On the Summability Problem of Multivariate Rational Functions in the Mixed Case

Shaoshi Chen, Lixin Du, Hanqian Fang and Yisen Wang

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences

July 15th, 2026
51st ISSAC in Oldenburg, Germany

Summability Problem

Problem. Determine whether $f(x_1, \ldots, x_n)$ can be written as

$$f = \Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n),$$

Application.

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2.$$

Summability Problem

Problem. Determine whether $f(x_1,\ldots,x_n)$ can be written as

$$f = \Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n),$$

where $f, g_1, \ldots, g_n \in \mathbb{F}(x_1,\ldots,x_n)$ with $\mathbb{F}$ a field of characteristic 0.

q-Summability Problem

Let $q \in \mathbb{F}$ such that $q^i \neq 1,\ \forall i \in \mathbb{Z}$.

Problem. Determine whether $f(x_1, \ldots, x_n)$ can be written as

$$f = \Delta_{q,x_1}(g_1) + \cdots + \Delta_{q,x_n}(g_n).$$

where Δ_{q,x_i} is the q-difference operator w.r.t. x_i.

Application. Evaluate

$$\sum_{(z_1,\ldots,z_n)\in\Omega} f(x_1, \ldots, x_n), \quad \text{where } x_i = q^{z_i} \text{ and } \Omega \subseteq \mathbb{Z}^n.$$

Mixed Summability Problem

Motivation. Let $n = k + m$. Evaluate a multiple sum:

$$\sum_{(y_1,\ldots,y_k,z_1,\ldots,z_m)\in\Omega} f(\underbrace{y_1,\ldots,y_k,q^{z_1},\ldots,q^{z_m}}_{x_1,\ldots,x_n}), \quad \text{where } \Omega \subseteq \mathbb{Z}^n.$$

Shift operators.

$$\theta_{x_i}(f(x_1,\ldots,x_n)) := \begin{cases} f(x_1,\ldots,x_i+1,\ldots,x_n), & \text{if } x_i \in \{y_1,\ldots,y_k\}; \\ f(x_1,\ldots,q\cdot x_i,\ldots,x_n), & \text{if } x_i \in \{q^{z_1},\ldots,q^{z_m}\}. \end{cases}$$

Problem. Determine whether f can be written as

$$f(x_1,\ldots,x_n) = \sum_{i=1}^{n} \Delta_{\theta_{x_i}}(g_i(x_1,\ldots,x_n)).$$

Mixed Summability Problem

Motivation. Let $n = k + m$. Evaluate a multiple sum:

$$\sum_{(y_1,\ldots,y_k,z_1,\ldots,z_m)\in\Omega} f(\underbrace{y_1,\ldots,y_k,q^{z_1},\ldots,q^{z_m}}_{x_1,\ldots,x_n}), \quad \text{where } \Omega \subseteq \mathbb{Z}^n.$$

Shift operators.

$$\theta_{x_i}(f(x_1,\ldots,x_n)) := \begin{cases} f(x_1,\ldots,x_i+1,\ldots,x_n), & \text{if } x_i \in \{y_1,\ldots,y_k\}; \\ f(x_1,\ldots,q\cdot x_i,\ldots,x_n), & \text{if } x_i \in \{q^{z_1},\ldots,q^{z_m}\}. \end{cases}$$

Problem. Determine whether

$$f(x_1,\ldots,x_n) \text{ is } (\theta_{x_1},\ldots,\theta_{x_n})\text{-summable in } \mathbb{F}(x_1,\ldots,x_n).$$

Univariate Rational Summation

Problem. Given a sequence $f(k) \in \mathbb{F}(k)$, find $g(k) \in \mathbb{F}(k)$ such that

$$f(k) = \Delta_k(g).$$

Univariate Rational Summation

Problem. Given a sequence $f(k) \in \mathbb{F}(k)$, find $g(k) \in \mathbb{F}(k)$ such that

$$f(k) = \Delta_k(g).$$

Note that if $f \in \mathbb{F}[k]$, then one can always find such g.

