Algorithms for Algebraic and Arithmetic Attributes of Hypergeometric Functions

joint work with X. Caruso (DOI: 10.1145/3815436.3815475)

Florian Fürnsinn

University of Vienna

51st International Symposium on Symbolic and Algebraic Computation, Oldenburg

July 14, 2026

Overview

Introduction

A **hypergeometric function** with **top parameters** $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Q}^n$ and **bottom parameters** $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m) \in \mathbb{Q}^m$ is defined as the power series

$$h(x) = \mathcal{H}\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}; x \end{bmatrix} := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_m)_k} \cdot x^k \in \mathbb{Q}[\![x]\!],$$

where $(\gamma)_k := \gamma(\gamma+1)\cdots(\gamma+k-1)$ denotes the **rising factorial.**

Introduction

A **hypergeometric function** with **top parameters** $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}^n$ and **bottom parameters** $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m) \in \mathbb{Q}^m$ is defined as the power series

$$h(x) = \mathcal{H}\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}; x \end{bmatrix} \coloneqq \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_m)_k} \cdot x^k \in \mathbb{Q}[\![x]\!],$$

where $(\gamma)_k \coloneqq \gamma(\gamma+1)\cdots(\gamma+k-1)$ denotes the **rising factorial.**

It satisfies the differential equation

$$x(\theta+\alpha_1)\cdots(\theta+\alpha_n)(\theta+1)F(x) = \theta(\theta+\beta_1-1)\cdots(\theta+\beta_m-1)F(x) \qquad (\theta = x\frac{\mathrm{d}}{\mathrm{d}x}).$$

Introduction

A **hypergeometric function** with **top parameters** $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Q}^n$ and **bottom parameters** $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m) \in \mathbb{Q}^m$ is defined as the power series

$$h(x) = \mathcal{H}\left[\begin{matrix}\boldsymbol{\alpha} \\ \boldsymbol{\beta}\end{matrix}; x\right] := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_m)_k} \cdot x^k \in \mathbb{Q}[\![x]\!],$$

where $(\gamma)_k := \gamma(\gamma+1)\cdots(\gamma+k-1)$ denotes the **rising factorial.**

It satisfies the differential equation

$$x(\theta+\alpha_1)\cdots(\theta+\alpha_n)(\theta+1)F(x) = \theta(\theta+\beta_1-1)\cdots(\theta+\beta_m-1)F(x) \qquad (\theta = x\frac{\mathrm{d}}{\mathrm{d}x}).$$

Classically, one requires $\beta_m = 1$ and writes

$$F(x) = {}_nF_{m-1}\left[\begin{matrix}\alpha_1, \dots, \alpha_n \\ \beta_1, \dots, \beta_{m-1}\end{matrix}; x\right].$$

Motivation

Hypergeometric functions are ideal test cases for many conjectures about algebraic and arithmetic properties of power series that satisfy a differential equation. For example:

- The p-**curvature conjecture** [Grothendieck, 1969].

Motivation

Hypergeometric functions are ideal test cases for many conjectures about algebraic and arithmetic properties of power series that satisfy a differential equation. For example:

- The **p-curvature conjecture** [Grothendieck, 1969].
- **Christol's conjecture** on diagonals [Christol, 1986].

Motivation

Hypergeometric functions are ideal test cases for many conjectures about algebraic and arithmetic properties of power series that satisfy a differential equation. For example:

- The p-**curvature conjecture** [Grothendieck, 1969].
- **Christol's conjecture** on diagonals [Christol, 1986].
- A conjecture relating **Galois groups** of reductions of solutions of differential equations across different primes. [Caruso, F., Vargas-Montoya 2025].

Motivation

Hypergeometric functions are ideal test cases for many conjectures about algebraic and arithmetic properties of power series that satisfy a differential equation. For example:

- The **p-curvature conjecture** [Grothendieck, 1969].
- **Christol's conjecture** on diagonals [Christol, 1986].
- A conjecture relating **Galois groups** of reductions of solutions of differential equations across different primes. [Caruso, F., Vargas-Montoya 2025].

