(x-3)^2}$$

$$a_1 = 1 + 5x + 6x^2 + x^3 + 6x^4 + 5x^5 + x^6 + \cdots = \frac{(x+1)(x^2+x-1)}{(x+3)(x-3)^2}$$

The algorithm in a nutshell

Input: a differential system $Y' = AY$, with $A \in \mathscr{M}_r(k[x][\frac{1}{f}])$
Output: its p-curvature $\mathbf{A}_p$ (def: $\mathbf{A}_{k+1} = \mathbf{A}_k' + \mathbf{A}_1 \cdot \mathbf{A}_k$, $\mathbf{A}_1 = -A$)

1. Choose $S \in k[x]$ separable and coprime with f
   Let $\ell = k[x]/S$ and $\varphi_S : k[x][\frac{1}{f}]/S^p \xrightarrow{\sim} \ell[t]/t^p, \quad x \mapsto t + x$
2. Compute a fundam. matrix $Y_S \in \mathscr{M}_r(\ell[t]/t^p)$ of $Y' = \varphi_S(A)Y$
   Cost: $\tilde{O}(pr^\omega)$ ops. in ℓ
3. Deduce $\mathbf{A}_p \bmod S^p$ as $\varphi_S^{-1}\left(Y_S \cdot \mathrm{coeff}(AY_S,\, p{-}1) \cdot Y_S^{-1}\right)$
   Cost: $\tilde{O}(pr^\omega)$ operations in ℓ
4. Use several S's and CRT to get $\mathbf{A}_p$ from the various $\mathbf{A}_p \bmod S^p$

Total cost: $\tilde{O}(pr^\omega d)$ operations in k, where $d = \deg(A)$

Experimental results

- For random linear differential operators in $k[x]\langle\partial\rangle$

(d, r)	p							
	157	**281**	**521**	**983**	**1 811**	**3 433**	**6 421**	**12 007**
(5, 5)	0.39 s	0.71 s	1.22 s	2.34 s	4.41 s	8.93 s	18.0 s	36.1 s
	0.26 s	0.76 s	2.69 s	9.05 s	32.6 s	145 s	593 s	2 132 s
(5, 11)	1.09 s	2.05 s	3.65 s	7.05 s	12.6 s	26.7 s	53.3 s	109 s
	1.25 s	3.70 s	12.8 s	45.5 s	163 s	725 s	2 942 s	–
(5, 20)	2.93 s	5.25 s	9.52 s	17.7 s	32.5 s	68.1 s	139 s	288 s
	4.29 s	12.4 s	42.5 s	153 s	548 s	2 460 s	–	–
(11, 20)	6.89 s	13.3 s	22.6 s	45.0 s	80.4 s	167 s	342 s	711 s
	11.6 s	34.7 s	121 s	486 s	1 943 s	–	–	–
(20, 20)	14.0 s	25.1 s	49.9 s	94.0 s	176 s	357 s	733 s	1 472 s
	27.0 s	84.5 s	314 s	1 283 s	–	–	–	–

Running times of the new algorithm, vs. Katz's algorithm

- For operators with physical relevance: e.g., $\phi_H^{(5)}$ in $(\mathbb{Z}/27449\,\mathbb{Z})[x]\langle\partial\rangle$, with $(d, r) = (108, 28)$ [Maillard et al. 2007]

$\longrightarrow$ (first column of) $\mathbf{A}_p(\phi_H^{(5)})$ in 19 hours, size $\approx$ 1GB

Conclusion

This work:

- Computation of p-curvature $\mathbf{A}_p(L)$ in quasi-optimal time $\tilde{\mathcal{O}}(p)$
- Basic tool: *evaluation/interpolation* on *Hurwitz series*

Next challenges:

- Compute invariant factors of $\mathbf{A}_p(L)$ in time $\tilde{\mathcal{O}}(\sqrt{p})$
- Factor L in time $\tilde{\mathcal{O}}(p)$

Thanks for your attention!