Solution. Abramov (1975), Paule (1995), ...

Examples.

$$\sum_{0 \le k < n} k = \frac{n(n-1)}{2} \qquad \sum_{1 \le k \le n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

$$\sum_{0 \le k < n} k^2 = \frac{n(n-1)(2n-1)}{6}$$

Univariate Rational Summation

Problem. Given a sequence $f(k) \in \mathbb{F}(k)$, find $g(k) \in \mathbb{F}(k)$ such that

$$f(k) = \Delta_k(g).$$

Note that if $f \in \mathbb{F}[k]$, then one can always find such g.

Solution. Abramov (1975), Paule (1995), ...

Examples.

$$\sum_{0 \le k < n} k = \frac{n(n-1)}{2} \qquad \sum_{1 \le k \le n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

$$\sum_{0 \le k < n} k^2 = \frac{n(n-1)(2n-1)}{6}$$

Abramov's Algorithm

Additive decomposition. Given $f \in \mathbb{F}(k)$, compute $g \in \mathbb{F}(k)$ and polynomials $a, d \in \mathbb{F}[k]$ such that

$$f = \Delta_k(g) + \frac{a}{d}, \text{ where } \deg_k(a) < \deg_k(d)$$

and d has the minimal degree among all possible choices, satisfying

$$\frac{d(k+i)}{d} \notin \mathbb{F}, \quad \text{for all } i \in \mathbb{Z}.$$

Summability criterion.

$$f \text{ is } \sigma_k\text{-summable in } \mathbb{F}(k) \quad \Leftrightarrow \quad a = 0.$$

Bivariate Rational Summation

Problem. Given $f \in \mathbb{F}(x,y)$, determine whether f can be written as

$$f = \Delta_x(g) + \Delta_y(h), \text{ where } g,h \in \mathbb{F}(x,y).$$

Additive decomposition w.r.t. y.

$$f = \Delta_x(g) + \Delta_y(h) + \sum_{j=0}^{J} \sum_{i=0}^{I} \frac{a_{i,j}}{d_i^j}, \text{ where } \deg_y(a_{i,j}) < \deg_y(d_i)$$

with d_i's being irreducible and not shift equivalent with each other.

Bivariate Rational Summation

Problem. Given $f \in \mathbb{F}(x,y)$, determine whether f can be written as

$$f = \Delta_x(g) + \Delta_y(h), \text{ where } g, h \in \mathbb{F}(x,y).$$

Additive decomposition w.r.t. y.

$$f = \Delta_x(g) + \Delta_y(h) + \sum_{j=0}^{J} \sum_{i=0}^{I} \frac{a_{i,j}}{d_i^j}, \text{ where } \deg_y(a_{i,j}) < \deg_y(d_i)$$

with d_i's being irreducible and not shift equivalent with each other.

Theorem (ChenSinger2014, HouWang2015).

$$f \text{ is } (\sigma_x, \sigma_y)\text{-summable in } \mathbb{F}(x,y)$$

$$\Updownarrow$$

$$\text{each } \frac{a_{i,j}}{d_i^j} \text{ is } (\sigma_x, \sigma_y)\text{-summable in } \mathbb{F}(x,y)$$

Bivariate Rational Summation

Problem. Given $f \in \mathbb{F}(x,y)$, determine whether f can be written as

$$f = \Delta_x(g) + \Delta_y(h), \text{ where } g, h \in \mathbb{F}(x,y).$$

Additive decomposition w.r.t. y.

$$f = \Delta_x(g) + \Delta_y(h) + \sum_{j=0}^{J} \sum_{i=0}^{I} \frac{a_{i,j}}{d_i^j}, \text{ where } \deg_y(a_{i,j}) < \deg_y(d_i)$$

with d_i's being irreducible and not shift equivalent with each other.