To test the last conjecture on hypergeometric functions, it is necessary to

- find the set of prime numbers appearing in the denominators of the series,

Motivation

Hypergeometric functions are ideal test cases for many conjectures about algebraic and arithmetic properties of power series that satisfy a differential equation. For example:

- The **p-curvature conjecture** [Grothendieck, 1969].
- **Christol's conjecture** on diagonals [Christol, 1986].
- A conjecture relating **Galois groups** of reductions of solutions of differential equations across different primes. [Caruso, F., Vargas-Montoya 2025].

To test the last conjecture on hypergeometric functions, it is necessary to

- find the set of prime numbers appearing in the denominators of the series,
- decide if their reductions modulo primes are algebraic, and

Motivation

Hypergeometric functions are ideal test cases for many conjectures about algebraic and arithmetic properties of power series that satisfy a differential equation. For example:

- The **p-curvature conjecture** [Grothendieck, 1969].
- **Christol's conjecture** on diagonals [Christol, 1986].
- A conjecture relating **Galois groups** of reductions of solutions of differential equations across different primes. [Caruso, F., Vargas-Montoya 2025].

To test the last conjecture on hypergeometric functions, it is necessary to

- find the set of prime numbers appearing in the denominators of the series,
- decide if their reductions modulo primes are algebraic, and
- find minimal polynomials of hypergeometric functions modulo prime numbers.

State of the Art: Valuations

Theorem (Christol 1986)

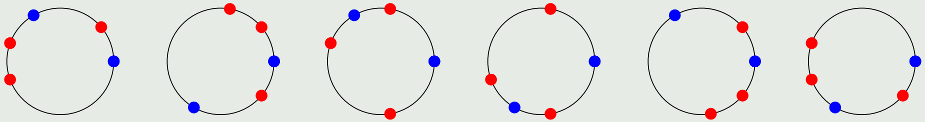
Let $h(x) = \mathcal{H}\left[\begin{matrix}\alpha_1,\dots,\alpha_n\\ \beta_1,\dots,\beta_m\end{matrix};x\right]$, *denote by* N *the least common denominator of all parameters.*

State of the Art: Valuations

Theorem (Christol 1986)

Let $h(x) = \mathcal{H}\left[\begin{matrix}\alpha_1,\ldots,\alpha_n\\ \beta_1,\ldots,\beta_m\end{matrix}; x\right]$, *denote by* N *the least common denominator of all parameters. Draw the sets* $\{\exp(2\pi i\lambda\alpha_j)\}$ *in red and* $\{\exp(2\pi i\lambda\beta_j)\}$ *in blue on the unit circle for all* $1 \leq \lambda \leq N$ *with* $\gcd(\lambda, N) = 1$. *Then the coefficients of* $h(x)$ *only have finitely many primes in the denominator if and only if there are always at least as many red as blue points going counter-clockwise starting after* 1 *(counted with multiplicity).*

Example ($\mathcal{H}([1/9, 4/9, 5/9], [1/3, 1, 1]; x)$)



State of the Art: Valuations

Theorem (Christol 1986)

Let $h(x) = \mathcal{H}\left[\begin{matrix}\alpha_1,\dots,\alpha_n \\ \beta_1,\dots,\beta_m\end{matrix}; x\right]$, *denote by* N *the least common denominator of all parameters. Draw the sets* $\{\exp(2\pi i\lambda\alpha_j)\}$ *in red and* $\{\exp(2\pi i\lambda\beta_j)\}$ *in blue on the unit circle for all* $1 \leq \lambda \leq N$ *with* $\gcd(\lambda, N) = 1$. *Then the coefficients of* $h(x)$ *only have finitely many primes in the denominator if and only if there are always at least as many red as blue points going counter-clockwise starting after* 1 *(counted with multiplicity).*

Algorithm (Caruso – F.)

