# Minimal Generating Sets of Syzygy Modules of Quadratic Ladder Determinantal Ideals

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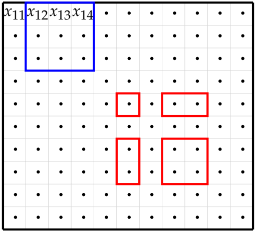
# Determinantal ideals

- $X$: generic matrix of size $r \times s$ with entries $x_{11}, \dots, x_{rs}$
- $\mathcal{K}[X]$: the polynomial ring $\mathcal{K}[x_{11}, \dots, x_{rs}]$

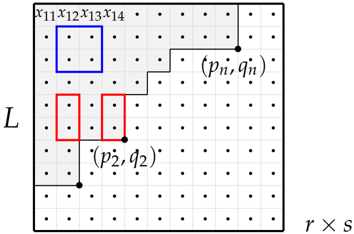



$r \times s$

- Take all the 3-minors of $X$: for example, $[1,2,3|2,3,4]$, $[5,7,8|5,7,8]$ $\Longrightarrow$ determinantal ideal $I_3(X)$: the ideal generated by them in $\mathcal{K}[X]$ [Bruns & Vetter 06]
- Results: the Gröbner basis of $I_3(X)$, syzygy module $\mathrm{Syz}(I_3(X))$, and first graded Betti numbers of $I_3(X)$ are all known [Sturmfels 90], [Kurano 89], [Ma 93]

# Generalization to quadratic ladder determinantal ideals

- $L$: a ladder-shape subset of $X$, represented by its corners
- $\mathcal{K}[L]$: the polynomial ring over $K$ with variables from $L$




- Take all the 2-minors of $L$: for example, $[2,3|2,3]$, $[5,6|2,4]$ $\Longrightarrow$ $J$: the ideal generated by them in $\mathcal{K}[L]$ [Abhyankar 84], [Conca & Herzog 97]
- Result: Gröbner basis of $J$ is known [Fulton 92], [Knutson & Miller 05]
- Goal: find the minimal generating set of the syzygy module $\mathrm{Syz}(J)$ and the first graded Betti numbers

# Syzygy modules of polynomial ideals

Let $S = \mathcal{K}[x_1, \ldots, x_n]$ equipped with **standard grading** $\deg(x_i) = 1$ and $I = \langle f_1, \ldots, f_r \rangle \subset S$. A syzygy of $I$ is a generalization of a "linear relation" over $S$:

$$s_1 f_1 + s_2 f_2 + \cdots + s_r f_r = 0, \quad s_i \in S.$$

To formalize this, let $\varepsilon_i$ is the $i$-th element of the natural basis of the free $S$-module $S^r$. Then we define the following $S$-linear map,

$$\varphi : S^r \to I, \quad \varphi(\sum_{i=1}^{r} s_i \varepsilon_i) = \sum_{i=1}^{r} s_i f_i, \quad s_i \in S,$$

and $\mathrm{Syz}(I) := \ker \varphi$ is called the syzygy module of $I$.

# First graded Betti numbers

**Definition**

The generating set $\{\tau_i\}_{i=1}^m$ of $\mathrm{Syz}(I)$ is said to be minimal if no $\tau_i$ is an $S$-combination of the other generators.

If $I$ is homogeneous, then the first graded Betti numbers $\beta_{1,j}(I)$ of $I$ are the numbers of the minimal generators of $\mathrm{Syz}(I)$ w.r.t. different degrees $j$.

- $I$ homogeneous: Iteratively computing Syzygy modules $\Longrightarrow$ minimal graded free resolution of $I$ [Peeva 11] $\Longrightarrow$ Betti table $\Longrightarrow$ various algebraic invariants of $I$: projective dimension, depth, Castelnuovo-Mumford regularity, Cohen-Macaulayness.
- Generators of $\mathrm{Syz}(I)$ can be constructed by using Schreyer's theorem: the induced syzygies computed with S-pairs form a generating set for $\mathrm{Syz}(I)$ [Schreyer 80]