Theorem (ChenSinger2014, HouWang2015).

$$f \text{ is } (\sigma_x, \sigma_y)\text{-summable in } \mathbb{F}(x,y)$$

$$\Updownarrow$$

$$d_i = P_i(\alpha_i x + \beta_i y),\ \alpha_i, \beta_i \in \mathbb{Z},\ P_i \in \mathbb{F}[z] \text{ and } \sigma_x^{\beta_i} \sigma_y^{-\alpha_i}(a_{i,j}) = a_{i,j}.$$

Multivariate Rational Summation

Additive decomposition w.r.t. x_1. Let $\mathbf{x} := x_1, \ldots, x_n$ and $f \in \mathbb{F}(\mathbf{x})$,

$$f = \Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n) + \sum_{i=1}^{I} \sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j}, \text{ where } \deg_{x_1}(a_{i,j}) < \deg_{x_1}(d_i)$$

with d_i's being irreducible and not shift equivalent with each other.

Multivariate Rational Summation

Additive decomposition w.r.t. x_1. Let $\mathbf{x} := x_1, \ldots, x_n$ and $f \in \mathbb{F}(\mathbf{x})$,

$$f = \Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n) + \sum_{i=1}^{I} \sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j}, \text{ where } \deg_{x_1}(a_{i,j}) < \deg_{x_1}(d_i)$$

with d_i's being irreducible and not shift equivalent with each other.

$$f \text{ is summable in } \mathbb{F}(\mathbf{x}) \quad \Leftrightarrow \quad \text{each } \frac{a_{i,j}}{d_i^j} \text{ is summable in } \mathbb{F}(\mathbf{x})$$

Multivariate Rational Summation

Additive decomposition w.r.t. x_1. Let $\mathbf{x} := x_1, \ldots, x_n$ and $f \in \mathbb{F}(\mathbf{x})$,

$$f = \Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n) + \sum_{i=1}^{I} \sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j}, \text{ where } \deg_{x_1}(a_{i,j}) < \deg_{x_1}(d_i)$$

with d_i's being irreducible and not shift equivalent with each other.

$$f \text{ is summable in } \mathbb{F}(\mathbf{x}) \quad \Leftrightarrow \quad \text{each } \frac{a_{i,j}}{d_i^j} \text{ is summable in } \mathbb{F}(\mathbf{x})$$

Isotropy group. Let $G = \langle \sigma_{x_1}, \ldots, \sigma_{x_n} \rangle$. For $p \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$, define

$$G_p := \{\sigma \in G \mid \sigma(p) = p\}.$$

Multivariate Rational Summation

Additive decomposition w.r.t. x_1. Let $\mathbf{x} := x_1, \ldots, x_n$ and $f \in \mathbb{F}(\mathbf{x})$,

$$f = \Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n) + \sum_{i=1}^{I} \sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j}, \text{ where } \deg_{x_1}(a_{i,j}) < \deg_{x_1}(d_i)$$

with d_i's being irreducible and not shift equivalent with each other.

$$f \text{ is summable in } \mathbb{F}(\mathbf{x}) \quad \Leftrightarrow \quad \text{each } \frac{a_{i,j}}{d_i^j} \text{ is summable in } \mathbb{F}(\mathbf{x})$$

Isotropy group. Let $G = \langle \sigma_{x_1}, \ldots, \sigma_{x_n} \rangle$. For $p \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$, define

$$G_p := \{\sigma \in G \mid \sigma(p) = p\}.$$

Theorem. Both G_p and G/G_p are free and $\text{rank}(G_p) \leq n-1$.

Summability Criteria

Theorem (ChenDuFang2025). Let $f = a/d^s$ with irreducible d and $\deg_{x_1}(a) < \deg_{x_1}(d)$. Let $\tau_1, \ldots, \tau_r$ be a basis of G_d with $r < n$.

$$f \text{ is } (\sigma_{x_1}, \ldots, \sigma_{x_n})\text{-summable in } \mathbb{F}(\mathbf{x})$$

$$\Updownarrow$$

$$a = \tau_1(b_1) - b_1 + \cdots + \tau_r(b_r) - b_r \text{ for some } b_i \in \mathbb{F}(\mathbf{x})$$