Given a hypergeometric function $h(x)$, and any prime number p, decides if p appears in the denominators of the coefficients of $h(x)$, and outputs the highest powers of the p by which any denominator is divisible.

Valuations

Let p be a prime. The **valuation** of $h(x) = \sum_{k\geq 0} h_k x^k \in \mathbb{Q}[\![x]\!]$ is defined as

$$\operatorname{val}_{p,0}(h(x)) = \min_{k\geq 0} \operatorname{val}_p h_k \qquad .$$

Valuations

Let p be a prime. The v-drifted **valuation** of $h(x) = \sum_{k \geq 0} h_k x^k \in \mathbb{Q}[\![x]\!]$ is defined as

$$\mathrm{val}_{p,v}(h(x)) = \min_{k \geq 0} \mathrm{val}_p\, h_k + vk.$$

Valuations

Let p be a prime. The v-drifted **valuation** of $h(x) = \sum_{k\geq 0} h_k x^k \in \mathbb{Q}[\![x]\!]$ is defined as

$$\mathrm{val}_{p,v}(h(x)) = \min_{k\geq 0} \mathrm{val}_p\, h_k + vk.$$

Extension of the Problem

Given a prime p, a number v, and parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$, determine

$$\mathrm{val}_{p,v}\left(\mathcal{H}\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix}; x\right]\right).$$

Valuations

Let p be a prime. The v-drifted **valuation** of $h(x) = \sum_{k\geq 0} h_k x^k \in \mathbb{Q}[\![x]\!]$ is defined as

$$\mathrm{val}_{p,v}(h(x)) = \min_{k\geq 0} \mathrm{val}_p\, h_k + vk.$$

Extension of the Problem

Given a prime p, a number v, and parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$, determine

$$\mathrm{val}_{p,v}\left(\mathcal{H}\begin{bmatrix}\boldsymbol{\alpha} \\ \boldsymbol{\beta}\end{bmatrix}; x\right).$$

Idea: Everything is determined by the relative position of the parameters modulo p^r for all r.

Our Algorithm in Action

Let us compute $\mathrm{val}_{p,0}\left(\mathcal{H}\left[\begin{matrix}\frac{1}{3},\frac{4}{3}\\ \frac{2}{3},1\end{matrix};x\right]\right)$ for $p = 11$.

$$w_r(k+1) = w_r(k) + \left|\{j : k+\alpha_j \equiv 0 \bmod p^r\}\right| - \left|\{j : k+\beta_j \equiv 0 \bmod p^r\}\right|.$$

Our Algorithm in Action

Let us compute $\mathrm{val}_{p,0}\left(\mathcal{H}\left[\begin{matrix}\frac{1}{3},\frac{4}{3}\\ \frac{2}{3},1\end{matrix};x\right]\right)$ for $p=11$.

$$w_r(k+1) = w_r(k) + \left|\{j : k+\alpha_j \equiv 0 \bmod p^r\}\right| - \left|\{j : k+\beta_j \equiv 0 \bmod p^r\}\right|.$$

The function w_r is p^r-periodic. Let $\xi_{r,i} \in \mathbb{N}$ be the sorted jump points of w_r.

r even

$$\xi_{r,0}=0, \quad \xi_{r,1}=\frac{p^r-1}{3},$$
$$\xi_{r,2}=\frac{p^r+2}{3}, \quad \xi_{r,3}=\frac{2p^r+1}{3}$$
$$w_r(k)=\begin{cases}0 & \text{if } k\in[0,\xi_{r,1})\\ 1 & \text{if } k\in[\xi_{r,1},\xi_{r,2})\\ 2 & \text{if } k\in[\xi_{r,2},\xi_{r,3})\\ 1 & \text{if } k\in[\xi_{r,3},p^r)\end{cases}.$$

r odd

$$\xi_{r,0}=0, \quad \xi_{r,1}=\frac{p^r+1}{3},$$
$$\xi_{r,2}=\frac{2p^r-2}{3}, \quad \xi_{r,3}=\frac{2p^r+2}{3}$$
$$w_r(k)=\begin{cases}0 & \text{if } k\in[0,\xi_{r,1})\\ -1 & \text{if } k\in[\xi_{r,1},\xi_{r,2})\\ 0 & \text{if } k\in[\xi_{r,2},\xi_{r,3})\\ 1 & \text{if } k\in[\xi_{r,3},p^r)\end{cases}.$$

