

# Removing Apparent Singularities of Linear Differential Systems with Rational Function Coefficients

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## Notation-Vocabulary

System of first order linear differential equations:

$$[A] \quad \frac{d}{dz}X = A(z)X,$$

where  $X = (x_1, \dots, x_n)^T$  is column-vector of length  $n$ .

$A(z)$  is an  $n \times n$  matrix with entries in  $\mathbf{K} = \mathbb{C}(z)$ .

The (finite) **singularities** of system  $[A]$  are the poles of the entries of  $A(z)$ .

Scalar linear differential equation of order  $n$ :  $L(y) = 0$

$$L = \partial^n + c_{n-1}(z)\partial^{n-1} + \dots + c_0(z) \in \mathbf{K}[\partial]$$

The (finite) **singularities** of  $L$  are the poles of the  $c_i$ 's.

## Apparent singularities

Consider linear complex differential equations  $L(x(z)) = 0$ ,  $L \in \mathbb{C}(z)[\frac{d}{dz}]$ .

- ▶ Singularities of solutions of  $L(x) = 0$  are necessarily singularities of the coefficients of  $L$ , but the converse is not always true.

**Def.** An **apparent singularity** of  $L$  is a singular point where the general solution of  $L(y) = 0$  is holomorphic.

**Example.**  $L(x) = \frac{dx}{dz} - \frac{\alpha}{z}x = 0$ ,  $\alpha \in \mathbb{C}$ .

- ▶ The general solution of  $L$  is

$$x(z) = cz^\alpha, \quad c \in \mathbb{C}.$$

- ▶ When  $\alpha \in \mathbb{N}$ , the general solution of  $L(x) = 0$  is holomorphic at  $z = 0$ .
- ▶ When  $\alpha \in \mathbb{N}$ , the point  $z = 0$  is an apparent singularity of  $L$ .

## A simple example

- ▶ Given the first-order differential system in the complex variable  $z$

$$[A] \quad \frac{d}{dz}X = A(z)X, \quad A(z) = \begin{bmatrix} 0 & 1 \\ \frac{-2}{z} & 1 + \frac{2}{z} \end{bmatrix}.$$

- ▶ The pole  $z = 0$  of  $A(z)$  is a singularity of system  $[A]$ .
- ▶ This system is equivalent to the second-order scalar differential equation:

$$L := \frac{d^2}{dz^2} - \frac{z+2}{z} \frac{d}{dz} + \frac{2}{z}.$$

- ▶ The general solution of  $L(x(z)) = 0$  is given by

$$c_1 e^z + c_2 \left( 1 + z + \frac{z^2}{2} \right) \quad c_1, c_2 \in \mathbb{C}$$

which is holomorphic (in a neighborhood of  $z = 0$ ).

- ▶ We thus say that  $z = 0$  is an **apparent singularity**.

## Desingularization of Scalar Equations

- ▶ consists classically of constructing another operator  $\tilde{L}$  of higher order such that
  - ▶ the solution space of  $\tilde{L}(x(z)) = 0$  contains that of  $L(x(z)) = 0$ ,
  - ▶ the singularities of  $\tilde{L}$  are exactly the *real* singularities of  $L$ .
- ▶ Several algorithms have been developed for linear differential (and more generally Ore) operators, e.g.
  - ▶ Abramov-Barkatou-van Hoeij'2006,
  - ▶ Chen-Jaroscsek-Kauers-Singer'2013, Chen-Kauers-Singer'2015

## A simple example

Consider the second order operator  $L := \partial^2 - \frac{(z+2)}{z}\partial + \frac{2}{z}$ .