Summability Criteria

Theorem (ChenDuFang2025). Let $f = a/d^s$ with irreducible d and $\deg_{x_1}(a) < \deg_{x_1}(d)$. Let $\tau_1, \ldots, \tau_r$ be a basis of G_d with $r < n$.

$$f \text{ is } (\sigma_{x_1}, \ldots, \sigma_{x_n})\text{-summable in } \mathbb{F}(\mathbf{x})$$

$$\Updownarrow$$

$$a = \tau_1(b_1) - b_1 + \cdots + \tau_r(b_r) - b_r \text{ for some } b_i \in \mathbb{F}(\mathbf{x})$$

Summability Criteria

Theorem (ChenDuFang2025). Let $f = a/d^s$ with irreducible d and $\deg_{x_1}(a) < \deg_{x_1}(d)$. Let $\tau_1, \ldots, \tau_r$ be a basis of G_d with $r < n$.

$$f \text{ is } (\sigma_{x_1}, \ldots, \sigma_{x_n})\text{-summable in } \mathbb{F}(\mathbf{x})$$

$$\Updownarrow$$

$$a \text{ is } (\tau_1, \ldots, \tau_r)\text{-summable in } \mathbb{F}(\mathbf{x})$$

Summability Criteria

Theorem (ChenDuFang2025). Let $f = a/d^s$ with irreducible d and $\deg_{x_1}(a) < \deg_{x_1}(d)$. Let $\tau_1, \ldots, \tau_r$ be a basis of G_d with $r < n$.

$$f \text{ is } (\sigma_{x_1}, \ldots, \sigma_{x_n})\text{-summable in } \mathbb{F}(\mathbf{x})$$

$$\Updownarrow$$

$$a \text{ is } (\tau_1, \ldots, \tau_r)\text{-summable in } \mathbb{F}(\mathbf{x})$$

$$\Updownarrow \quad \text{via a difference transformation } \phi$$

$$\phi(a) \text{ is } (\sigma_{x_1}, \ldots, \sigma_{x_r})\text{-summable in } \mathbb{F}(\mathbf{x})$$

Summability Criteria

Theorem (ChenDuFang2025). Let $f = a/d^s$ with irreducible d and $\deg_{x_1}(a) < \deg_{x_1}(d)$. Let $\tau_1, \ldots, \tau_r$ be a basis of G_d with $r < n$.

$$f \text{ is } \boxed{(\sigma_{x_1}, \ldots, \sigma_{x_n})}\text{-summable in } \mathbb{F}(\mathbf{x})$$

$$\Updownarrow$$

$$a \text{ is } (\tau_1, \ldots, \tau_r)\text{-summable in } \mathbb{F}(\mathbf{x})$$

$$\Updownarrow \quad \text{via a difference transformation } \phi$$

$$\phi(a) \text{ is } \boxed{(\sigma_{x_1}, \ldots, \sigma_{x_r})}\text{-summable in } \mathbb{F}(\mathbf{x})$$

Strategy. Reduce the problem to the case of fewer variables.

Summability Criteria

Theorem (ChenDuFang2025). Let $f = a/d^s$ with irreducible d and $\deg_{x_1}(a) < \deg_{x_1}(d)$. Let $\tau_1, \ldots, \tau_r$ be a basis of G_d with $r < n$.

$$f \text{ is } (\sigma_{x_1}, \ldots, \sigma_{x_n})\text{-summable in } \mathbb{F}(\mathbf{x})$$

$$\Updownarrow$$

$$a \text{ is } (\tau_1, \ldots, \tau_r)\text{-summable in } \mathbb{F}(\mathbf{x})$$

Example. Let $f := 1/d$ with $d := x_1^s + \cdots + x_n^s$ where $n > 1, s \geq 1$.

