Error-Correcting Sparse Interpolation in the Chebyshev Basis

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Introduction

Let $T_n(x)$ be the nth Chebyshev polynomial of the first kind:

$$T_0 = 1, \quad T_1 = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad \text{for } n \geq 2$$

E.g. $T_2 = 2x^2 - 1, \quad T_3 = 4x^3 - 3x, \quad T_4 = 8x^4 - 8x^2 + 1, \quad \ldots$

Useful Properties:

- $T_n(T_m(x)) = T_{mn}(x)$
- $T_m(x)T_n(x) = \frac{1}{2}(T_{m+n}(x) + T_{|m-n|}(x))$
- $T_n(\frac{x+x^{-1}}{2}) = \frac{1}{2}(x^n + x^{-n})$
- Over $\mathbb{R}$, for $|\xi| > 1$, $T_n(\xi) \neq 0$

We are interested in the interpolation and error-correcting properties of sparse Chebyshev polynomials (i.e. polynomials that are sparse in the Chebyshev basis).

Contributions

Covered in this talk

An error-correcting black-box interpolation algorithm for sparse Chebyshev polynomials.

Not covered in this talk

An alternative sparse Chebyshev interpolation algorithm for $f \in \mathsf{K}[x]$, $\mathrm{char}(\mathsf{K}) \neq 2$, that reduces the problem to sparse interpolation in the power basis, i.e. $1, x, x^2, x^3, \ldots$.

- Allows for early termination (Kaltofen, Lee; 2003), such that we can (probabilistically) interpolate f with t terms, with cost sensitive to t even when bounds for t are not supplied as input.

The Problem

- Suppose $f \in \mathbb{R}[x]$ is of the form

$$f = \sum_{i=1}^{t} \underbrace{c_i T_{\delta_i}(x)}_{c_i \neq 0 \text{ and with } \delta_1 > \delta_2 > \ldots > \delta_t}$$

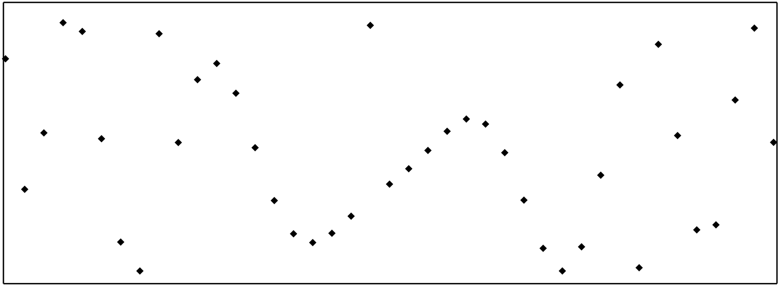
- We are given a black box $\blacksquare$ for f. For $j = 0, 1, \ldots, L-1$, we query $\blacksquare$ with x_j and get back a_j, where either $a_j = f(x_j)$ or a_j is an erroneous evaluation.

- Problem: reconstruct f and identify errors, given $\blacksquare$ and bounds

$$B \geq t, \quad D \geq \deg(f), \quad E \geq \#\{\text{errors}\}$$

while minimizing, L the number of queries to $\blacksquare$.

Interpolating with Errors - Example



E.g. $\deg(f) \le D = 19, \quad \le B = 3$ terms, $\quad \le E = 5$ errors

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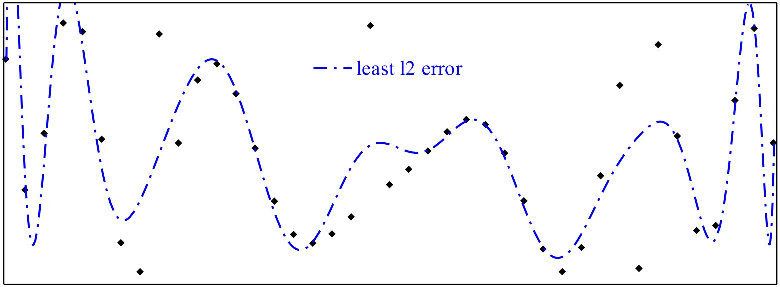

E.g. $\deg(f) \leq D = 19, \quad \leq B = 3$ terms, $\quad \leq E = 5$ errors

- Minimizing ℓ_2-error gives a dense approximation for the model,

$$0.786462\,T_{19} - 0.253808\,T_{19} - 0.270838\,T_{18} + 0.101009\,T_{16} + 0.206344\,T_{15} - 0.135857\,T_{15} - 0.076361\,T_{14} +$$
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$$0.753950\,T_{5} - 0.112232\,T_{5} - 1.388821\,T_{4} + 1.025795\,T_{2} + 1.364547\,T_{1} + 3.325460\,T_{0}$$