- ▶  $z = 0$  is a singularity of  $L$ .
- ▶ The general solution of  $L(y) = 0$  is given by

$$c_1 e^z + c_2 \left(1 + z + \frac{z^2}{2}\right) \quad c_1, c_2 \in \mathbb{C}.$$

- ▶  $L$  has an apparent singularity at  $z = 0$ .
- ▶ The desingularization computed by ABH method is **of order 4**

$$\tilde{L} = \partial^4 + \left(-1 + \frac{z}{4}\right) \partial^3 + \left(-\frac{1}{4} - \frac{3z}{8}\right) \partial^2 + \left(\frac{1}{2} + \frac{z}{8}\right) \partial - \frac{1}{4}$$

- ▶ The apparent singularity of  $L$  at  $z = 0$  can be removed by computing a *gauge equivalent* first-order differential system with coefficient in  $\mathbb{C}(z)$  of **size**  $\text{ord}(L) = 2$ .

- ▶ Consider the first-order differential system associated with  $L$

$$[A] \quad \frac{d}{dz} X = A(z)X, \quad A(z) = \begin{bmatrix} 0 & 1 \\ \frac{-2}{z} & 1 + \frac{2}{z} \end{bmatrix}.$$

- ▶ Set

$$X = T(z) Y, \quad \text{where} \quad T(z) = \begin{bmatrix} 1 & 0 \\ 1 & z^2 \end{bmatrix}.$$

- ▶ The new variable  $Y$  satisfies the **gauge equivalent** first-order differential system of the same dimension given by

$$[B] \quad \frac{d}{dz} Y = B Y$$

where

$$B := T^{-1}AT - T^{-1}\frac{d}{dz}T = \begin{bmatrix} 1 & z^2 \\ 0 & 0 \end{bmatrix}.$$

# What I am going to talk about

## Main goal:

- ▶ Given any system  $[A]$  with rational coefficients, it can be reduced to a *gauge equivalent* system  $[B]$  with rational coefficients, such that the finite singularities of  $[B]$  coincide with the non-apparent singularities of  $[A]$ .

## Outline:

1. Detecting and removing the apparent singularities
2. Application to desingularization of scalar equations
3. The rational version of the new algorithm
4. Examples and Conclusion

## General Framework

- ▶  $\mathbb{Q} \subseteq k \subset \bar{k} \subset \mathbb{C}$
- ▶ For simplicity,  $k = \mathbb{C}$ .

Given a System of first order linear differential equations:

$$[A] \quad \frac{d}{dz}X = A(z)X,$$

where

- ▶  $X = (x_1, \dots, x_n)^T$  is column-vector of length  $n$ ;
- ▶  $A(z)$  is an  $n \times n$  matrix with entries in  $\mathbb{C}(z)$ ;

The poles of  $A(z)$  are the finite singularities of system  $[A]$ .

## Classification of Singularities

- ▶ If  $z_0$  is not a pole of  $A(z)$  then the point  $z_0$  is an **ordinary point** of system  $[A]$ . There exists a fund soln matrix  $W$  whose entries are holomorphic in some neighborhood of  $z_0$ .
- ▶ The point  $z_0$  is a **regular singular point** for  $[A]$  if it is a simple pole of  $A(z)$  or it can be reduced to a simple pole by a gauge transformation. Every fund soln matrix  $W$  of  $[A]$  has the form:

$$W(z) = \Phi(z)(z - z_0)^\Lambda$$

where  $\Phi(z)$  is holomorphic and  $\Lambda$  is a constant matrix. (cf. Wasow)

- ▶ Otherwise  $z_0$  is called an **irregular singular point**.
- ▶ In particular, if  $z_0$  is a point of singularity of system  $[A]$  but there exists a fund soln matrix  $W$  whose entries are holomorphic in some neighborhood of  $z_0$ , then  $z_0$  is an **apparent singularity** (cf. intro example).
- ▶ The change of variable  $z \mapsto \frac{1}{z}$  permits to classify the point  $z = \infty$  as an ordinary, regular singular or irregular singular point for  $[A]$ .

