Nonlinear Differential Guessing in Practice

ISSAC 2026 - Software presentation

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What is D-algebraic guessing?

Find a differential equation satisfied by the sequence $f := \sum_{n=0}^{\infty} \zeta(2n+2)x^n$:

$$(\zeta(2n+2))_{n\in\mathbb{N}} = \frac{\pi^2}{6}, \frac{\pi^4}{90}, \frac{\pi^6}{945}, \frac{\pi^8}{9450}, \frac{\pi^{10}}{93555}, \frac{691\pi^{12}}{638512875}, \frac{2\pi^{14}}{18243225}, \ldots$$

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$$p := -2y^2 + 5y' - 4xy'y + 2xy'', \;\; p(f) = 0.$$

What is D-algebraic guessing?

Algorithm

1. INPUT: $s_0, s_1, \ldots, s_N$, $N > 0$, of some $f := \sum_{n=0}^{\infty} s_n x^n$.

2. Search
$$\sum_{j=0}^{r_{\delta_k}} \boldsymbol{C_j(x)\delta_k(y,j)},$$
$C_j(x) \in \mathbb{K}[x]$, $d := \deg(C_j(x))$, $\delta_k(y,j) \equiv$"jth up-to-k-product of derivatives of $y(x)$".

3. OUTPUT: If successful, a differential polynomial p such that $p(\sum_{n=0}^{N} s_n x^n) = 0$.

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Applications

- Combinatorics [Bostan, 2017].
- Physics [Guttmann et al., 2016].

Related work and applications

1. Holonomic guessing

   - `listtodiffeq` [Salvy and Zimmermann, 1994] (Maple).
     - `Gfun` (latest version 4.18, March 2026).
     - Implements [Beckermann and Labahn, 1994].
     - Implements modular computations and reconstruction.

   - `GuessMinDE` [Kauers, 2009] (Mathematica).
     - `Guess.m` (latest version: 2022).
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   - `guess` [Kauers et al., 2015] (SageMath). Preferred for univariate guessing.

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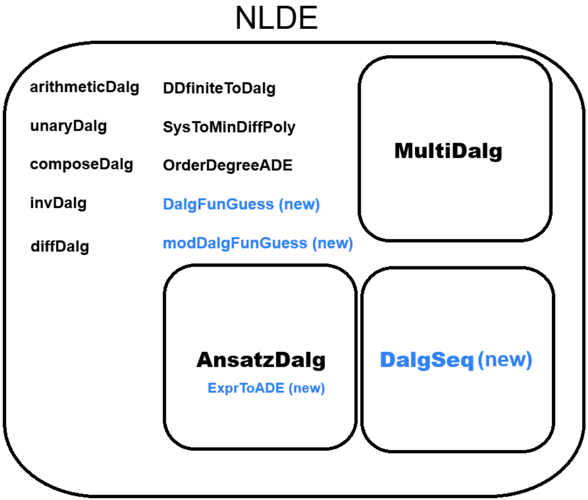
2. Nonlinear D-algebraic guessing
   - `guessADE` [Hebisch and Rubey, 2011] (FriCAS).
     - Implements [Beckermann and Labahn, 1994].
     - Implements modular computations and reconstruction.

This software presentation

Within the **NLDE** package [T., 2023].

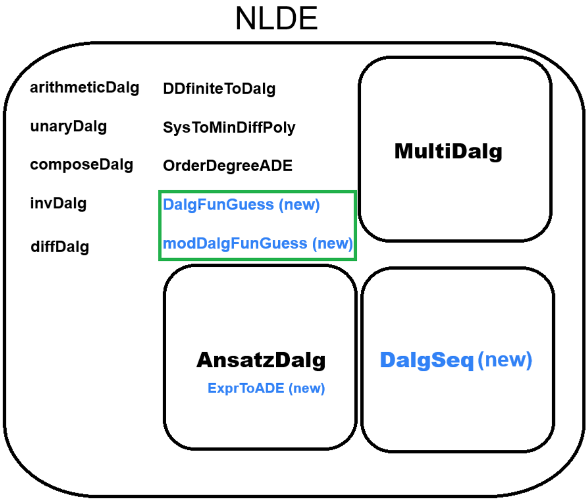
https://github.com/T3gu1a/D-algebraic-functions



This software presentation

Dedicated GitHub page:

https://github.com/T3gu1a/D-algebraic-Guessing



Outline

1. D-algebraic guessing

2. Practical observations

3. Performance

Recurrence approach vs Polynomial substitution

Remark 3 from [T., 2025]: $\approx$ 15 minutes to instantiate a recurrence!

$$\sum_{k_2=0}^{n-2}\sum_{k_1=0}^{k_2}\sum_{k_0=0}^{k_1}(k_0+1)(k_2-k_1+1)(n-1-k_2)s_{k_0+1}s_{k_1-k_0+1}s_{k_2-k_1+1}s_{n-1-k_2}.$$

Recurrence approach vs Polynomial substitution

$$\underbrace{\sum_{k_2=0}^{n-2}\sum_{k_1=0}^{k_2}\sum_{k_0=0}^{k_1}(k_0+1)(k_2-k_1+1)(n-1-k_2)s_{k_0+1}s_{k_1-k_0+1}s_{k_2-k_1+1}s_{n-1-k_2}.}_{\mathcal{O}\left(n^3\right)}$$

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$$\underbrace{\prod_{i=1}^{4} y^{(j_i)}}_{\text{FFT: } \mathcal{O}(n\log n)},\ y := \sum_{n=0}^{N} s_0 x^n.$$

Recurrence approach vs Polynomial substitution

Example $\left(\frac{x}{\tan(x)} + \frac{x}{\sec(x)}\right)$

```
DalgFunGuess(L,degADE=4,degPoly=8, startfromord=2)
```

- Polynomial substitution: 29.36 seconds
- Recurrence approach: Cancelled after 15 minutes!