Our Algorithm in Action

Then $\mathrm{val}_p(h_k) = \sum_{r=1}^{\infty} w_r(k)$, and $w_r(k) = 0$ for "large" k.

Our Algorithm in Action

Then $\mathrm{val}_p(h_k) = \sum_{r=1}^{\infty} w_r(k)$, and $w_r(k) = 0$ for "large" k. Let $\sigma_r = \sum_{s=1}^{r} w_r$. Let $\mu_{r,i} = \min_{k \in [\xi_{r,i}, \xi_{r,i+1})} \sigma_r(k)$ and $\boldsymbol{\mu}_r = (\mu_{r,0}, \mu_{r,1}, \mu_{r,2}, \mu_{r,3})$. In particular, $\boldsymbol{\mu}_1 = (0, -1, 0, 1)$.

Our Algorithm in Action

Then $\mathrm{val}_p(h_k) = \sum_{r=1}^{\infty} w_r(k)$, and $w_r(k) = 0$ for "large" k. Let $\sigma_r = \sum_{s=1}^{r} w_r$. Let $\mu_{r,i} = \min_{k \in [\xi_{r,i}, \xi_{r,i+1})} \sigma_r(k)$ and $\boldsymbol{\mu}_r = (\mu_{r,0}, \mu_{r,1}, \mu_{r,2}, \mu_{r,3})$. In particular, $\boldsymbol{\mu}_1 = (0, -1, 0, 1)$.

Over the tropical semiring $\mathbb{Q} \cup \{\infty\}$ with $\oplus = \min$, $\odot = +$, we have the recurrences $\boldsymbol{\mu}_{2r} = \boldsymbol{\mu}_{2r-1} \odot T_0$, and $\boldsymbol{\mu}_{2r+1} = \boldsymbol{\mu}_{2r} \odot T_1$, where

$$T_0 := \begin{pmatrix} 0 & +\infty & 2 & 1 \\ 0 & +\infty & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & +\infty & 2 & 1 \end{pmatrix} \qquad T_1 := \begin{pmatrix} 0 & -1 & +\infty & 1 \\ 0 & -1 & +\infty & 1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & +\infty & 1 \end{pmatrix}.$$

We set $T := T_0 \odot T_1$, and $T^+ := T \oplus T^{\odot 2} \oplus T^{\odot 3} \oplus \dots$

Our Algorithm in Action

Then $\mathrm{val}_p(h_k) = \sum_{r=1}^{\infty} w_r(k)$, and $w_r(k) = 0$ for "large" k. Let $\sigma_r = \sum_{s=1}^{r} w_r$. Let $\mu_{r,i} = \min_{k \in [\xi_{r,i}, \xi_{r,i+1})} \sigma_r(k)$ and $\boldsymbol{\mu}_r = (\mu_{r,0}, \mu_{r,1}, \mu_{r,2}, \mu_{r,3})$. In particular, $\boldsymbol{\mu}_1 = (0, -1, 0, 1)$.