## Desingularization of a First Order System

The nature of a singular point  $z_0$ , whether regular, irregular, or apparent, is thus based upon the knowledge of fundamental solution matrix and hence is not immediately checkable for a given system.

$$[A] \quad \frac{d}{dz} X = A(z)X$$

A system

$$[B] \quad \frac{d}{dz} Y = B(z)Y$$

with  $B \in \mathbb{C}(z)^{n \times n}$  is called a **desingularization** of  $[A]$  if:

- (i) there exists a polynomial matrix  $T(z)$  with  $\det T(z) \neq 0$  such that  $B = T[A]$ ,
- (ii) The singularities of  $[B]$  are the singularities of  $[A]$  that are not apparent.

## Existence of Desingularization

**Prop.0** If  $z = z_0$  is a finite apparent singularity of  $[A]$  then there exists a polynomial matrix  $T(z)$  with

$$\det T(z) = c(z - z_0)^\alpha, \quad c \in \mathbb{C}^*, \alpha \in \mathbb{N}$$

such that  $[B] := T[A]$  has no pole at  $z = z_0$ .

### Proof.

- ▶ Every fund soln matrix  $W$  of  $[A]$  has the form:  $W(z) = \Phi(z)$  where  $\Phi(z)$  is holomorphic (in a neighborhood of  $z_0$ );
- ▶ Since  $\mathbb{C}[[z - z_0]]$  is a Principal Ideal Domain, there exists unimodular transformations  $P(z) \in GL_n(\mathbb{C}[z])$ , and  $Q(z) \in GL_n(\mathbb{C}[[z - z_0]])$  such that

$$P(z)\Phi(z)Q(z) = \text{Diag}((z - z_0)^{\alpha_1}, \dots, (z - z_0)^{\alpha_n})$$

where  $\alpha_1, \dots, \alpha_n$  are nonnegative integers.

- ▶  $T(z) = P^{-1}(z) \text{Diag}((z - z_0)^{\alpha_1}, \dots, (z - z_0)^{\alpha_n})$

## How to detect and remove an apparent singularity?

**Prop.1:** If  $z = z_0$  is a finite apparent singularity of  $[A]$  then one can construct a polynomial matrix  $T(z)$  with  $\det T(z) = c(z - z_0)^\alpha$ ,  $c \in \mathbb{C}^*$  and  $\alpha \in \mathbb{N}$  such that  $T[A]$  has at worst a simple pole at  $z = z_0$ .

- ▶ We use the fact that a system with a regular singularity at  $z_0$  is equivalent to a system with a simple pole at  $z_0$ . This system can be constructed using the so called *Moser rational algorithm* (Bar'1995).

**Prop.2:** Suppose that  $A(z)$  has simple pole at  $z = z_0$  and let

$$A(z) = \frac{A_0}{(z - z_0)} + \sum_{k \geq 1} A_k (z - z_0)^{k-1}, \quad A_k \in \mathbb{C}^{n \times n}.$$

If  $z_0$  is an apparent singularity then the eigenvalues of  $A_0$  are nonnegative integers and  $A_0$  is diagonalizable.

**Remark:**

- ▶ When  $A_0$  is not diagonalizable, the local solution at  $z_0$  involve logarithmic terms.

**Prop.3:** Suppose that  $z = z_0$  is a simple pole of  $A(z)$  and that its residue matrix  $A_0$  has only nonnegative integer eigenvalues. Then one can construct a polynomial matrix  $T(z)$  with

$$\det T(z) = c(z - z_0)^\alpha$$

for some  $c \in \mathbb{C}^*$  and  $\alpha \in \mathbb{N}$  such that

$$B := T[A] = B_0(z - z_0)^{-1} + \dots$$

has at worst a simple pole at  $z = z_0$  with

$$B_0 = mI_n + N$$

where  $m \in \mathbb{N}$  and  $N$  nilpotent.

- ▶ Moreover  $z_0$  is an apparent singularity iff  $N = 0$ .
- ▶ In this case the gauge transformation  $Y = (z - z_0)^m \tilde{Y}$  leads to a system for which  $z = z_0$  is an ordinary point.