We have

$$f \text{ is } (\sigma_{x_1}, \ldots, \sigma_{x_n})\text{-summable in } \mathbb{F}(\mathbf{x}) \quad \Leftrightarrow \quad s = 1.$$

Summability Criteria

Theorem (ChenDuFang2025). Let $f = a/d^s$ with irreducible d and $\deg_{x_1}(a) < \deg_{x_1}(d)$. Let $\tau_1, \ldots, \tau_r$ be a basis of G_d with $r < n$.

$$f \text{ is } (\sigma_{x_1}, \ldots, \sigma_{x_n})\text{-summable in } \mathbb{F}(\mathbf{x})$$

$$\Updownarrow$$

$$a \text{ is } (\tau_1, \ldots, \tau_r)\text{-summable in } \mathbb{F}(\mathbf{x})$$

Example. Let $f := 1/d$ with $d := x_1^s + \cdots + x_n^s$ where $n > 1$, $s \geq 1$.

If $s > 1$, then $G_d = \{\mathbf{1}\}$.

$$\text{So } f \text{ is not } (\sigma_{x_1}, \ldots, \sigma_{x_n})\text{-summable in } \mathbb{F}(\mathbf{x}).$$

Summability Criteria

Theorem (ChenDuFang2025). Let $f = a/d^s$ with irreducible d and $\deg_{x_1}(a) < \deg_{x_1}(d)$. Let $\tau_1, \ldots, \tau_r$ be a basis of G_d with $r < n$.

$$f \text{ is } (\sigma_{x_1}, \ldots, \sigma_{x_n})\text{-summable in } \mathbb{F}(\mathbf{x})$$

$$\Updownarrow$$

$$a \text{ is } (\tau_1, \ldots, \tau_r)\text{-summable in } \mathbb{F}(\mathbf{x})$$

Example. Let $f := 1/d$ with $d := x_1^s + \cdots + x_n^s$ where $n > 1$, $s \geq 1$.

If $s = 1$, then

$$G_d = \langle \tau_1, \ldots, \tau_{n-1} \rangle \quad \text{and } 1 = \tau_1(x_1) - x_1,$$

where $\tau_i = \sigma_{x_i}\sigma_{x_{i+1}}^{-1}$. So f is summable in $\mathbb{F}(\mathbf{x})$.

Summability Criteria

Theorem (ChenDuFang2025). Let $f = a/d^s$ with irreducible d and $\deg_{x_1}(a) < \deg_{x_1}(d)$. Let $\tau_1, \ldots, \tau_r$ be a basis of G_d with $r < n$.

$$f \text{ is } (\sigma_{x_1}, \ldots, \sigma_{x_n})\text{-summable in } \mathbb{F}(\mathbf{x})$$

$$\Updownarrow$$

$$a \text{ is } (\tau_1, \ldots, \tau_r)\text{-summable in } \mathbb{F}(\mathbf{x})$$

Example. Let $f := 1/d$ with $d := x_1^s + \cdots + x_n^s$ where $n > 1$, $s \geq 1$.

If $s = 1$, then

$$\frac{1}{x_1 + \cdots + x_n} = \Delta_{x_1}\left(\frac{x_1}{x_1 + \cdots + x_n}\right) + \Delta_{x_2}\left(\frac{-x_1 - 1}{x_1 + \cdots + x_n}\right).$$

Multivariate Rational Summation in the Mixed Case

Goal.

- Reduction to the case of fewer variables.
- Unified framework for mixed shift case.

Solution (What is new?).

- Generalize the problem: Given $f \in \mathbb{F}(\mathbf{x})$ and $\mathbf{c} \in \mathbb{F}^n$, determine whether f is $(c_1\theta_{x_1}, \ldots, c_n\theta_{x_n})$-summable in $\mathbb{F}(\mathbf{x})$.
- Distinguish the normal and special cases of the denominator d.
- Direct decomposition of the isotropy group $G_d = G_d^\sigma \oplus G_d^\tau$.
- Construct a combined difference transformation ϕ.

Multivariate Rational Summation in the Mixed Case

Goal.

- Reduction to the case of fewer variables.
- Unified framework for mixed shift case.

Normal polynomials. Let p denote a polynomial in x_1 over $\mathbb{F}(\check{\mathbf{x}})$. Define that p is normal w.r.t. θ_{x_1} if $\gcd(p, \theta_{x_1}^{\ell}(p)) = 1, \ \forall \ell \in \mathbb{Z} \setminus \{0\}$.