Over the tropical semiring $\mathbb{Q} \cup \{\infty\}$ with $\oplus = \min$, $\odot = +$, we have the recurrences $\boldsymbol{\mu}_{2r} = \boldsymbol{\mu}_{2r-1} \odot T_0$, and $\boldsymbol{\mu}_{2r+1} = \boldsymbol{\mu}_{2r} \odot T_1$, where

$$T_0 := \begin{pmatrix} 0 & +\infty & 2 & 1 \\ 0 & +\infty & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & +\infty & 2 & 1 \end{pmatrix} \qquad T_1 := \begin{pmatrix} 0 & -1 & +\infty & 1 \\ 0 & -1 & +\infty & 1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & +\infty & 1 \end{pmatrix}.$$

We set $T := T_0 \odot T_1$, and $T^+ := T \oplus T^{\odot 2} \oplus T^{\odot 3} \oplus \dots$

The matrix T^+ can be computed with the **Floyd-Warshall algorithm**, to diverge.

Our Algorithm in Action

Then $\mathrm{val}_p(h_k) = \sum_{r=1}^{\infty} w_r(k)$, and $w_r(k) = 0$ for "large" k. Let $\sigma_r = \sum_{s=1}^{r} w_r$. Let $\mu_{r,i} = \min_{k\in[\xi_{r,i},\xi_{r,i+1})} \sigma_r(k)$ and $\boldsymbol{\mu}_r = (\mu_{r,0}, \mu_{r,1}, \mu_{r,2}, \mu_{r,3})$. In particular, $\boldsymbol{\mu}_1 = (0, -1, 0, 1)$.

Over the tropical semiring $\mathbb{Q} \cup \{\infty\}$ with $\oplus = \min$, $\odot = +$, we have the recurrences $\boldsymbol{\mu}_{2r} = \boldsymbol{\mu}_{2r-1} \odot T_0$, and $\boldsymbol{\mu}_{2r+1} = \boldsymbol{\mu}_{2r} \odot T_1$, where

$$T_0 := \begin{pmatrix} 0 & +\infty & 2 & 1 \\ 0 & +\infty & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & +\infty & 2 & 1 \end{pmatrix} \qquad T_1 := \begin{pmatrix} 0 & -1 & +\infty & 1 \\ 0 & -1 & +\infty & 1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & +\infty & 1 \end{pmatrix}.$$

We set $T := T_0 \odot T_1$, and $T^+ := T \oplus T^{\odot 2} \oplus T^{\odot 3} \oplus \dots$

The matrix T^+ can be computed with the **Floyd-Warshall algorithm**, to diverge.

We have $\mathrm{val}_{p,0}(h(x)) = \lim_{r\to\infty} \mu_{r,0} = (\mu_0 \odot T^+)_1$. Thus $\mathrm{val}_{p,0}(h(x)) = -\infty$.

Remarks and Applications

Drifted valuations are used to compute the valuation in case of parameters not in $\mathbb{Z}_{(p)}$.

Remarks and Applications

Drifted valuations are used to compute the valuation in case of parameters not in $\mathbb{Z}_{(p)}$.

For large enough primes, the result of the algorithm only depends on the **congruence class** of p modulo the common denominator N of all the parameters.

Remarks and Applications

Drifted valuations are used to compute the valuation in case of parameters not in $\mathbb{Z}_{(p)}$.

For large enough primes, the result of the algorithm only depends on the **congruence class** of p modulo the common denominator N of all the parameters.

One can use the algorithm to compute **evaluations** of hypergeometric functions in the **p-adics**, up to a prescribed precision.

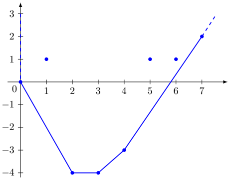
Remarks and Applications

Drifted valuations are used to compute the valuation in case of parameters not in $\mathbb{Z}_{(p)}$.

For large enough primes, the result of the algorithm only depends on the **congruence class** of p modulo the common denominator N of all the parameters.

One can use the algorithm to compute **evaluations** of hypergeometric functions in the **p-adics**, up to a prescribed precision.

The algorithm can be extended to the tropical semi-ring of polygons with $\oplus$ representing the convex hull, and $\otimes$ the Minkowski sum, to compute the **Newton polygon** of the hypergeometric series over the p-adics.



