Complete Reductions for Symbolic Integration

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Outline

I. Complete reductions (C.R.)

II. C.R. in the rational case

III. Transcendental Liouvillian extensions

IV. C.R. in the transcendental Liouvillian case

V. Elementary integration via C.R.

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I. Example 1: polynomial division

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The K-linear operator

$$\begin{array}{rccc} \Phi: & K[x] & \to & K[x] \\ & f & \mapsto & \mathrm{rem}(f, p) \end{array}$$

satisfies

- $\forall f \in K[x]$, $f - \Phi(f) \in (p)$,
- $\ker(\Phi) = (p)$.

I. Example 2: normal forms w.r.t. a Gröbner basis

Let

- $I \subset K[x_1, \ldots, x_n]$ be an ideal with $I \neq \{0\}$,
- G be a Gröbner basis of I w.r.t. a given term ordering.

The K-linear operator

$$\begin{array}{rccl} \Phi : & K[x_1, \ldots, x_n] & \to & K[x_1, \ldots, x_n] \\ & f & \mapsto & \text{the normal form of } f \text{ w.r.t. } G \end{array}$$

satisfies

- $\forall f \in K[x_1, \ldots, x_n]$, $f - \Phi(f) \in I$,
- $\ker(\Phi) = I$.

I. C.R. in terms of linear algebra

Let V be a linear space and $U \subset V$ be a subspace.

Definition. A linear operator $\Phi : V \to V$ is a C.R. for U if

- for all $v \in V$, $v - \Phi(v) \in U$,
- $\ker(\Phi) = U$.

(See Definition 5.67 in *D-Finite Functions* by M. Kauers)

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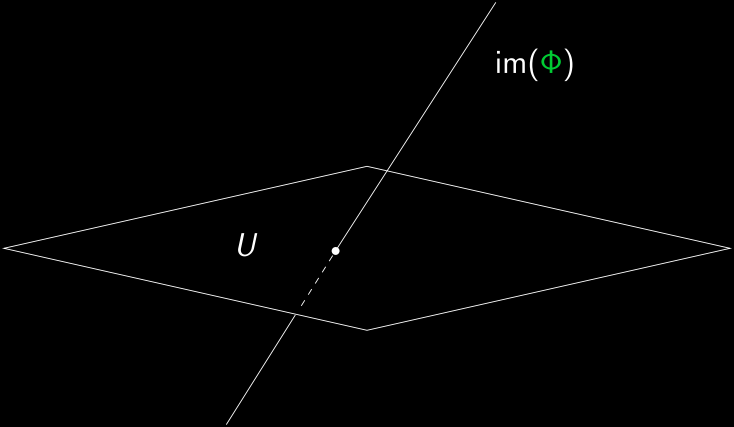
Proposition. A linear operator $\Phi : V \to V$ is a C.R. for U iff

(i) $V = U \oplus \operatorname{im}(\Phi)$,

(ii) Φ is the projection from V to $\operatorname{im}(\Phi)$ w.r.t. (i).



$$V = U \oplus \operatorname{im}(\Phi), \quad U = \ker(\Phi)$$

I. C.R. for symbolic integration

Let

- C be a field of characteristic 0,
- V be a C-linear space of some differentiable functions in x,
- $' = d/dx$ be a C-linear operator on V.

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$$\underbrace{V' := \{f' \mid f \in V\}}_{\text{integrable subspace}} \quad \subset \quad V.$$

Construct a C.R. for V'.

1. Fix a subspace R s.t. $V = V' \oplus R$.
2. For all $f \in V$, compute $g \in V$ and $r \in R$ s.t.

$$f = g' + r$$

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$$C' = \{0\} \quad \text{and} \quad C = \{0\} \oplus C.$$

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2. Let $V = C[x]$. Then

$$C[x]' = C[x] \quad \text{and} \quad C[x] = C[x] \oplus \{0\}.$$

   The C.R. for $C[x]'$ is the zero map.

I. Motivation: in-space integrability

Problem. Given a differential space V and $f \in V$,

- decide if $\exists\ g \in V$ s.t. $f = g'$,
- find such a g if there exists one.

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Idea. With a C.R. Φ for V', compute $g, \Phi(f) \in V$ s.t.

$$f = g' + \Phi(f).$$

Then

$$f \in V' \iff \Phi(f) = 0.$$

I. Motivation: a parametric version

Problem. Given $f_1, \ldots, f_k \in V$,

- decide if $\exists\ c_1, \ldots, c_k \in C$, not all zero, s.t.

$$y' = c_1 f_1 + \cdots + c_k f_k \qquad (*)$$

  has a solution $y \in V$,

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  has a solution $y \in V$,

- find such a solution if there exists one.

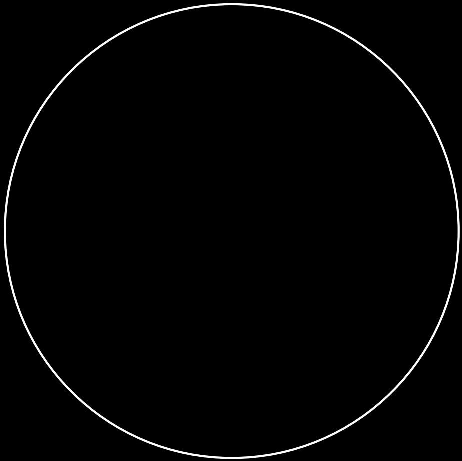
Idea. With a C.R. Φ for V', decompose

$$f_i = g_i' + \Phi(f_i), \quad i = 1, \ldots, k.$$

Then

$$(*) \iff \begin{cases} c_1 \Phi(f_1) + \cdots + c_k \Phi(f_k) = 0, \\ y = c_1 g_1 + \cdots + c_k g_k. \end{cases}$$

I. Motivation: additive decompositions



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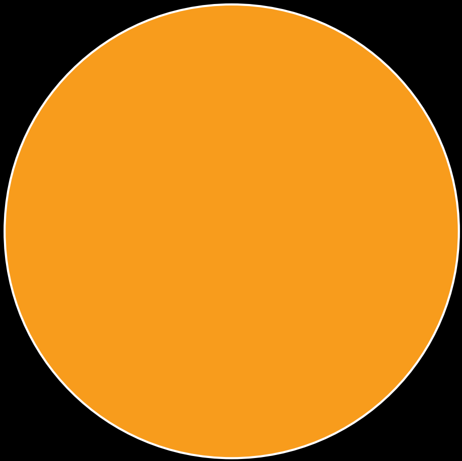

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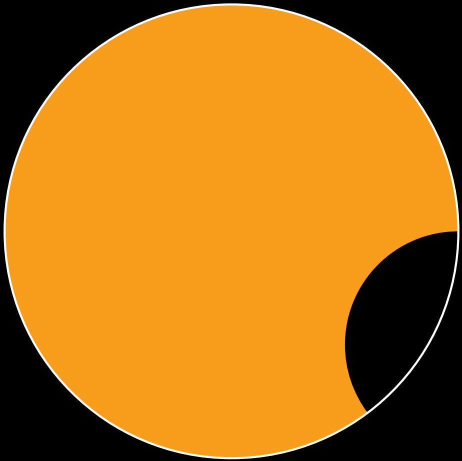

I. Additive decompositions for derivatives

Let V be a differential space over C.