## Syzygy modules of determiantal ideals: Type-I syzygies

Let $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$ and consider the determinantal ideal $I_2(X) = \langle f_1, f_2, f_3 \rangle$, with

$$f_1 = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}, \quad f_2 = \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix}, \quad f_3 = \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix}.$$

Instead of computing the generators of $\mathrm{Syz}(I_2(X))$ with the Schreyer's theorem, we present the result by Kurano [Kurano 89]

- Construct the following matrices constructed from $X$

$$X_1 = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_{21} & x_{22} & x_{23} \\ x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}.$$

# Syzygy modules of determiantal ideals: Type-I syzygies

- Then it is obvious that $\det X_1 = 0$ and $\det X_2 = 0$. We expand them along the first row to get

$$\det X_1 = x_{11}f_3 - x_{12}f_2 + x_{13}f_1 = 0, \quad \det X_2 = x_{21}f_3 - x_{22}f_2 + x_{23}f_1 = 0.$$

The above equations yield syzygies of $I_2(X)$,

$$\tau_1 = x_{11}\varepsilon_3 - x_{12}\varepsilon_2 + x_{13}\varepsilon_1, \quad \tau_2 = x_{21}\varepsilon_3 - x_{22}\varepsilon_2 + x_{23}\varepsilon_1.$$

$\Longrightarrow$ We call $\tau_1, \tau_2$ **type-I** syzygies.

# Syzygy modules of determiantal ideals: Type-II syzygies

Consider the generic matrix $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ and the determinantal ideal $I_2(X)$.

We expand $X$ along the **3rd row** and the **3rd column**, then **subtract** the two expressions and get

$$\begin{aligned} 0 = &(x_{31}[1,2 \mid 2,3] - x_{32}[1,2 \mid 1,3] + x_{33}[1,2 \mid 1,2]) - (x_{13}[2,3 \mid 1,2] - x_{23}[1,3 \mid 1,2] + x_{33}[1,2 \mid 1,2]) \\ = &\ x_{31}[1,2 \mid 2,3] - x_{32}[1,2 \mid 1,3] - x_{13}[2,3 \mid 1,2] + x_{23}[1,3 \mid 1,2]. \end{aligned}$$

The above equation gives a syzygy of $I_2(X)$.
$\Longrightarrow$ We call it a **type-II** syzygy: $I_2(X)$ has $3 \times 3 = 9$ syzygies of type-II.

# Generating set of $\mathrm{Syz}(I_t(X))$: Kurano generators

**Kurano's Theorem** [Kurano 89]

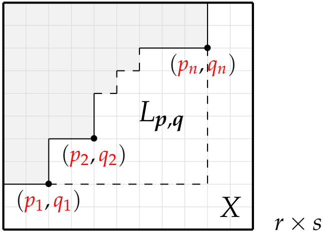
Let $X = [x_{ij}]_{r\times s}$ be a generic matrix of size $r \times s$ and $S = \mathcal{K}[X]$. Given $t \le \min(r,s)$, Then type-I and type-II syzygies generate $\mathrm{Syz}(I_t(X))$ for the determinantal ideal $I_t(X)$.

Remarks:

- Type-I and type-II syzygies are NOT necessarily minimal
- In [Ma 93], a subset of type-I and type-II syzygies that minimally generates $\mathrm{Syz}(I_t(X))$ are identified $\Longrightarrow \mathbb{S}_K(I_t)$, Kurano generators
- For the first graded Betti numbers: $\beta_{1,t+1}(I_t(X)) = \#\mathbb{S}_K(I_t(X))$, and $\beta_{1,t+\ell} = 0$ for all $\ell > 1$.

# Generalization to quadratic ladder determinantal ideals

- $L_{\boldsymbol{p},\boldsymbol{q}}$: ladder submatrix of a generic matrix $X$, represented with corners $(p_1, q_1), \ldots, (p_n, q_n) \Longrightarrow L_{\boldsymbol{p},\boldsymbol{q}} = \cup_{k=1}^{n} X_{p_k,q_k}$




- $J_k = I_2(X_{p_k \times q_k})$, the ideal generated by 2-minors of $X_{p_k \times q_k}$;
- $J = \sum_{k=1}^{n} J_k$, the ideal generated by 2-minors of $L_{\boldsymbol{p},\boldsymbol{q}}$: quadratic ladder determinantal ideal

# Generalization to quadratic ladder determinantal ideals

Let $J$ be the quadratic ladder determinantal ideal of $L_{p,q}$.