## Main idea of the proof:

- ▶ The eigenvalues of  $A_0$  are nonnegative integers:

$$m_1 > m_2 > \dots > m_s, \quad m_i - m_{i+1} = \ell_i \in \mathbb{N}^*, \quad i = 1, \dots, s-1.$$

- ▶ By applying a constant gauge transformation we can assume that:

$$A_0 = \begin{pmatrix} A_0^{11} & 0 \\ 0 & A_0^{22} \end{pmatrix},$$

where  $A_0^{11}$  is an  $\nu_1$  by  $\nu_1$  matrix having one single eigenvalue  $m_1$

$$A_0^{11} = m_1 I_{\nu_1} + N_1$$

$N_1$  being a nilpotent matrix.

- ▶ By applying polynomial transformations of the form  $\text{Diag}((z - z_0)I_{\nu_1}, I_{n-\nu_1})$  one can decrease  $m_1$  by 1 successively until  $m_1 = m_2$ . The same can then be repeated for  $m_2, \dots, m_s$ :



- Due to the form of its determinant, the gauge transformation  $T(z)$  in the above proposition **does not affect the other finite singularities** of  $[A]$ . We have:

**Theorem** One can construct a polynomial matrix  $T(z)$  which is invertible in  $\mathbb{C}(z)$  such that the finite poles of  $B := T[A]$  are exactly the poles of  $A$  that are not apparent singularities for  $[A]$ .

## Remark

- ▶ If the point at infinity of the original system is singular regular then it will be also singular regular of the the computed desingularization but the order of the pole at infinity may increase.

## Algorithm

1. Let  $\mathcal{P}(A)$  be the set of poles of  $A$ .
2. Compute a polynomial matrix  $T(z)$  such that
  - ▶ the zeros of  $\det T(z)$  are in  $\mathcal{P}(A)$
  - ▶  $T[A]$  has the same poles as  $A$  with minimal orders.
3. For each simple pole  $z_0$  compute  $A_{0,z_0}$  the residue matrix of  $A(z)$  at  $z = z_0$  and its eigenvalues.
4. Let  $\mathcal{App}(A)$  denote the set of singularities  $z_0$  such that  $A_{0,z_0}$  has only nonnegative integer eigenvalues.
5. For each  $z_0 \in \mathcal{App}(A)$  compute a polynomial matrix  $T_{z_0}(z)$  with  $\det T_{z_0}(z) = c(z - z_0)^\alpha$  such that  $T_{z_0}[A]$  has at worst a simple pole at  $z = z_0$  with residue matrix of the form  $R_{z_0} = m_{z_0} I_n + N_{z_0}$  where  $m_{z_0} \in \mathbb{N}$  and  $N_{z_0}$  nilpotent.
6. Keep in  $\mathcal{App}(A)$  only the points  $z_0$  for which  $N_{z_0} = 0$ .
7. The scalar transformation  $T = \prod_{z_0 \in \mathcal{App}(A)} (z - z_0)^{m_{z_0}} I_n$  yields a desingularization of the original system  $[A]$ .

# Application to Desingularization of Scalar Differential Equations

## Example 5

- ▶ Let  $\partial = \frac{d}{dz}$  and consider

$$L = \partial^2 - \frac{(z^2 - 3)(z^2 - 2z + 2)}{(z - 1)(z^2 - 3z + 3)z} \partial + \frac{(z - 2)(2z^2 - 3z + 3)}{(z - 1)(z^2 - 3z + 3)z}.$$

- ▶  $L$  has apparent singularities at  $z = 0$  and the roots of  $z^2 - 3z + 3 = 0$ .
- ▶ A desingularization computed by the classical algorithm\* is given by:

$$\begin{aligned} \tilde{L}_{\text{Classical}} = & (z - 1)(z^4 - z^3 + 3z^2 - 6z + 6)\partial^4 \\ & - (z^5 - 2z^4 + z^3 - 12z^2 + 24z - 24)\partial^3 \\ & - (3z^3 + 9z^2)\partial^2 + (6z^2 + 18z)\partial - (6z + 18). \end{aligned}$$