Definition. An additive decomposition for V' is an algorithm that, for every $f \in V$, computes $g, r \in V$ s.t.

- $f = g' + r$,
- r is minimal in some sense,
- $f \in V'$ iff $r = 0$.

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Definition. For $u, v \in V$, $u \succeq_B v$ if $\mathrm{supp}_B(u) \supseteq \mathrm{supp}_B(v)$.

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Definition. For $u, v \in V$, $u \succeq_B v$ if $\mathrm{supp}_B(u) \supseteq \mathrm{supp}_B(v)$.

Proposition. For $f, r \in V$

$$f \equiv r \mod V' \implies r \succeq_B \Phi(f).$$

I. Example

Let

$$f(x) = \frac{x \log(x)^2}{x+3}.$$

- By Risch's algorithm (Trans. of the AMS, 1969),

$$\int f(x)\,dx = \int \frac{x \log(x)^2}{x+3}\,dx.$$

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- By a C.R.,

$$\int f\, dx = \left(x \log(x)^2 - 2x \log(x) + 2x\right) - \int \frac{3 \log(x)^2}{x+3}\, dx.$$

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II. Rational C.R. for derivatives

Let $V = C(x)$ and $' = d/dx$. For $f \in C(x)$,

the Hermite-Ostrogradsky reduction finds $g, r \in C(x)$ s.t.

(i) $f = g' + r$,

(ii) r is $\underbrace{x\text{-proper with a sqfr denominator}}_{x\text{-simple}}$.

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(ii) r is $\underbrace{x\text{-proper with a sqfr denominator}}_{x\text{-simple}}$.

Let $S_x = \{s \mid s \text{ is } x\text{-simple}\}$. Then

$$V = V' \oplus S_x$$

and

$$\begin{array}{rccc} \Phi: & V & \to & S_x, \\ & f & \mapsto & r \end{array}$$

is a C.R. for V'.

II. Ingredients

1. Let $p \in C[x]$, $\deg(p) > 0$ and $\gcd(p, p') = 1$.

$$\forall\ k > 1,\ \frac{\star}{p^k} = (\cdots)' + \frac{\star}{p^{k-1}}.$$

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2. Polynomials are derivatives.

II. Example: in-field integration

Let

$$f = \frac{x^4}{(x-1)^2(x+1)} \in \mathbb{Q}(x).$$

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Apply the rational C.R.:
$$f = \left(\frac{1}{2}\frac{x^3+x^2-2x-1}{x-1}\right)' + \underbrace{\frac{1}{2}\frac{4x+3}{(x-1)(x+1)}}_{\text{remainder}}.$$

$$\text{remainder} \neq 0 \implies f \text{ is not a derivative.}$$

II. Rational creative telescoping

Let $V = C(x, y)$. Set $D_x := \dfrac{\partial}{\partial x}$ and $D_y := \dfrac{\partial}{\partial y}$.

Problem. Given $f \in C(x, y)$, find a nonzero elt $\mathcal{T} \in C(x)[D_x]$ s.t.

$$\mathcal{T}(f) = D_y(g) \quad \text{for some } g \in C(x, y).$$

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Call

- $\mathcal{T}$ a telescoper for f,
- g a certificate corresponding to $\mathcal{T}$.

II. Reduction-based creative telescoping

Let Φ_y be a C.R. for $\mathrm{im}(D_y)$ in $C(x, y)$.

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OUTPUT: $\mathcal{T}$ and $g \in C(x, y)$ s.t. $\mathcal{T}(f) = D_y(g)$

1. $k \leftarrow 0$
2. Compute g_k and $r_k := \Phi_y(D_x^k(f))$ s.t. $D_x^k(f) = D_y(g_k) + r_k$
3. Solve $\sum_{i=0}^{k} c_i r_i = 0$ for $c_0, \ldots, c_k \in C(x)$, not all zero
4. If such $c_0, \ldots, c_k$ are found, then

$$\mathcal{T} \leftarrow \sum_{i=0}^{k} c_i D_x^i, \quad g \leftarrow \sum_{i=0}^{k} c_i g_i, \quad \text{return } (\mathcal{T}, g)$$

5. Go to step 2 with $k \leftarrow k + 1$

II. Reduction-based creative telescoping (certificate-free)

Let Φ_y be a C.R. for $\operatorname{im}(D_y)$ in $C(x, y)$.

INPUT: $f \in C(x, y)$

OUTPUT: $\mathcal{T}$ s.t. $\mathcal{T}(f) \in \operatorname{im}(D_y)$

1. $k \leftarrow 0$
2. Compute $r_k := \Phi_y(D_x^k(f))$
3. Solve $\sum_{i=0}^{k} c_i r_i = 0$ for $c_0, \ldots, c_k \in C(x)$, not all zero
4. If such $c_0, \ldots, c_k$ are found, then

$$\mathcal{T} \leftarrow \sum_{i=0}^{k} c_i D_x^i, \quad \text{return } \mathcal{T}$$

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II. Example

Let $f = \dfrac{y^4 x}{(y^2 - xy + x)^2 (y + x)} \in \mathbb{Q}(x, y)$.