Special polynomials. p is called special w.r.t. θ_{x_1} if p is not normal. For example, we have x_1 is special w.r.t. τ_{q,x_1} since $\tau_{q,x_1}(x_1) = q \cdot x_1$.

Isotropy group. Let $G = \langle \theta_{x_1}, \ldots, \theta_{x_n} \rangle$. For $p \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$, define

$$G_p := \{\sigma \in G \mid \sigma(p) = c \cdot p \text{ for some } c \in \mathbb{F}\}.$$

Multivariate Rational Summation in the Mixed Case

Goal.

- Reduction to the case of fewer variables.
- Unified framework for mixed shift case.

Normal polynomials. Let p denote a polynomial in x_1 over $\mathbb{F}(\check{\mathbf{x}})$. Define that p is normal w.r.t. θ_{x_1} if $\gcd(p, \theta_{x_1}^{\ell}(p)) = 1, \ \forall \ell \in \mathbb{Z} \setminus \{0\}$.

Special polynomials. p is called special w.r.t. θ_{x_1} if p is not normal. For example, we have x_1 is special w.r.t. τ_{q,x_1} since $\tau_{q,x_1}(x_1) = q \cdot x_1$.

Isotropy group. Let $G = \langle \theta_{x_1}, \ldots, \theta_{x_n} \rangle$. For $p \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$, define

$$G_p := \{\sigma \in G \mid \sigma(p) = c \cdot p \text{ for some } c \in \mathbb{F}\}.$$

A Recursive Strategy

Let $\tilde{\theta}_{x_i} = c_i \cdot \theta_{x_i}$, the $(\tilde{\theta}_{x_1}, \ldots, \tilde{\theta}_{x_n})$-summability of f

$\Downarrow$ Additive decomposition of f w.r.t. x_1

$(\tilde{\theta}_{x_1}, \ldots, \tilde{\theta}_{x_n})$-summability of $\dfrac{a}{d^j}$

if d is normal w.r.t. θ_{x_1}
$G_d := \langle \tau_1, \ldots, \tau_r \rangle$

$(\tau_1, \ldots, \tau_r)$-summability of a

$\Downarrow$ via difference transformation ϕ

$(\widehat{\theta}_{x_1}, \ldots, \widehat{\theta}_{x_r})$-summability of $\phi(a)$ $\longrightarrow$ **Reduction to fewer variables**

if d is special w.r.t. θ_{x_1}
and $c_1 = q^{-\nu}$

$(\tilde{\theta}_{x_2}, \ldots, \tilde{\theta}_{x_n})$-summability of $\dfrac{a}{d^j}$ $\longrightarrow$ **Reduction to fewer variables**

Example

Determine whether f is $(\tau_{q,x_1}, \ldots, \tau_{q,x_n})$-summable in $\mathbb{F}(\mathbf{x})$, where

$$f := \frac{1}{x_1^s + \cdots + x_n^s} \text{ whith } s, n \in \mathbb{N}^+.$$

Let $G := \langle \tau_{q,x_1}, \ldots, \tau_{q,x_n} \rangle$ and $d := x_1^s + \cdots + x_n^s$.

Example

Determine whether f is $(\tau_{q,x_1},\ldots,\tau_{q,x_n})$-summable in $\mathbb{F}(\mathbf{x})$, where

$$f := \frac{1}{x_1^s + \cdots + x_n^s} \text{ whith } s, n \in \mathbb{N}^+.$$

Let $G := \langle \tau_{q,x_1},\ldots,\tau_{q,x_n} \rangle$ and $d := x_1^s + \cdots + x_n^s$. When $n = 1$,

$$f = \frac{1}{x_1^s} = \Delta_{\tau_{q,x_1}}\left(\frac{1/(q^{-s}-1)}{x_1^s}\right).$$