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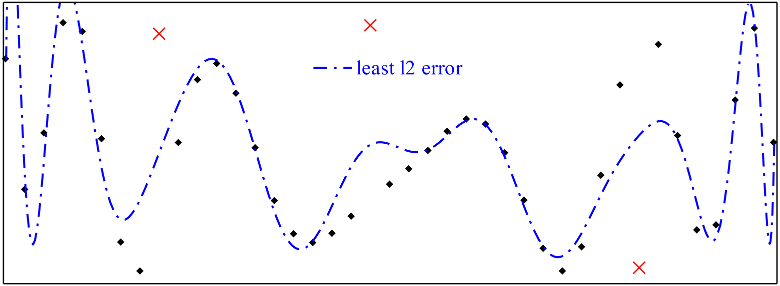

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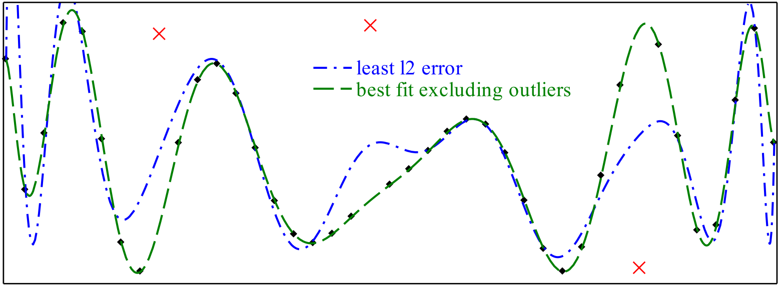

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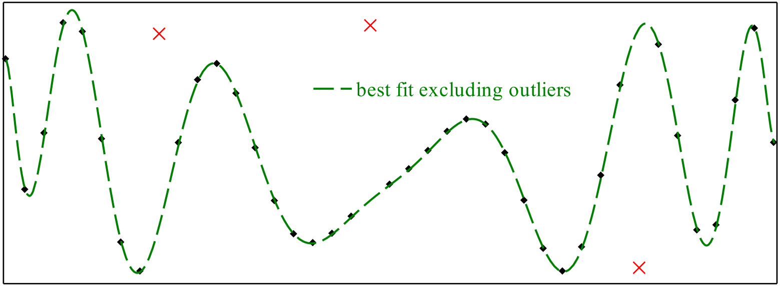

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Related Work - Sparse Interpolation in Power Basis

Theorem (Prony; 1795)

Let K *be a field,* $f(x) = \sum_{i=1}^{t} c_i x^{e_i} \in \mathsf{K}[x]$, $\omega \in \mathsf{K}$, *and*

$$\Phi(y) = \prod_{i=1}^{t}(y - \omega^{e_i}) = y^t + \sum_{i=0}^{t-1} \phi_i y^i.$$

Then for $a_j = f(\omega^j)$,

$$\underbrace{\begin{bmatrix} a_0 & a_1 & \cdots & a_{t-1} \\ a_1 & a_2 & \cdots & a_t \\ \vdots & \vdots & \ddots & \vdots \\ a_{t-1} & a_t & \cdots & a_{2t-2} \end{bmatrix}}_{=\mathcal{H}_t} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{t-1} \end{bmatrix} = \begin{bmatrix} a_t \\ a_{t+1} \\ \vdots \\ a_{2t-1} \end{bmatrix}.$$

- i.e., exponents of f are encoded in a solution to a Hankel system.
- Corollary: $\mathcal{H}_{t'}$ is singular for $t' > t$.

Prony's Method for Interpolation in the Power Basis[1]

Inputs: Error-free $\blacksquare$ for t-sparse $f \in \mathsf{K}[x]$, bounds $B \geq t$, $D \geq \deg(f)$

1. Choose $\omega \in \mathsf{K}$ of order $> D$. Let $x_j = \omega^j$.
   Evaluate $a_j = \blacksquare(x_j)$ for $j = 0, 1, \ldots, 2B - 1$.

[1]See also Ben-Or–Tiwari; 1988

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5. Obtain the coefficients c_i as the solution to the Vandermonde system

$$\begin{bmatrix} x_0^{e_1} & x_0^{e_2} & \ldots & x_0^{e_t} \\ x_1^{e_1} & x_1^{e_2} & \ldots & x_1^{e_t} \\ \vdots & \vdots & \ddots & \vdots \\ x_{t-1}^{e_1} & x_{t-1}^{e_2} & \ldots & x_{t-1}^{e_t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_t \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{t-1} \end{bmatrix},$$

   and output $f = \sum_{i=1}^{t} c_i x^{e_i}$.