A telescoper for f is

$$\mathcal{T} = \frac{x^3(8x^3 - 30x^2 - 9x + 4)}{2x^2 + 19x - 3} D_x^3 + \frac{x^2(32x^3 - 26x^2 + 67x - 10)}{2x^2 + 19x - 3} D_x^2 - 6x D_x + 6.$$

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A certificate g is

$$-\frac{1}{3}\frac{(28x^3 - 46x^2 - 31x + 13)\,x^5}{(y+x)^2(2x^2 + 19x - 3)(2x+1)^2} - \frac{1}{3}\frac{x^2(520x^7 + 320x^6y - 384x^6 + 160x^5y - 812x^5 - 140x^4y - 53x^4 - 59x^3y + 169x^3 + 31x^2y + 40x^2 + 4yx - 3x - y + 1)}{(-yx + y^2 + x)(2x+1)^3(2x^2 + 19x - 3)}$$
$$- \frac{(4x^6y - 4x^6 - 37x^5y + 33x^5 + 113x^4y - 84x^4 - 118x^3y + 59x^3 + 3x^2y + 18x^2 + 33yx - 15x - 7y + 2)\,x^5}{(-yx + y^2 + x)^4(2x^2 + 19x - 3)} + \frac{1}{3}\frac{(64x^4 + 32x^3 - 28x^2 - 11x + 15)\,x^4}{(y+x)(2x+1)^3(2x^2 + 19x - 3)}$$
$$+ \frac{1}{3}\frac{(4x^2 - 17x + 4)\,x^6}{(y+x)^3(2x^2 + 19x - 3)(2x+1)} - \frac{1}{3}\frac{(232x^7 + 444x^6y - 1174x^6 - 936x^5y + 547x^5 - 354x^4y + 1085x^4 + 352x^3y + 32x^3 + 5x^2y - 95x^2 - 58yx + 11x - 2y + 1)\,x^3}{(-yx + y^2 + x)^2(2x^2 + 19x - 3)(2x+1)^2}$$
$$- \frac{1}{3}\frac{(24x^7 + 140x^6y - 302x^6 - 769x^5y + 920x^5 + 964x^4y - 490x^4 + 201x^3y - 519x^3 - 348x^2y + 113x^2 - 5yx + 38x + 24y - 9)\,x^4}{(-yx + y^2 + x)^3(2x^2 + 19x - 3)(2x+1)}.$$

II. Hyperexponential functions

Definition. A function $H(x)$ is hyperexponential over $C(x)$ if

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Example. Let $f \in C(x)$ with $f \neq 0$.

- Rational: $f = \exp\left(\int \frac{f'}{f}\right)$.
- Exponential: $\exp(f) = \exp\left(\int f'\right)$.
- Radical: $\sqrt[n]{f} = \exp\left(\int \frac{f'}{nf}\right)$.

II. In-space integration

Let $H = \exp(\int h)$ for some $h \in C(x)$ and $V = \operatorname{span}_{C(x)}\{H\}$.

Remark. $V' \subset V$.

Goal. *Construct a C.R. for V'.*

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Example. Let $u \in C(x)$. Then

$$uH \in V' \iff \exists\, y \in C(x) \text{ s.t. } \underbrace{y' + hy = u}_{\text{Risch diff. eqn}}.$$

II. Risch operators

Definition. For $h \in C(x)$, the C-linear map

$$\begin{array}{rccc} \mathcal{R}_h : & C(x) & \to & C(x) \\ & y & \mapsto & y' + hy \end{array}$$

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Proposition. Let $H = \exp(\int h)$ and $V = \mathrm{span}_{C(x)}\{H\}$.
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Goal. *Develop a C.R. for im($\mathcal{R}_h$) on $C(x)$.*

II. Normalization

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Proposition. Let $h \in C(x)$.

(i) $\exists\, \xi, \eta \in C(x)$ s.t. ξ is normalized and $h = \xi + \frac{\eta'}{\eta}$.

(ii) If Φ_ξ is a C.R. for $\mathrm{im}(\mathcal{R}_\xi)$, then

$$\eta^{-1} \circ \Phi_\xi \circ \eta$$

is a C.R. for $\mathrm{im}(\mathcal{R}_h)$.

II. Good and bad news

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Good. For $k > 1$,

- Let $p \in C[x]$ with $\gcd(p, p') = 1$. If $\gcd(p, b) = 1$, then

$$\frac{\star}{p^k} = \mathcal{R}_h(\cdots) + \frac{\star}{b} + \frac{\star}{p^{k-1}}.$$

- $\dfrac{\star}{b^k} = \mathcal{R}_h(\cdots) + \dfrac{\star}{b^{k-1}}.$

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- $\dfrac{\star}{b^k} = \mathcal{R}_h(\cdots) + \dfrac{\star}{b^{k-1}}.$

Bad.

$$\frac{\star}{b} \in \mathrm{im}(\mathcal{R}_h)\,?$$

II. Generalized rational reduction

Let $h = a/b \in C(x)$ be normalized. Set

$$S_{x,h} := \{s \in S_x \mid \text{the denominator of } s \text{ is coprime with } b\}.$$

Proposition. For $f \in C(x)$,

$$f = \mathcal{R}_h(\cdots) + \frac{u}{b} + s,$$

for some $u \in C[x]$ and a unique $s \in S_{x,h}$.

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Chen's observation. $\exists\ u \in C[x]$

$$\text{whose number of terms} \leq \max(\deg(a), \deg(b) - 1).$$

II. Example

Let $h = -\frac{20(x^3+2)}{x^4+1}$. Then

$$x = \mathcal{R}_h(\cdots) + \frac{\frac{73224716}{54755473} - \frac{91943182\,x}{54755473}}{x^4+1}.$$

$$x^{10} = \mathcal{R}_h(\cdots) + \frac{-\frac{29097914828}{711821149} + \frac{31412792776\,x}{711821149}}{x^4+1}.$$

$$x^{23} = \mathcal{R}_h(\cdots) + \frac{\frac{-3559105745\,x^{23}}{711821149} - \frac{5812238365969920\,x}{711821149} - \frac{7355567628109550}{711821149}}{x^4+1}.$$

Remark. $* \in \mathrm{span}_C\{x^{23}, x, 1\}$.

II. Companion operators

Let $h = a/b \in C(x)$ be normalized.

Definition. The companion operator of $\mathcal{R}_h$ is

$$\begin{array}{rccc} \mathcal{P}_h : & C[x] & \to & C[x] \\ & p & \mapsto & bp' + ap. \end{array}$$

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$$C(x) = \operatorname{im}(\mathcal{R}_h) \oplus \underbrace{b^{-1}\operatorname{im}(\Psi_h) \oplus S_{x,h},}_{R} \qquad (*)$$

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Goal. *Construct a C.R. for im($\mathcal{P}_h$) in $C[x]$.*

II. Basis transformations

Let $h = a/b \in C(x)$ be nonzero and normalized.

$$
\begin{array}{c}
C[x] \\
\vdots \\
x^i \\
\vdots \\
x \\
1
\end{array}
$$

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\begin{array}{ccc}
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\vdots & & \vdots \\
x^i & \xrightarrow{\mathcal{P}_h,\ \text{inj}} & ibx^{i-1} + ax^i \\
\vdots & & \vdots \\
x & & b + ax \\
1 & & a
\end{array}
$$