**Theorem (Gröbner basis of $J$)**

All the 2-minors of $L_{p,q}$ form a Gröbner basis for $J$ w.r.t. any diagonal order.

Problems:

- What is the generating set of $\mathrm{Syz}(J)$?
- How can we minimize it? (the first graded Betti numbers $\beta_{1,j}(J)$)

First observation:

$$\tau \in \mathbb{S}_K(J_k) \implies \tau \in \mathrm{Syz}(J) \implies \bigcup_{k=1}^{n} \mathbb{S}_K(J_k) \subset \mathrm{Syz}(J).$$

We denote $\bigcup_k \mathbb{S}_K(J_k)$ as $\mathbb{S}_K(J)$: Part I of our generating set

# Type-L syzygies of quadratic ladder determinantal ideals

**Example (Type-L syzygies (inspired by type-II syzygies))**

Give an L-shape matrix $L_{\boldsymbol{p},\boldsymbol{q}} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & * \end{bmatrix}$, where $\boldsymbol{p} = (3,2)$ and $\boldsymbol{q} = (2,3)$. We have

$$\begin{aligned} 0 =& (x_{31}[1,2 \mid 2,3] - x_{32}[1,2 \mid 1,3] + *[1,2 \mid 1,2]) - (x_{13}[2,3 \mid 1,2] - x_{23}[1,3 \mid 1,2] + *[1,2 \mid 1,2]) \\ =& \ x_{31}[1,2 \mid 2,3] - x_{32}[1,2 \mid 1,3] - x_{13}[2,3 \mid 1,2] + x_{23}[1,3 \mid 1,2]. \end{aligned}$$

The above equation gives a syzygy of $J$. And we call it a **type-L** syzygy.

- the terms involving the empty entry would be eliminated
- we denote this type of syzygies as $\mathbb{S}_L(J)$: Part II of our generating set

# Type-L syzygies of quadratic ladder determinantal ideals

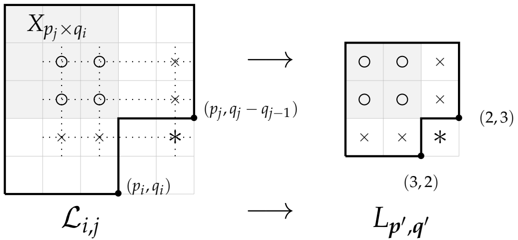



Figure: An illustration of construction of Type-L syzygies

ladder matrix $L_{p,q} \longrightarrow$ ladder submatrix $\mathcal{L}_{i,j} \longrightarrow$ simplest $L_{p',q'}$ case

# Trivial syzygies of quadratic ladder determinantal ideals

## Example (Trivial syzygies)

Give an L-shape matrix $L_{p,q} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & * & * \\ x_{41} & x_{42} & * & * \end{bmatrix}$, where $\boldsymbol{p} = (4,2)$ and $\boldsymbol{q} = (2,4)$. Consider the following minors

$$\Delta_1 = [3,4 \mid 1,2], \quad \Delta_2 = [1,2 \mid 3,4].$$

Then the syzygy $\gamma$ induced by $\mathcal{S}(\Delta_1, \Delta_2)$ is called a trivial syzygy.

- Intuitively speaking, a trivial syzygy determines an "empty block" of the ladder matrix.

We denote all the trivial syzygies as $\mathbb{S}_\Gamma(J)$: Part III of our generating set

# Main result: a minimal generating set of $\mathrm{Syz}(J)$

**Theorem (Minimal generators)**

$\mathbb{S}_K(J) \cup \mathbb{S}_L(J) \cup \mathbb{S}_\Gamma(J)$ forms a minimal generating set of $\mathrm{Syz}(J)$. Moreover,

- $\beta_{1,3}(J) = \#\mathbb{S}_K(J) + \#\mathbb{S}_L(J)$;
- $\beta_{1,4}(J) = \#\mathbb{S}_\Gamma(J)$;
- $\beta_{1,\ell} = 0$, $\forall \ell > 4$.