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Related Work - Sparse Chebyshev Interpolation

Theorem (Lakshman, Saunders; 1995)

Let $f(x) = \sum_{i=1}^{t} c_i T_{\delta_i}(x) \in \mathbb{R}[x], \quad \xi > 1$, *and*

$$\Phi(y) = \prod_{i=1}^{t}(y - T_{\delta_i}(\xi)) = y^t + \sum_{i=0}^{t-1} \phi_i y^i.$$

Then for $a_j = f(T_j(\xi))$,

$$\Big(\underbrace{\left[a_{i+j}\right]_{i,j=0}^{t-1}}_{\mathcal{H}_t} + \underbrace{\left[a_{|i-j|}\right]_{i,j=0}^{t-1}}_{\mathcal{T}_t}\Big) \left[\phi_i\right]_{i=0}^{t} = \left[a_{t+i} + a_{t-i}\right]_{i=0}^{t-1}.$$

- *i.e., indices* δ_i *are encoded in solution to a Hankel+Toeplitz system.*
- *One can show* $\mathcal{H}_{t'} + \mathcal{T}_{t'}$ *is nonsingular for* $t' > t$.

Lakshman–Saunders Sparse Chebyshev Interpolation

Inputs: Error-free ■ for t-sparse $f \in \mathbb{R}[x]$; bounds $B \geq t$, $D \geq \deg(f)$

1. Choose $\xi > 1$. Let $x_j = T_j(\xi)$.
   Evaluate $a_j = ■(x_j)$ for $j = 0, \ldots, 2B - 1$.

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4. Factor $\Phi(y)$ to get $T_{\delta_i}(\xi)$. Determine δ_i from T_{δ_i} for $1 \leq i \leq t$.
5. Obtain coefficients c_i as a solution to the linear system

$$\left[T_{\delta_i}(x_{j-1})\right]_{i,j=1}^{t} \left[c_i\right]_{i=1}^{t} = \left[a_{i-1}\right]_{i=1}^{t},$$

   output $f = \sum_{i=1}^{t} c_i T_{\delta_i}$.

Block-Decoding Error Correction

We can use Prony (or Lakshman–Saunders) as in order to employ error-correcting interpolation in the power (or Chebyshev) basis.

Block Majority-Vote Decoding

- Run Prony or Lakshman–Saunders on $(2E + 1)$ blocks of $2B$ evaluations.
- A majority of blocks will produce the true interpolant f.

Block List Decoding

- Run Prony or Lakshman–Saunders on $(E + 1)$ blocks of $2B$ evaluations.
- One block must produce the true interpolant.

Can we uniquely decode or list decode f with fewer evaluations than block decoding?

Subsequence List Decoding (Kaltofen, Pernet; ISSAC '14)

- Prony's algorithm generalizes for evaluations $\blacksquare(\rho\omega^j), 0 \leq j < 2B$.

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- We can query over $a_j = \blacksquare(\omega^j)$, $j = 0, 1, 2, \ldots, L-1$, and run Prony's algorithm over every geometric subsequence of length $2B$, i.e., $a_r, a_{r+s}, a_{r+2s}, \ldots, a_{r+2Bs}$ for any $r \geq 0, s \geq 1$ with $r + 2Bs < L$.

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- Kaltofen & Pernet showed that this outperforms block list decoding. i.e., we can always choose $L \leq (E+1)2B$.
- For $f \in \mathbb{R}[x], \omega > 0$, and $L \geq 2B + 2E$, if a t-sparse g satisfies $g(\omega^j) = \blacksquare(\omega^j)$ for $0 \leq j < L$ with $\leq E$ exceptions, then by Descartes' rule of signs, g must be f.
  $\Rightarrow$ list decoder uniquely determines f if we use at least $2B + 2E$ evaluations.

Generalization of Lakshman–Saunders Algorithm

In order to extend subsequence list decoding to the Chebyshev basis, we need to be able to interpolate f over choices of subsequences.

Lemma (A, Kaltofen; 2015)

Let $f(x) = \sum_{i=1}^{t} c_i T_{\delta_i}(x) \in \mathbb{R}[x]$ *and fix* $r, s \in \mathbb{Z}$, *and let*

$$\Phi(y) = \prod_{i=1}^{t} (y - T_{s\delta_i}(\xi)) = y^t + \sum_{i=0}^{t-1} \phi_i y^i.$$

Then the sequence $a_j = f(T_j(\xi)), j = 0, 1, \ldots,$ *where* $\xi \in \mathbb{R}$ *satisfies*

$$\underbrace{\left[a_{|r+(i+j)s|} + a_{|r+(i-j)s|}\right]_{i,j=0}^{t-1}}_{A^{(r,s)}} \left[\phi_i\right]_{i=0}^{t-1} = \left[a_{|r+(i+t)s|} + a_{|r+(i-t)s|}\right]_{i=0}^{t-1}$$

This requires evaluations $a_{|rj+s|}$ for $-B \leq j < 2B$.