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Let $h = a/b \in C(x)$ be nonzero and normalized.

$$
\begin{array}{ccccc}
C[x] & & \mathrm{im}(\mathcal{P}_h) & & \text{Echelon basis} \\
 & & & & \vdots \\
\vdots & & \vdots & & x^{d+2} + \cdots \\
 & & & & x^{d+1} + \cdots \\
x^i & \xrightarrow{\mathcal{P}_h,\ \text{inj}} & ibx^{i-1} + ax^i & \xrightarrow{\text{elim}} & x^{d} + \cdots \\
\vdots & & \vdots & & x^{e_k} + \cdots \\
 & & & & x^{e_{k-1}} + \cdots \\
x & & b + ax & & \vdots \\
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\vdots & & \vdots & & x^{e_k} + \cdots \\
 & & & & x^{e_k - 1} + \cdots \\
x & & b + ax & & \vdots \\
1 & & a & & x^{e_1} + \cdots
\end{array}
$$

$$\dim\left(C[x]/\mathrm{im}(\mathcal{P}_h)\right) = d - k \le \max(\deg(a), \deg(b) - 1).$$

II. Example

Let $h = -\frac{20(x^3+2)}{x^4+1}$. Verify $\dim\left(C[x]/\mathrm{im}(\mathcal{P}_h)\right) \leq 3$.

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$$
\begin{array}{r}
C[x] \\
\vdots \\
x^{21} \\
x^{20} \\
x^{19} \\
\vdots \\
x \\
1
\end{array}
$$

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$$
\begin{array}{ccc}
C[x] & & \mathrm{im}(\mathcal{P}_h) \\
\vdots & & \vdots \\
x^{21} & & x^{24} + \cdots \\
x^{20} & \xrightarrow{\mathcal{P}_h,\ \text{monic}} & x^{20} + \cdots \\
x^{19} & & x^{22} + \cdots \\
\vdots & & \vdots \\
x & & x^4 + \cdots \\
1 & & x^3 + \cdots
\end{array}
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$C[x]$		$\mathrm{im}(\mathcal{P}_h)$		Echelon basis
$\vdots$		$\vdots$		$\vdots$
x^{21}		$x^{24} + \cdots$		$x^{24} + \cdots$
x^{20}	$\xrightarrow{\mathcal{P}_h,\ \text{monic}}$	$x^{20} + \cdots$	$\xrightarrow{\text{elim}}$	
x^{19}		$x^{22} + \cdots$		$x^{22} + \cdots$
$\vdots$		$\vdots$		$\vdots$
x		$x^4 + \cdots$		$x^4 + \cdots$
1		$x^3 + \cdots$		$x^3 + \cdots$
				$x^2 + \cdots$

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x^{21} & & x^{24}+\cdots & & x^{24}+\cdots \\
x^{20} & \xrightarrow{\mathcal{P}_h,\ \text{monic}} & x^{20}+\cdots & \xrightarrow{\text{elim}} & \\
x^{19} & & x^{22}+\cdots & & x^{22}+\cdots \\
\vdots & & \vdots & & \vdots \\
x & & x^4+\cdots & & x^4+\cdots \\
1 & & x^3+\cdots & & x^3+\cdots \\
 & & & & x^2+\cdots
\end{array}
$$

$$C[x] = \mathrm{im}(\mathcal{P}_h) \oplus \mathrm{span}_C\{1, x, x^{23}\}.$$

II. Summary for rational C.R.

C.R. for $\mathrm{im}(\mathcal{R}_h)$ on $C(x)$ with $h \neq 0$

$\downarrow$

C.R. for $\mathrm{im}(\mathcal{R}_h)$ $C(x)$ with normalized h

$\downarrow$

C.R. for $\mathrm{im}(\mathcal{P}_h)$ on $C[x]$

$\downarrow$

$C[x] = \mathrm{im}(\mathcal{P}_h) \oplus W_h$, $\dim(W_h) < \infty$

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$$\boxed{\text{C.R. for } \operatorname{im}(\mathcal{P}_h) \text{ on } C[x]}$$

$$\downarrow$$

$$\boxed{C[x] = \operatorname{im}(\mathcal{P}_h) \oplus W_h,\ \dim(W_h) < \infty}$$

Theorem (ISSAC 2013. Bostan, Chen, Chyzak, L and Xin)

For every $f \in C(x)$, one can construct a C.R. for $\operatorname{im}(\mathcal{R}_f)$.

II. Related publications in the differential case

- Algebraic (Chen–Kauers–Koutschan 2016;
  Chen–Du–Kauers 2021)
- Differential fractions
  (Boulier–Lemaire–Lallemand–Regensburger–Rosenkranz 2016)
- Fuchsian D-finite (Chen–van Hoeij–Kauers–Koutschan 2018)
- D-finite (van der Hoeven 2017, 2018, 2021;
  Bostan–Chyzak–Lairez–Salvy 2018;
  Chen–Du–Kauers 2023)
- Primitive towers (Du–Gao–Li-L 2025)

Outline

I. Complete reductions (C.R.)

II. C.R. in the rational case

III. Transcendental Liouvillian extensions

IV. C.R. in the transcendental Liouvillian case

V. Elementary integration via C.R.

III. Differential fields (D-fields)

Let F be a field.

Definition. An additive map $' : F \rightarrow F$ is a derivation on F if

$$\forall\ a, b \in F,\ (ab)' = a'b + ab'.$$

Let $'$ be a derivation on F. Call $(F, ')$ a D-field.

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Example. $(C,\ d/dx)$ and $(C(x),\ d/dx)$ are D-fields.

III. Constants

Let $(F, ')$ be a D-field.

Definition. An element $c \in F$ is a constant if $c' = 0$.

The set of all constants in F is denoted by $\mathrm{Const}(F)$.

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Example.

- In $(C(x), d/dx)$, $C = \mathrm{Const}(C(x))$.
- In $(F, ')$, $F' = \{f' \mid f \in F\}$ is a subspace over $\mathrm{Const}(F)$.

III. Monomials

Let $(F, ')$ and (E, δ) be two D-fields.

If F is a subfield of E and $\delta|_F = '$, call E a D-field extension of F.

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Call t a monomial over F if

- t is transcendental over F,
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Call t a monomial over F if

- t is transcendental over F,
- $t' \in F[t]$.

Example. $\log(x)$ and $\exp(x)$ are monomials over $\mathbb{Q}(x)$.