Hence, $\beta_1(J) = \beta_{1,3}(J) + \beta_{1,4}(J) = \#\mathbb{S}_K(J) + \#\mathbb{S}_L(J) + \#\mathbb{S}_\Gamma(J)$.

Main idea of the proof:

- induction on the scale of the ladder
- divide syzygies into different sets

# Main result: combinatorial formulas

## Theorem (Counting Formulas)

- $\#\mathbb{S}_K(J) = f(p_1, q_1, 2) + \sum_{k=2}^{n} f(p_k, q_k, 2) - f(p_k, q_{k-1}, 2)$,
- $\#\mathbb{S}_L(J) = \sum_{i=2}^{n} \sum_{j=1}^{i-1} \binom{p_i}{2}\binom{q_j}{2} \, (q_i - q_{i-1})(p_j - p_{j+1})$,
- $\#\mathbb{S}_\Gamma(J) = \sum_{k=2}^{n} (A_k + B_k)$

where

- $f(r,s,t) = t\left(\binom{r}{t}\binom{s}{t+1} + \binom{r}{t+1}\binom{s}{t}\right) + 2t\binom{r}{t+1}\binom{s}{t+1}$;
- $A_k = \binom{q_k - q_{k-1}}{2}\binom{p_k}{2} C_k$;
- $B_k = (q_k - q_{k-1})\binom{p_k}{2} \sum_{\ell=2}^{k-1} (q_\ell - q_{\ell-1}) C_\ell$;
- $C_k = \sum_{i=2}^{k} \left[\binom{p_{i-1} - p_i}{2}\binom{q_{i-1}}{2} + (p_{i-1} - p_i) \sum_{j=1}^{i-2} (p_j - p_{j+1}) \binom{q_j}{2}\right]$.

- We only need to count to have the first graded Betti numbers of $J$!

# Experimental results for computing $\mathrm{Syz}(J)$

Table: Our algorithms versus those in MACAULAY2

| $(p_1, \ldots, p_n)$ | $(q_1, \ldots, q_n)$ | $\beta_{1,3}(J)$ | $\beta_{1,4}(J)$ | Ours | MACAULAY2 |
|---|---|---|---|---|---|
| 9, 6, 3 | 3, 6, 9 | 5055 | 1377 | 2.04 s | 8.87 s |
| 12, 7, 2 | 2, 7, 12 | 9655 | 4800 | 3.78 s | 42.98 s |
| 8, 7, 6, 5 | 5, 6, 7, 8 | 10425 | 100 | 4.46 s | 24.54 s |
| 11, 8, 5, 3 | 3, 5, 8, 11 | 11857 | 6264 | 6.78 s | 904.36 s |
| 12, 9, 6, 3 | 3, 6, 9, 12 | 26260 | 11421 | 11.57 s | O-M |
| 9, 8, 6, 5, 4 | 3, 4, 5, 6, 7 | 4875 | 354 | 2.25 s | 5.19 s |
| 10, 8, 6, 4, 2 | 2, 4, 6, 8, 10 | 6540 | 1299 | 2.65 s | 18.56 s |

- We only need to construct the generators! $\Longrightarrow$ much faster
- If only the first Betti numbers are needed, then $<$0.1s with our formulas

# Concluding Remarks

Our contributions (for quadratic ladder determinantal ideals):

- Minimal generating sets for syzygies of different degrees
- Combinatorial formulas for the first graded Betti numbers
- Algorithms for the above

Future works

- 2-minors $\rightarrow$ **t-minors** $\rightarrow$ mixed size...
- ladder determinantal ideals $\rightarrow$ Schubert ones (Schubert calculus)
- the first syzygies $\rightarrow$ higher order syzygies $\rightarrow$ **free resolutions**...

Thanks for your attention!

Vielen Dank für Ihre Aufmerksamkeit.