Generalization of Lakshman–Saunders Algorithm

In order to interpolate f from evaluation points $a_{|rj+s|}, -B \le j < 2B$, we require that $A^{(r,s)} = \left[a_{|r+(i+j)s|} + a_{|r+(i-j)s|}\right]_{i,j=0}^{t-1}$ is nonsingular.

Definition

We say r, s are valid if $|r + is|, 0 \le i < B$, are distinct.

Fact

$A^{(r,s)} = \mathcal{U}\mathcal{B}\mathcal{V}$, *where* $\mathcal{B} = \mathrm{diag}(2c_1, 2c_2, \ldots, 2c_t)$ *and*

$$\mathcal{U} = \left[T_{|r+si|}(T_{\delta_{j+1}}(\xi))\right]_{i,j=0}^{t-1}, \qquad \mathcal{V} = \left[T_{sj}(T_{\delta_{i+1}}(\xi))\right]_{i,j=0}^{t-1},$$

Observation: If $w\mathcal{U} = 0$ for a row vector $w \ne 0$, then

$$\sum_{i=1}^{t} w_i T_{|r+si|}(T_{\delta_{j+1}}(\xi)) = 0 \quad \text{for} \quad 0 \le j < B.$$

i.e., $\sum_{i=1}^{t} w_i T_{|r+si|}$ is nonzero (for valid r, s) and t-sparse with t roots $T_{\delta_1}(\xi), \ldots, T_{\delta_t}(\xi)$.

A Rule of Signs for Chebyshev Basis

Theorem (Obrechkoff, 1918[2])

Let $f = \sum_{i=1}^{t} c_i T_{\delta_i}(x) \in \mathbb{R}[x]$*, and let* s *be the number of sign changes in* $(c_1, \ldots, c_t)$*. Then* f *has at most* s *real roots* ≥ 1*.*

Corollary

If f *is* B*-sparse, and* $f(x) = 0$ *for* $x = \underbrace{x_1, \ldots, x_B}_{\text{distinct}, \geq 1}$*, then* $f = 0$*.*

- $\Rightarrow \mathcal{U}, \mathcal{V}$ are nonsingular $\Rightarrow A^{(r,s)} = \mathcal{U}\mathcal{B}\mathcal{V}$ is nonsingular for valid r, s
- Generalized Lakshman–Saunders uses $\in \{2B, \ldots, 3B\}$ evaluations.

Corollary

A list decoder decodes $f \in \mathbb{R}[x]$ *uniquely for* $L \geq 2B + 2E$ *evaluations at points* $x_i > 1$*.*

[2] Thanks to the anonymous referee for bringing this reference to our attention

Subsequence List Decoding Sparse Chebyshev Interpolation

- We evaluate $a_j = \blacksquare(T_j(\xi))$, for $j = 0, 1, \ldots, L_{B,E} - 1$, where $L_{B,E}$ is the least value such that $a_0, \ldots, a_{L_{B,E}-1}$ must have valid r, s such that $a_{|r+is|}, -B \leq i < 2B$ is error-free.
- We run Lakshman–Saunders algorithm over evaluation points $a_{|r+is|}$, $-B \leq i < 2B$, for all valid (r, s), and check each resulting polynomial g if it agrees with $\blacksquare$ for all but at most E exceptions.
- We would like, for each $B \geq 1$, that subsequence list decoding requires fewer evaluations than list decoding for sufficiently large E
- Problem: $L_{B,E}$ is exponentially costly to compute.

Combining Block and Subsequence List Decoding

Fact (Kaltofen, Pernet; 2014)

$L_{1,8} \leq 17, \quad L_{2,8} \leq 34$

- i.e., We can correct 8 errors for B-sparse f and $B = 1, 2$ with $17B$ evaluations.
- If $B = 1, 2$, and we want to correct for E errors, then we can choose $\lceil E/8 \rceil$ blocks of $17B$ evaluations, and run the subsequence list decoder on each block.
  $\Rightarrow$ We can correct E errors with $17B\lceil E/8 \rceil$ evaluations.

Corollary

Let $L_{B,E}^{\min}$ be the least number of evaluations needed to interpolate a B-sparse polynomial with E errors. Then

$$L_{B,E}^{\min} < (E+1)2 \text{ for } E \geq 57, \qquad L_{B,E}^{\min} < (E+1)4 \text{ for } E \geq 86.$$

$\Rightarrow$ We can do better than block list decoding for $B = 1, 2$.

Conjecture: for every $B > 0$, $\exists E_B$ such that $L_{B,E}^{\min} < (E+1)B$ for $E > E_B$.

Thank You