III. Types of monomials

Definition. Let t be a monomial over $(F, ')$.

$$t \text{ is } \begin{cases} \text{primitive} & \text{if } t' = u \text{ for some } u \in F, \text{ i.e., } t = \int u, \\ \text{logarithmic} & \text{if } t' = \dfrac{u'}{u} \text{ for some } u \in F, \text{ i.e., } t = \log(u), \\ \text{hyperexponential} & \text{if } \dfrac{t'}{t} = u \text{ for some } u \in F, \text{ i.e., } t = \exp(\int u), \\ \text{exponential} & \text{if } \dfrac{t'}{t} = u' \text{ for some } u \in F, \text{ i.e., } t = \exp(u). \end{cases}$$

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\end{cases}
$$

Remark.

- logarithmic $\Rightarrow$ primitive,
- exponential $\Rightarrow$ hyperexponential.

III. Transcendental Liouvillian extensions

Definition. Let

$$\begin{array}{ccccccccc} F_0 & \subset & F_1 & \subset & \cdots & \subset & F_{n-1} & \subset & F_n \\ \shortparallel & & \shortparallel & & & & \shortparallel & & \shortparallel \\ C & \subset & F_0(t_1) & \subset & \cdots & \subset & F_{n-2}(t_{n-1}) & \subset & F_{n-1}(t_n). \end{array}$$

Call F_n a transcendental Liouvillian extension of C if t_i is either a primitive or a hyperexponential monomial over F_{i-1} for each i, and

$$\mathrm{Const}(F_n) = C.$$

III. C.R. on transcendental Liouvillian extensions F_n

Let F_n be a transcendental Liouvillian extension.

Definition. For $h \in F_n$, the Risch operator associated to h is

$$\begin{array}{rccc} \mathcal{R}_h : & F_n & \to & F_n \\ & y & \mapsto & y' + hy. \end{array}$$

Goal. *Construct a C.R. for im$(\mathcal{R}_h)$.*

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B. $n = 0$. Trivial.

H. $n > 0$. Set $F := F_{n-1}$, and assume that,

$$\boxed{\text{for all } \alpha \in F, \text{ there is a C.R. } \Phi_\alpha \text{ for } \mathrm{im}(\mathcal{R}_\alpha)}$$

I. Set $t := t_n$. For all $h \in F(t)$, construct a C.R. for $\mathrm{im}(\mathcal{R}_h)$.

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p: normal	$\gcd(p, p') = 1$	p is sqfr
h: t-simple	t-proper and b normal	x-simple
h: t-normalized	$\gcd(a - ib', b) = 1,\ i \in \mathbb{Z}$	x-normalized
h: normal form	$h = \xi + \dfrac{\eta'}{\eta}$	the same

IV. Generalized Hermite reduction

Let $h = a/b \in F(t)$ be t-normalized and

$S_{t,h} := \{s \in F(t) \mid t\text{-simple whose the denom is coprime with } b\}.$

Proposition. For every $f \in F(t)$, $f = \mathcal{R}_h(\cdots) + \dfrac{u}{b} + s$, where

$$u \in F\langle t\rangle := \begin{cases} F[t] & \text{if } t \text{ is primitive,} \\ F[t, t^{-1}] & \text{if } t \text{ is hyperexponential,} \end{cases}$$

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Remark. Call $F\langle t\rangle$ the special subring.

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and $s \in S_{t,h}$ is unique.

Remark. Call $F\langle t\rangle$ the special subring.

Question. Reduce $\dfrac{u}{b}$ modulo $\mathrm{im}(\mathcal{R}_h)$?

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Challenge. $C \subset C[x]$ vs. $C \subset F \subset F\langle t\rangle$.

IV. Guideline

C.R. for $\mathrm{im}(\mathcal{P}_h)$ on $F\langle t\rangle$

Primitive
$F\langle t\rangle = F[t]$

Hyperexponential
$F\langle t\rangle = F[t, t^{-1}]$

simplification of coeffs

Auxiliary subspace A_h s.t. $F\langle t\rangle = \mathrm{im}(\mathcal{P}_h) + A_h$

echelon basis of $\mathrm{im}(\mathcal{P}_h) \cap A_h$

C-subspace W_h s.t. $F\langle t\rangle = \mathrm{im}(\mathcal{P}_h) \oplus W_h$

IV. Notation

Let $h \in F(t)$ be t-normalized, and write

$$h = \frac{a_m t^m + a_{m-1} t^{m-1} + \cdots + a_0}{b_m t^m + b_{m-1} t^{m-1} + \cdots + b_0},$$

where

- $a_i, b_i \in F$,
- $a_m \neq 0$ or $b_m \neq 0$,
- $b_m = 1$ if $b_m \neq 0$.

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Remark. Call m the degree of h.

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- $a_m \neq 0$ or $b_m \neq 0$,
- $b_m = 1$ if $b_m \neq 0$.

Remark. Call m the degree of h.

Notation. $F[t]_{<k} := \{p \in F[t] \mid \deg(p) < k\}$.

IV. Simplification of coefficients: t primitive

Let $\alpha \in F$.

Easy case. $b_m = 0$ and $k \geq m$.

$$\alpha t^k \equiv \cdots \mod \operatorname{im}(\mathcal{P}_h),$$

where $\cdots \in F[t]_{<k}$.

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Easy case. $b_m = 0$ and $k \geq m$.

$$\alpha t^k \equiv \cdots \mod \operatorname{im}(\mathcal{P}_h),$$

where $\cdots \in F[t]_{<k}$.

Hard case. $b_m = 1$ and $k \geq m$.

$$\alpha t^k \equiv \Phi_{a_m}(\alpha) t^k + \cdots \mod \operatorname{im}(\mathcal{P}_h),$$

where $\cdots \in F[t]_{<k}$.

IV. Auxiliary subspaces and reduction: t primitive

Definition. The auxiliary subspace associated to $(F[t], \mathcal{P}_h)$ is

$$A_h := \begin{cases} F[t]_{<m} & \text{if } b_m = 0, \\ F[t]_{<m} + \sum_{i \geq m} \mathrm{im}(\Phi_{a_m}) t^i & \text{if } b_m = 1. \end{cases}$$

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Proposition. $F[t] = \operatorname{im}(\mathcal{P}_h) + A_h$.

IV. Example

Let $F = \mathbb{Q}(x)$, $t = \log(x)$ and $h = 0$. Then $A_0 = \sum_{i=0}^{\infty} S_x t^i$.