State of the Art: Algebraicity

Theorem (Christol 1986; Vargas-Montoya 2021, 2024; Caruso–F.-Vargas-Montoya 2025)

Let $h(x)$ *be a hypergeometric function, let* p *be a large enough prime number, and assume* $\mathrm{val}_{p,0}(h(x)) \geq 0$. *Then* $h(x) \bmod p$ *satisfies a polynomial equation of the form*

$$A_0(x)h(x) + A_1(x)h(x)^q + \ldots + A_w(x)h(x)^{q^w} = 0,$$

where $q = p^\ell$, *for some* ℓ, w *bounded independently of* p.

State of the Art: Algebraicity

Theorem (Christol 1986; Vargas-Montoya 2021, 2024; Caruso–F.-Vargas-Montoya 2025)

Let $h(x)$ be a hypergeometric function, let p be a large enough prime number, and assume $\mathrm{val}_{p,0}(h(x)) \geq 0$. Then $h(x) \bmod p$ satisfies a polynomial equation of the form

$$A_0(x)h(x) + A_1(x)h(x)^q + \ldots + A_w(x)h(x)^{q^w} = 0,$$

where $q = p^\ell$, for some ℓ, w bounded independently of p.

Vargas-Montoya gave a construction for an annihilating polynomial for “large” primes.

State of the Art: Algebraicity

Theorem (Christol 1986; Vargas-Montoya 2021, 2024; Caruso–F.-Vargas-Montoya 2025)

Let $h(x)$ be a hypergeometric function, let p be a large enough prime number, and assume $\mathrm{val}_{p,0}(h(x)) \geq 0$. Then $h(x) \bmod p$ satisfies a polynomial equation of the form

$$A_0(x)h(x) + A_1(x)h(x)^q + \ldots + A_w(x)h(x)^{q^w} = 0,$$

where $q = p^\ell$, for some ℓ, w bounded independently of p.

Vargas-Montoya gave a construction for an annihilating polynomial for "large" primes.

Algorithm (Caruso – F.)

Given a hypergeometric function $h(x)$ and **any** prime p, with $\mathrm{val}_{p,0}(h(x)) \geq 0$, outputs an annihilating polynomial of $h(x) \bmod p$.

Sections and Dwork Relations of Hypergeometric Functions

r-th section operator:

$$\begin{aligned}\Lambda_r : \qquad \mathbb{Q}[\![x]\!] &\to \mathbb{Q}[\![x]\!] \\ \sum_{k=0}^{\infty} h_k x^k &\mapsto \sum_{k=0}^{\infty} h_{kp+r} x^k.\end{aligned}$$

Sections and Dwork Relations of Hypergeometric Functions

r-th section operator:

$$\begin{aligned}\Lambda_r : \quad \mathbb{Q}[\![x]\!] &\to \mathbb{Q}[\![x]\!] \\ \textstyle\sum_{k=0}^{\infty} h_k x^k &\mapsto \textstyle\sum_{k=0}^{\infty} h_{kp+r} x^k.\end{aligned}$$

$$h(x) \equiv \sum_{r=0}^{p-1} x^r \Lambda_r(h(x))^p.$$

Sections and Dwork Relations of Hypergeometric Functions

r-th section operator:

$$\begin{aligned} \Lambda_r : \quad \mathbb{Q}[\![x]\!] &\to \mathbb{Q}[\![x]\!] \\ \sum_{k=0}^{\infty} h_k x^k &\mapsto \sum_{k=0}^{\infty} h_{kp+r} x^k. \end{aligned}$$

$$h(x) \equiv \sum_{r=0}^{p-1} x^r \Lambda_r(h(x))^p.$$

Dwork map: $\mathfrak{D}_p : \mathbb{Q} \to \mathbb{Q}$ defined by

$$p\mathfrak{D}_p(\gamma) - \gamma \in \{0, 1, \ldots, p-1\}.$$

Sections and Dwork Relations of Hypergeometric Functions

r-th section operator:

$$\begin{array}{rccl} \Lambda_r : & \mathbb{Q}[\![x]\!] & \to & \mathbb{Q}[\![x]\!] \\ & \sum_{k=0}^{\infty} h_k x^k & \mapsto & \sum_{k=0}^{\infty} h_{kp+r} x^k. \end{array}$$

$$h(x) \equiv \sum_{r=0}^{p-1} x^r \Lambda_r(h(x))^p.$$

Dwork map: $\mathfrak{D}_p : \mathbb{Q} \to \mathbb{Q}$ defined by

$$p\mathfrak{D}_p(\gamma) - \gamma \in \{0, 1, \ldots, p-1\}.$$

Multiplicative congruence:

$a \equiv_\times b :\Leftrightarrow \frac{a}{b} \in 1 + p\mathbb{Z}_{(p)}$.

Sections and Dwork Relations of Hypergeometric Functions

r-th section operator:

$$\begin{aligned}\Lambda_r: \quad \mathbb{Q}[\![x]\!] &\to \mathbb{Q}[\![x]\!] \\ \textstyle\sum_{k=0}^{\infty} h_k x^k &\mapsto \textstyle\sum_{k=0}^{\infty} h_{kp+r} x^k.\end{aligned}$$

$$h(x) \equiv \sum_{r=0}^{p-1} x^r \Lambda_r(h(x))^p.$$

Dwork map: $\mathfrak{D}_p : \mathbb{Q} \to \mathbb{Q}$ defined by

$$p\mathfrak{D}_p(\gamma) - \gamma \in \{0, 1, \ldots, p-1\}.$$

Multiplicative congruence:

$a \equiv_\times b :\Leftrightarrow \frac{a}{b} \in 1 + p\mathbb{Z}_{(p)}$.

Proposition

$\Lambda_r(h(x)) \equiv_\times h_r \cdot \mathcal{H}\left[\begin{matrix}\mathfrak{D}_p(\boldsymbol{\alpha}+r) \\ \mathfrak{D}_p(\boldsymbol{\beta}+r)\end{matrix}; (-p)^\nu x\right] \bmod p$, for some $\nu \in \mathbb{Z}$.

Sections and Dwork Relations of Hypergeometric Functions

r-th section operator:

$$\Lambda_r : \begin{array}{rl} \mathbb{Q}[\![x]\!] & \to \mathbb{Q}[\![x]\!] \\ \sum_{k=0}^{\infty} h_k x^k & \mapsto \sum_{k=0}^{\infty} h_{kp+r} x^k. \end{array}$$

$$h(x) \equiv \sum_{r=0}^{p-1} x^r \Lambda_r(h(x))^p.$$

Dwork map: $\mathfrak{D}_p : \mathbb{Q} \to \mathbb{Q}$ defined by

$$p\mathfrak{D}_p(\gamma) - \gamma \in \{0, 1, \ldots, p-1\}.$$

Multiplicative congruence:
$a \equiv_\times b :\Leftrightarrow \frac{a}{b} \in 1 + p\mathbb{Z}_{(p)}$.

Proposition

$\Lambda_r(h(x)) \equiv_\times h_r \cdot \mathcal{H}\left[\frac{\mathfrak{D}_p(\boldsymbol{\alpha}+r)}{\mathfrak{D}_p(\boldsymbol{\beta}+r)}; (-p)^\nu x\right] \bmod p$, for some $\nu \in \mathbb{Z}$.

Thus, $h(x)$ is a $\mathbb{F}_p[x]$-linear combination of p-th powers of hypergeometric functions, an identity we call **Dwork relation**.

Algebraicity and Annihilating Polynomial

Let X be the smallest set of hypergeometric functions, containing $h(x)$, and "closed under taking Dwork relations".

Proposition

X is finite.