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\*Exm 1, Chen-Kauers-Singer'14

- ▶ A desingularization computed by the probabilistic method of CKS14<sup>†</sup> is given by:

$$\begin{aligned} \tilde{L}_{CKS} = & (z - 1)(z^6 - 3z^5 + 3z^4 - z^3 + 6) \partial^4 \\ & - (2z^6 - 9z^5 + 15z^4 - 11z^3 + 3z^2 - 24) \partial^3 \\ & - (z^7 - 4z^6 + 6z^5 - 4z^4 + z^3 + 6z - 6) \partial \\ & + (2z^6 - 9z^5 + 15z^4 - 11z^3 + 3z^2 - 24). \end{aligned}$$

- ▶ The removal of one apparent singularity introduces new singularities. The latter can then be removed by using a trick introduced in ABH algorithm.

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<sup>†</sup>Exm 7(1), Chen-Kauers-Singer'14

- The desingularization computed by ABH method is:

$$\begin{aligned} \tilde{L}_{ABH} = & \partial^4 + \frac{(16z^4 - 55z^3 + 63z^2 - 42z + 36)}{9(z-1)} \partial^3 \\ & - \frac{(64z^5 - 316z^4 + 591z^3 - 468z^2 + 123z + 42)}{9(z-1)^2} \partial^2 \\ & - \frac{96z^5 - 570z^4 + 1333z^3 - 1597z^2 + 993z - 219}{9(z-1)^3} \\ & + \frac{\beta}{9(z-1)^3} \partial, \end{aligned}$$

where

$$\beta = (48z^6 - 197z^5 + 148z^4 + 488z^3 - 1162z^2 + 999z - 288).$$

- ▶ The companion matrix of  $L$  is

$$A = \begin{bmatrix} 0 & 1 \\ \frac{(z-2)(2z^2-3z+3)}{(z-1)(z^2-3z+3)z} & \frac{(z^2-3)(z^2-2z+2)}{(z-1)(z^2-3z+3)z} \end{bmatrix}$$

- ▶ Our new algorithm computes the following gauge transformation  $T$

$$T = \begin{bmatrix} 1 & 0 \\ 1 & (-z^2 + 3z - 3)z^2 \end{bmatrix}$$

- ▶ The matrix of the new equivalent system is

$$B = T^{-1}(AT - T') = \begin{bmatrix} 1 & -z^2(z^2 - 3z + 3) \\ 0 & \frac{2}{1-z} \end{bmatrix}$$

- ▶ It has  $z = 0$  and roots of  $z^2 - 3z + 3 = 0$  as ordinary points.
- ▶ No new apparent singularities are introduced.

## Comments

- ▶ The desingularization algorithms developed specifically for scalar equations are based on computing a least common left multiple of the operator in question and an appropriately chosen operator.
- ▶ This outputs in general an equation whose solution space contains strictly the solution space of the input equation.
- ▶ The new algorithm is based on an adequate choice of a gauge transformation.
- ▶ The desingularized output system is always equivalent to the input system and the dimension of the solution space is preserved.
- ▶ However, a scalar differential equation equivalent to the desingularized system would generally feature apparent singularities.
- ▶ The transformations and the equivalent systems computed by our algorithm, have rational function coefficients.

## Key component of the “rational” algorithm

- ▶ Given a system  $\frac{d}{dz}X = A(z)X$  with  $A \in k(z)^{n \times n}$  ( $\mathbb{Q} \subseteq k \subset \bar{k} \subset \mathbb{C}$ ).
- ▶  $\Omega = \{\alpha_1, \dots, \alpha_d\} \subset \mathbb{C}$ : a set of conjugate apparent singularities.
- ▶  $p(z) = \prod_{i=1}^d (z - \alpha_i) \in k[z]$  irreducible polynomial.