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Decompose $f = \frac{x+1}{x} t + \frac{x^2+x+1}{x}$ as:

$$f = \left(x' + \frac{1}{x} \right) t + \frac{x^2+x+1}{x}$$

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$$\begin{aligned} f &= \left(x' + \frac{1}{x}\right) t + \frac{x^2+x+1}{x} \\ &= (xt)' + \frac{t}{x} + \frac{x^2+1}{x} \end{aligned}$$

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$$
\begin{array}{rcl}
f & = & \left(x' + \dfrac{1}{x}\right) t \quad + \quad \dfrac{x^2+x+1}{x} \\
 & = & (xt)' \; + \; \dfrac{t}{x} \quad + \quad \dfrac{x^2+1}{x} \\
 & = & (xt)' \; + \; \dfrac{t}{x} \quad + \quad \left(\dfrac{x^2}{2}\right)' + \dfrac{1}{x} \\
 & = & \left(xt + \dfrac{x^2}{2}\right)' \quad + \quad \dfrac{t}{x} + \dfrac{1}{x} \\
 & & \qquad \Cap \qquad\qquad\qquad \Cap \\
 & & \quad \mathrm{im}(\mathcal{P}_0) \qquad\qquad\quad A_0
\end{array}
$$

IV. Simplification of coefficients: t hyperexponential

Let $\alpha \in F$.

Easy case 1. $b_m = 0$ and $k \geq m$.

$$\alpha t^k \equiv \cdots \mod \operatorname{im}(\mathcal{P}_h),$$

where $\cdots \in F[t]_{<k}$.

Easy case 2. $b_0 = 0$ and $\ell < 0$.

$$\alpha t^\ell \equiv \star\star\star \mod \operatorname{im}(\mathcal{P}_h),$$

where

$$\star\star\star \in \operatorname{span}_F\{t^{m-1}, t^{m-2}, \ldots, t^{\ell+2}, t^{\ell+1}\}.$$

IV. Hard cases

1. $b_m = 1$ and $k \geq m$. Set $\lambda_k := a_m + k\frac{t'}{t}$.

$$\alpha t^k \equiv \Phi_{\lambda_{k-m}}(\alpha) t^k + \cdots \quad \mod \operatorname{im}(\mathcal{P}_h),$$

where $\cdots \in F[t]_{<k}$.

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where $\cdots \in F[t]_{<k}$.

2. $b_0 \neq 0$ and $\ell < 0$. Set $\mu_\ell := \dfrac{a_0}{b_0} + \ell\dfrac{t'}{t}$.

$$\alpha t^\ell \equiv \star\star\star + b_0\Phi_{\mu_\ell}\left(b_0^{-1}\alpha\right)t^\ell \quad \text{mod im}(\mathcal{P}_h),$$

where $\star\star\star \in \text{span}_F\{t^{m-1}, t^{m-2}, \ldots, t^{\ell+2}, t^{\ell+1}\}$.

IV. Auxiliary subspaces: t hyperexponential

Definition. The auxiliary subspace associated to $(F[t, t^{-1}], \mathcal{P}_h)$ is

$$A_h = A_h^+ + F[t]_{<m} + A_h^-,$$

where

$$A_h^+ = \begin{cases} \{0\} & \text{if } b_m = 0, \\ \sum_{i \geq m} \operatorname{im}\left(\Phi_{\lambda_{i-m}}\right) t^i & \text{if } b_m = 1, \end{cases}$$

and

$$A_h^- = \begin{cases} \{0\} & \text{if } b_0 = 0, \\ b_0 \sum_{j<0} \operatorname{im}\left(\Phi_{\mu_j}\right) t^j & \text{if } b_0 \neq 0. \end{cases}$$

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Proposition. $F[t, t^{-1}] = \operatorname{im}(\mathcal{P}_h) + A_h$.

IV. Determine the intersection $\mathrm{im}(\mathcal{P}_h) \cap A_h$

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Let $I_h = \mathrm{im}(\mathcal{P}_h) \cap A_h$.

Proposition. If t is hyperexponential, then

$$\dim_C (I_h) \leq 2.$$

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Let $I_h = \mathrm{im}(\mathcal{P}_h) \cap A_h$.

Proposition. If t is hyperexponential, then

$$\dim_C (I_h) \leq 2.$$

Moreover, one can

- decide if $I_h = \{0\}$,
- construct a basis of I_h.

IV. Key lemmas in the hyperexponential case

Set $f := f_k t^k + \cdots + f_0 + f_{-1} t^{-1} + \cdots + f_\ell t^\ell \in F[t, t^{-1}]$.

Lemma 1. Assume $k \geq 0$ and $f_k \neq 0$.

$$f \in P_h^{-1}(I_h) \implies f_k \in \ker(\mathcal{R}_{\lambda_k}).$$

Lemma 2. There exists at most one $k \geq 0$ s.t. $\ker(\mathcal{R}_{\lambda_k}) \neq \{0\}$.

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Lemma 1. Assume $k \geq 0$ and $f_k \neq 0$.

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Lemma 3. Assume $\ell < 0$ and $f_\ell \neq 0$.

$$f \in P_h^{-1}(I_h) \implies f_\ell \in \ker(\mathcal{R}_{\mu_\ell}).$$

Lemma 4. There exists at most one $\ell < 0$ s.t. $\ker(\mathcal{R}_{\mu_\ell}) \neq \{0\}$.

IV. Example

Let $F = \mathbb{Q}(x)$, $t = \exp(x)$ and $h = \dfrac{1 + (1-x)t}{(1+t)^2}$. Then

m	a_m	b_m	a_0	b_0	λ_k	μ_ℓ
2	0	1	1	1	k	$\ell + 1$

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Auxiliary subspace:

$$A_h = \underbrace{\sum_{k\geq 2} \operatorname{im}(\Phi_{k-2})t^k}_{A_h^+} + F[t]_{<2} + \underbrace{\sum_{\ell<0} \operatorname{im}(\Phi_{\ell+1})t^\ell}_{A_h^-}.$$

Determine I_h.

IV. Example (continued)

By $\lambda_k = k$, $\ker(\mathcal{R}_{\lambda_0}) = C$. Thus,

$$\mathcal{P}_h(1) = (1-x)t + 1 \in A_h \Longrightarrow (1-x)t + 1 \in I_h.$$

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$$\mathcal{P}_h(t^{-1}) = -t - x - 1 \in A_h \Longrightarrow -t - x - 1 \in I_h.$$

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Remark. Such k and ℓ can by found by Raab's thesis (Chap. 4).