Choice of monomial ordering

- (bounded-degree) Ordering from [Hebisch and Rubey, 2011]:

$$y, y^2, y', y^3, y\,y', y'', y^4, y^2\,y', {y'}^2, y\,y''$$

- bounded-degree lex ordering:

$$y, y^2, y^3, y^4, y^5, y^6, y', y\,y', y^2\,y', y^3\,y'$$

Choice of monomial ordering

1. (bounded-degree) Ordering from [Hebisch and Rubey, 2011]:

$$y, y^2, y', y^3, y\,y', y'', y^4, y^2\,y', y'^2, y\,y''$$

2. bounded-degree lex ordering:

$$y, y^2, y^3, y^4, y^5, y^6, y', y\,y', y^2\,y', y^3\,y'$$

Choice of monomial ordering

Example ($\sum_{n=0}^{\infty} \frac{(an+b)^n}{n!} x^n$)

It satisfies a third-order ADE of degree 3 with polynomial coefficients of degree at most 2.

- Linear algebra with ordering 1: 77 terms.
- Linear algebra with ordering 2: 74 terms.
- However, **`guessADE`** (FriCAS) only needs 33 terms!

3. Performance

With `guessADE` from `FriCAS 1.3.13`

$$\frac{x}{\sin(x)} + \frac{1}{\cos(x)},$$

$$\frac{x}{\log(1+x)} + \frac{1}{1+\exp(x)},$$

$$\log(1+\sin(x)).$$

FriCAS guessADE for $\frac{x}{\sin(x)} + \frac{1}{\cos(x)}$

The following differential equation is obtained with 38 terms.

$$
\begin{aligned}
&(3\,x^2+6)\,{y''}^2+(9\,x^2+4)\,y^2+(22\,x^2+40)\,{y'}^2-6\,x\,y''\,y'\\
&\quad+(-12\,x^2-10)\,y''\,y-20\,x\,y'\,y+(-2\,x^2-4)\,y^{(3)}\,y'+2\,x\,y^{(3)}\,y=0
\end{aligned}
$$

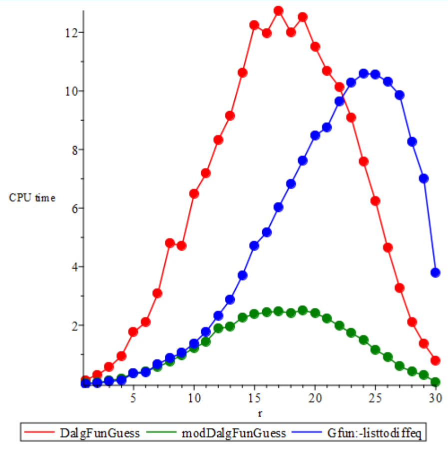
`FriCAS guessADE` for $\frac{x}{\sin(x)} + \frac{1}{\cos(x)}$

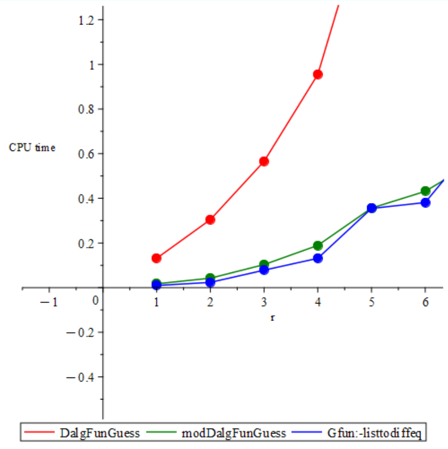
```
                1698159967378800978344092789866016479403179761657128736 36_
                 66083386326645986624410741634268593490986726766464233762_
                 09747128144942616046835628586365528230318385777801523956_
                 54229531221890864423705599001118555750527717376913289389_
                 42944439461866148803229101204010112118165303040447325702_
                 12145363117717084610327003776789307709133835540835767455_
                 293483446585758226632918 4
              *
                   4
                 x
         +
             18606458205030921099485790245870128862066084967401473444815 3_
              06294690697253976300465935798873131283189141741272789908559_
              37501987458427077810904236462537068640406197413175716409183_
              25139255198479595285896794652702151380415919406152716215734_
              49816236721045923035554738901444481079164046776377974819723_
              31214657699208767168442003092948799779421311737251566494438_
              90673553
           *
                2
              x
         +
           -
              50469163176692008841642602120038016571514722856196273160938_
               52829154835787222774047039935107202740273566358109296113 27_
               30790661768524823552227773545458362544867778109145815234 45_
               22732250207278485623925780760821337579075980069045058755 17_
               23697063364760896757753290534795962816493507541192595065 98_
               03642261287127468128782105634330374012020525329172560878 79_
               44955501026870
        *
                ' ' '
           f(x)f     (x)

        +
```

With **Gfun** (holonomic guessing)

NLDE (fast linear algebra) and **Gfun** (Beckermann-Labahn, 1997)





Thank You very much!

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