- ▶  $p$ -adic expansion:

$$A(z) = \frac{1}{p} (A_{0,p} + pA_{1,p} + \dots)$$

- ▶  $\alpha_i$ -Laurent expansion:

$$A(z) = \frac{1}{(z - \alpha_i)} (A_{0,\alpha_i} + (z - \alpha_i)A_{1,\alpha_i} + \dots), \quad 1 \leq i \leq d.$$

- ▶ Then we have,

$$\frac{1}{\frac{dp}{dz}(\alpha_i)} A_{0,p}(\alpha_i) = A_{0,\alpha_i}, \quad 1 \leq i \leq d.$$

## Definition

- ▶ The matrix given by

$$\frac{A_{0,p}(z)}{\frac{dp}{dz}} \in (k[z]/(p))^{n \times n}$$

is called the **residue matrix** of  $A(z)$  at  $p$ .

- ▶  $R_{0,p}(z)$  is its representative in  $k[z]^{n \times n}$ .
- ▶ The latter is of degree strictly less than  $d$  and can be computed as

$$u A_{0,p} \bmod p$$

where  $u$  denotes the inverse of  $\frac{dp}{dz} \bmod p$ .

## Example 6

$$\frac{d}{dz}X = A(z)X = \frac{1}{1+z^2} \begin{bmatrix} 1-z & z \\ -z & 1+z \end{bmatrix} X.$$

- ▶  $p = 1 + z^2$  is an irreducible polynomial over  $\mathbb{Q}[z]$  and its roots are given by  $\pm i$  over  $\mathbb{Q}(i)$ .
- ▶  $u = -\frac{z}{2}$  is the inverse of  $\frac{dp}{dz} \bmod p$ .
- ▶  $R_{0,p}(z)$  is given by  $u A_{0,p} \bmod p$ :

$$R_{0,p}(z) = \frac{1}{2} \begin{bmatrix} 1-z & -1 \\ -1 & -1-z \end{bmatrix}.$$

- ▶ Indeed, one can verify that the residue matrices at  $\pm i$ , are given by

$$A_{0,i} = \begin{bmatrix} \frac{1-i}{2i} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1+i}{2i} \end{bmatrix} \quad \text{and} \quad A_{0,-i} = \begin{bmatrix} \frac{-1-i}{2i} & \frac{1}{2} \\ -\frac{1}{2} & \frac{-1+i}{2i} \end{bmatrix}.$$

## Summary

- ▶ We gave a method for detecting and removing the apparent singularities of linear differential systems via a rational algorithm, i.e. an algorithm which avoids the computations with individual conjugate singularities.
- ▶ Our method can be used, in particular, for the desingularization of differential operators in the scalar case.
- ▶ Maple Package available for download at:

*[http://www.unilim.fr/pages\\_perso/suzy.maddah/Research.html](http://www.unilim.fr/pages_perso/suzy.maddah/Research.html)*

- ▶ More examples can be found there:
  - ▶ Desingularization at polynomial of degree 4: The Ising Model<sup>‡</sup> in statistical physics.
  - ▶ Desingularization at polynomial of degree 37.

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<sup>‡</sup>Bostan-Boukraa-Hassani-van Hoeij-Maillard-Weil-Zenine

## Further investigations

- ▶ The complexity study of the various algorithms existing for the scalar case, as well as this new algorithm which can be applied to the companion system, so that their efficiency can be compared. Partial results are given in BP'09.
- ▶ The generalization of our algorithm to treat more general systems, e.g. systems with parameters as well as investigating the case of difference systems. First steps in this direction, namely reductions in the parameter and the partial desingularization, are established in BBP'07 (Regular Systems of Linear Functional Equations and Applications) and ABM'14 (Reduction of Singularly-Perturbed Linear Differential Systems) respectively.