IV. A C-basis of $F\langle t\rangle$

1. Let $\mathbb{B}_F$ be a C-basis of F. Then $F\langle t\rangle$ has a C-basis

$$\Omega := \begin{cases} \{bt^i \mid b \in \mathbb{B}_F,\ i \in \mathbb{N}_0\} & \text{if } t \text{ is primitive,} \\ \{bt^i \mid b \in \mathbb{B}_F,\ i \in \mathbb{Z}\} & \text{if } t \text{ is hyperexponential.} \end{cases}$$

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2. For $\omega \in \Omega$, the C-linear function ω^* is defined by

$$\omega^*(z) = \begin{cases} 1 & \text{if } z = \omega, \\ 0 & \text{if } z \in \Omega \setminus \{\omega\}. \end{cases}$$

IV. Find a complementary subspace in the hyperexp case

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- If $\dim(I_h) = 1$, then
  - find $p \in F[t, t^{-1}]$ s.t. $I_h = \mathrm{span}_C\{p\}$
  - find $\omega \in \Omega$ s.t. $\omega^*(p) \neq 0$.

Hence,

$$F[t, t^{-1}] = \mathrm{im}(\mathcal{P}_h) \oplus \underbrace{A_h \cap \ker(\omega^*)}_{W_h}.$$

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IV. Find a complementary subspace in the hyperexp case

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- If $\dim(I_h) = 2$, then
  - find $p_1, p_2 \in F[t, t^{-1}]$ s.t. $I_h = \mathrm{span}_C\{p_1, p_2\}$, and $\omega_1, \omega_2 \in \Omega$ s.t.

$$\begin{array}{llllll} p_2 = & \omega_2 & + & \cdots & + & \cdots \\ p_1 = & & & \omega_1 & + & \cdots \end{array} \quad \text{with } \omega_2^*(\cdots) = 0.$$

Hence,

$$F[t, t^{-1}] = \mathrm{im}(\mathcal{P}_h) \oplus \underbrace{A_h \cap \ker(\omega_1^*) \cap \ker(\omega_2^*)}_{W_h}.$$

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Let

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Then

$$I_h = \mathrm{span}_{\mathbb{Q}}\{\underbrace{t + x + 1}_{p_1}, \underbrace{(1-x)t + 1}_{p_2}\}.$$

Set $\omega_1 = t$ and $\omega_2 = xt$. Then

$$p_1 = \omega_1 + x + 1, \quad p_2 = -\omega_2 + \omega_1 + 1$$

form an echelon basis of I_h with pivots ω_1 and ω_2. Thus,

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Hard case. $b_m = 1$.

- $I_h = \{0\} \iff \ker(\mathcal{R}_{a_m}) = \{0\}$.
- For $\lambda \in \ker(\mathcal{R}_{a_m})$ with $\lambda \neq 0$, $f \in \mathcal{P}_h^{-1}(I_h)$ is of the form

$$c\lambda t^k + \cdots ,$$

where $c \in C$ and $k \in \mathbb{N}_0$.

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Let $b_m = 1$ and $\ker(\mathcal{R}_{a_m}) \neq \{0\}$. Fix $\lambda \in \ker(\mathcal{R}_{a_m})$ with $\lambda \neq 0$.

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$$
\begin{array}{ccc}
\text{basis of } \mathcal{P}_h^{-1}(I_h) & & \text{generators of } I_h \\
\vdots & & \vdots \\
\lambda t^i + \cdots & \xrightarrow{\mathcal{P}_h} & (i\sigma + \tau)t^{m+i-1} + \cdots \\
\vdots & & \vdots \\
\lambda t + \cdots & & (\sigma + \tau)t^m + \cdots \\
\lambda & & \mathcal{P}_h(\lambda)
\end{array}
$$

where $\sigma := \Phi_{a_m}(\lambda t')$ and $\tau := \Phi_{a_m}(b_{m-1}\lambda' + a_{m-1}\lambda)$.

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where $\sigma := \Phi_{a_m}(\lambda t')$ and $\tau := \Phi_{a_m}(b_{m-1}\lambda' + a_{m-1}\lambda)$.

$$
\text{Moreover, basis of } I_h = \begin{cases} \text{generators} & \text{if } h \notin F \\ \text{generators without } \mathcal{P}_h(\lambda) & \text{otherwise.} \end{cases}
$$

IV. Example

Let $F = \mathbb{Q}(x)$, $t = \log(x)$ and $h = 0$ Then $1 \in \ker(\mathcal{R}_0)$.

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\begin{array}{ccc}
\text{basis of } \mathcal{P}_0^{-1}(I_0) & & \text{generators of } I_0 \\
\vdots & & \vdots \\
t^i & & i x^{-1} t^{i-1} \\
 & \xrightarrow{\mathcal{P}_0 = '} & \\
\vdots & & \vdots \\
t & & x^{-1} \\
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t & & x^{-1} \\
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\end{array}
$$

A basis of I_0 is $\{x^{-1}t^{i-1} \mid i \in \mathbb{N}\}$. Thus,

$$F(t) = \operatorname{im}(\mathcal{P}_0) \oplus \left(A_0 \cap \left(\cap_{i=1}^{\infty} \ker(\omega_i^*)\right)\right),$$

where $A_0 = \sum_{j=0}^{\infty} S_x t^j$.

IV. Conclusion

Proposition. Let $h \in F(t)$ be t-normalized. Then one can construct a C-subspace W_h s.t.

$$F\langle t\rangle = \mathrm{im}(\mathcal{P}_h) \oplus W_h.$$

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Theorem. (Chen, Du, Gao, Huang, Li and L, submitted)

Let F_n be a transcendental Liouvillian extension of C. For every $f \in F_n$, one can construct a C.R. Φ_f for $\mathrm{im}(\mathcal{R}_f)$ on F_n.

In particular, Φ_0 is a C.R. for F_n'.

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Let

$$f = \frac{x}{1+\exp(x)} \cdot \exp\left(\frac{x}{1+\exp(x)}\right).$$

Compute $\int f \, dx$.

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Setting. Let $t_1 = \exp(x)$ and $t_2 = \exp\left(\frac{x}{1+\exp(x)}\right)$. Then

$$f = \frac{x}{1+t_1} t_2 \in \mathbb{Q}(x, t_1, t_2).$$

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With Φ_0 on $\mathbb{Q}(x, t_1, t_2)$, we have $f = \left((-1 - t_1^{-1}) t_2\right)'$.

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With Φ_0 on $\mathbb{Q}(x, t_1, t_2)$, we have $f = \left((-1 - t_1^{-1})t_2\right)'$.

In functional notation,

$$\int \frac{x}{1+\exp(x)} \cdot \exp\left(\frac{x}{1+\exp(x)}\right) dx = -(1+\exp(-x)) \cdot \exp\left(\frac{x}{1+\exp(x)}\right).$$

IV. Experiments[1]

Let $t_1 = x$, $t_2 = \log(x^2+1)$, $t_3 = \exp(\frac{x^2}{2})$, $t_4 = \exp(x \log(x^2+1))$.