Example

Determine whether f is $(\tau_{q,x_1},\dots,\tau_{q,x_n})$-summable in $\mathbb{F}(\mathbf{x})$, where

$$f := \frac{1}{x_1^s+\cdots+x_n^s} \text{ whith } s,n\in\mathbb{N}^+.$$

Let $G := \langle \tau_{q,x_1},\dots,\tau_{q,x_n}\rangle$ and $d := x_1^s+\cdots+x_n^s$. When $n=1$,

$$f = \frac{1}{x_1^s} = \Delta_{\tau_{q,x_1}}\left(\frac{1/(q^{-s}-1)}{x_1^s}\right).$$

When $n>1$, $G_d =< \tau := \tau_{q,x_1}\cdots\tau_{q,x_n} >$ with $\tau(d) = q^s\cdot d$. Since

$$1 = q^{-s}\tau\left(\frac{1}{q^{-s}-1}\right) - \frac{1}{q^{-s}-1},$$

we have $f = \sum_{i=1}^n \Delta_{\tau_{q,x_i}}(g_i)$, where

$$g_i := \tau_{q,x_{i+1}}\circ\cdots\circ\tau_{q,x_n}\left(\frac{1}{(q^{-s}-1)(x_1^s+\cdots+x_n^s)}\right).$$

Applications of Evaluating Sums

We show that $S(q)$ converges for $0<|q|<1$, where

$$S(q) := \sum_{n,k,m\geq 1}\left(\frac{q^{k+m}}{[n+k+m]_q} - (1-q)q^{k+m}\right) \text{ with } [N]_q := \frac{1-q^N}{1-q}.$$

Let $(x,y,z) := (q^n, q^k, q^m)$. Then considering the first sum we have

$$\frac{(1-q)yz}{1-xyz} = \Delta_{\tau_{q,x}}\left(\frac{yz}{1-q^{-1}xyz}\right) - \Delta_{\tau_{q,y}}\left(\frac{yz}{1-q^{-1}xyz}\right).$$

In conclusion, we have

$$S(q) = 2\left(\frac{q}{1-q}\right)^2 - \sum_{k,m\geq 1}\frac{q^{k+m}}{1-q^{k+m}} - q\sum_{m\geq 1}\frac{q^m}{1-q^m}.$$

Conclusion and Future Work

Conclusion. Determine whether $f \in \mathbb{F}(x_1, \ldots, x_n)$ can be written as

$$f = \Delta_{\theta_{x_1}}(g_1) + \cdots + \Delta_{\theta_{x_n}}(g_n),$$

where $g_i \in \mathbb{F}(x_1, \ldots, x_n)$. If this is the case, compute the g_i's.

Conclusion and Future Work

Conclusion. Determine whether $f \in \mathbb{F}(x_1, \ldots, x_n)$ can be written as

$$f = \Delta_{\theta_{x_1}}(g_1) + \cdots + \Delta_{\theta_{x_n}}(g_n),$$

where $g_i \in \mathbb{F}(x_1, \ldots, x_n)$. If this is the case, compute the g_i's.

Future work.

- Additive decomposition:

$$f = \Delta_{\theta_{x_1}}(g_1) + \cdots + \Delta_{\theta_{x_n}}(g_n) + r,$$

  where $g_i, r \in \mathbb{F}(x_1, \ldots, x_n)$ and r is minimal in some sense.
- Multivariate hypergeometric case.
- Existence and construction of telescoper.

Conclusion and Future Work

Conclusion. Determine whether $f \in \mathbb{F}(x_1, \ldots, x_n)$ can be written as

$$f = \Delta_{\theta_{x_1}}(g_1) + \cdots + \Delta_{\theta_{x_n}}(g_n),$$

where $g_i \in \mathbb{F}(x_1, \ldots, x_n)$. If this is the case, compute the g_i's.

Future work.

- Additive decomposition:

  $$f = \Delta_{\theta_{x_1}}(g_1) + \cdots + \Delta_{\theta_{x_n}}(g_n) + r,$$

  where $g_i, r \in \mathbb{F}(x_1, \ldots, x_n)$ and r is minimal in some sense.
- Multivariate hypergeometric case.
- Existence and construction of telescoper.

Thank you!