Let $n := |X|$. The Dwork relations for all elements of X give n polynomial congruence relations between n hypergeometric functions. By taking appropriate powers, one can eliminate all of them but $h(x)$ to obtain an annihilating polynomial of $h(x) \bmod p$.

Algebraicity and Annihilating Polynomial

Let X be the smallest set of hypergeometric functions, containing $h(x)$, and "closed under taking Dwork relations".

Proposition

X is finite.

Let $n := |X|$. The Dwork relations for all elements of X give n polynomial congruence relations between n hypergeometric functions. By taking appropriate powers, one can eliminate all of them but $h(x)$ to obtain an annihilating polynomial of $h(x) \bmod p$.

The resulting polynomial will be of the shape

$$A_0(x)f(x) + A_1(x)f(x)^q + \ldots + A_w(x)f(x)^{q^w} = 0,$$

for $q = p^\ell$, and some explicit $w, \ell \in \mathbb{N}$ that can be read off the Dwork relations.

Example

We determine an annihilating polynomial modulo $p = 17$ for $h(x)$:

$$h(x) = \mathcal{H}\left[\begin{matrix}\frac{1}{9}, \frac{4}{9}, \frac{5}{9} \\ \frac{1}{3}, 1, 1\end{matrix}; x\right]. \qquad \text{Set } h_1(x) := \mathcal{H}\left[\begin{matrix}\frac{4}{9}, \frac{5}{9}, \frac{8}{9} \\ \frac{2}{3}, 1, 1\end{matrix}; x\right], \qquad h_2(x) := \mathcal{H}\left[\begin{matrix}\frac{4}{9}, \frac{5}{9}, \frac{10}{9} \\ \frac{4}{3}, 1, 1\end{matrix}; x\right].$$

Then $X = \{h(x), h_1(x), h_2(x)\}$, as the Dwork relations read

$$\begin{aligned} h(x) &\equiv (5x^7 + 12x^6 + 12x^5 + 5x^4 + 6x^3 + 6x^2 + 4x + 1) \cdot h_1(x)^{17}, \\ h_1(x) &\equiv (16x + 1) \cdot h(x)^{17} + (12x^7 + 12x^6) \cdot h_2(x)^{17}, \\ h_2(x) &\equiv (13x^7 + 7x^6 + 9x^5 + 9x^4 + 10x^3 + 9x^2 + 10x + 1) \cdot h_1(x)^{17}. \end{aligned}$$

We obtain $A(x)h(x)^{17^2} + B(x)h(x) \equiv 0$, where $A(x)$ has degree 990 and $B(x)$ has degree 864.

The End

Thank you for your attention!

```
sage: S.<x> = GF(19)[]
sage: f = hypergeometric([1/9, 4/9, 5/9], [1/3, 1], x)
sage: %time f.annihilating_ore_polynomial()
CPU times: user 128 ms, sys: 3.74 ms, total: 131 ms
Wall time: 172 ms
(18*x^76 + 13*x^57 + 6*x^38 + 17*x^19 + 12)*Frob^2 + (12*x^38 + 11*x^32 + 10*x^31 + 7*x^30 + 17*x^29 + 6
*x^28 + x^27 + 16*x^26 + 9*x^25 + 11*x^24 + 5*x^23 + 4*x^22 + 7*x^21 + 17*x^20 + 9*x^19 + x^18 + 16*x^17
 + 9*x^16 + 11*x^15 + 5*x^14 + 18*x^13 + 18*x^12 + 7)*Frob + x^30 + 16*x^29 + 9*x^28 + 11*x^27 + 5*x^26
+ 4*x^25 + 7*x^24 + 17*x^23 + 6*x^22 + x^21 + 16*x^20 + 9*x^19 + 11*x^18 + 5*x^17 + 4*x^16 + 7*x^15 + 17
*x^14 + 6*x^13 + x^12
```

Xavier presents our SageMath implementation tomorrow at 11 a.m.