[1] MAPLE 2026 on a computer with iMac CPU 3.6GHz, Intel Core i9, 16G

IV. Experiments[1]

Let $t_1 = x$, $t_2 = \log(x^2+1)$, $t_3 = \exp(\frac{x^2}{2})$, $t_4 = \exp(x\log(x^2+1))$.

Derivatives of sparse fractions

total degree	6	7	8	9	10
`CR`	0.19	20.45	0.14	3.43	518.15
`int`	0.88	41.39	1.44	7.67	1449.17

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Derivatives of dense polynomials

total degree	11	12	13	14	15
`CR`	0.40	0.54	0.67	0.81	1.04
`int`	10.06	14.66	21.34	31.84	51.05

[1] MAPLE 2026 on a computer with iMac CPU 3.6GHz, Intel Core i9, 16G

Outline

I. Complete reductions (C.R.)

II. C.R. in the rational case

III. Transcendental Liouvillian extensions

IV. C.R. in the transcendental Liouvillian case

V. Elementary integration via C.R.

V. Elementary extensions

Let $(F, ')$ be a D-field.

Definition. A D-field $F(u_1, \ldots, u_k)$ is an elementary extension of F if, for all $j \in \{1, \ldots, k\}$,

$$u_j \text{ is } \begin{cases} \text{algebraic} \\ \text{logarithmic} \\ \text{exponential} \end{cases} \quad \text{over } F(u_1, \ldots, u_{j-1}).$$

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Definition. An element $f \in F$ has an elementary integral over F if f has an integral in some elementary extension of F.

V. Residues of t-simple fractions

Let t be a monomial over F and $f = \frac{a}{b}$ be t-simple.

Definition. For $\alpha \in \overline{F}$ with $b(\alpha) = 0$,

$$\text{Residue}_t(f, \alpha) := \frac{a(\alpha)}{b'(\alpha)}.$$

V. Simple elements and their residues

Let $F_n = C(t_1, \ldots, t_n)$ be a transcendental D-field extension.

Set $S_i := \{s \in C(t_1, \ldots, t_i) \mid \ s \text{ is } t_i\text{-simple}\}$, $i = 1, \ldots, n$.

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Definition. For a simple element

$$s = s_1 + \cdots + s_n, \quad s_i \in S_i,$$

its residues are exactly the residues of the individual s_i.

V. Property of remainders

Let F_n be a transcendental Liouvillian extension of C.

- $\mathbb{P} := \{i \in \{1, \ldots, n\} \mid t_i \text{ is primitive over } F_{i-1}\}$.
- $\mathbb{H} := \{i \in \{1, \ldots, n\} \mid t_i \text{ is hyperexponential over } F_{i-1}\}$.

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- $$R_i := \begin{cases} t_i F_{i-1}[t_i] & \text{if } i \in \mathbb{P}, \\ t_i F_{i-1}[t_i] + t_i^{-1} F[t_i^{-1}] & \text{if } i \in \mathbb{H}. \end{cases}$$
- $R := C \oplus R_1 \oplus \cdots \oplus R_n$ and $S := S_1 \oplus \cdots \oplus S_n$.

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- $R := C \oplus R_1 \oplus \cdots \oplus R_n$ and $S := S_1 \oplus \cdots \oplus S_n$.

Proposition. Let Ψ_n be a C.R. for F_n'. Then

$$\operatorname{im}(\Psi_n) \subset R \oplus S.$$

V. Elementary integrals of remainders

Definition. Let Ψ_i be a C.R. for F_i' on F_i, $i = 0, 1, \ldots, n$.

The ith significant remainder is $\begin{cases} \Psi_{i-1}(t_i') & \text{if } i \in \mathbb{P}, \\ \Psi_{i-1}(t_i'/t_i) & \text{if } i \in \mathbb{H}. \end{cases}$

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Theorem. Let C be algebraically closed. Then $f \in F_n$ has an elementary integral over F_n iff $\exists$ a simple element s s.t.

(i) $\Psi_n(f) - s$ is a C-linear combination of significant remainders,

(ii) all residues of s belong to C.

V. Outline of an algorithm

Given $f \in F_n$,

- decide if f has an elementary integral over F_n,
- compute such an integral if there exists one.

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remainders:	$\Psi_n(f)$	$\Psi_{i-1}(t_i'),\ i \in \mathbb{P}$	$\Psi_{j-1}(t_j'/t_j),\ j \in \mathbb{H}$
proj to R:	r_f	r_i	r_j
proj to S:	s_f	s_i	s_j

$$\Downarrow$$

$$f \text{ has an elem. int.} \iff \begin{cases} r_f = \sum_{k=1}^n c_k r_k \\ s_f - \sum_{k=1}^n c_k s_k \text{ has only constant residues} \end{cases}$$

for some $c_1, \ldots, c_n \in C$, which leads to a linear system in the c_k's by Raab's thesis (Chap. 3).

V. Example

Integrate

$$f = \frac{(\log x + 1)\,\mathrm{Li}(x^c)}{x^{c+1}},$$

where c is a constant indeterminate, and $\mathrm{Li}(x)' = 1/\log x$.

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1. [Construct a D-field] Set

$$t_1 := x,\ \ t_2 := x^c,\ \ t_3 := \log(x),\ \ t_4 := \mathrm{Li}(x^c).$$

$$\Downarrow$$

$$f \in \mathbb{Q}(t_1, t_2, t_3, t_4).$$

2. [C.R.]

$$f = \Big(\underbrace{\frac{ct_2 t_3 - ct_3 t_4 - (c+1)t_4}{c^2 t_2}}_{g}\Big)' + \underbrace{\frac{c+1}{c^2 t_1 t_3}}_{\Psi_4(f)}.$$

V. Example (continued)

3. [Form a linear system]

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 \end{array}\right).$$

4. [Compute the residues]

$$\Psi_4(f) = \frac{c+1}{c^2}\frac{t_3'}{t_3}.$$

5. [Backtrack]

$$f = g' + \frac{c+1}{c^2}\frac{t_3'}{t_3}$$

$$\Downarrow$$

$$\int f = \frac{1}{c}\log(x) - \frac{c\log x + (c+1)}{c^2 x^c}\mathrm{Li}(x^c) + \frac{c+1}{c^2}\log(\log(x)).$$

Summary

Let F_n be a transcendental Liouvillian extension.

Results.

1. A C.R. Ψ_n for F_n' on F_n, which decomposes every $f \in F_n$ as
$$f = (\cdots)' + \Psi_n(f).$$
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Thank you for listening!