Fast and Reliable Evaluation of the Distribution of Quadratic Forms of Gaussian Random Variables

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Finite-Precision CDF evaluation for Gaussian Quadratic Forms

Cumulative Distribution Function (CDF) for Noncentral Quadratic Sums

$$Q = \sum_{i=1}^{d} X_i^2 \qquad X_i \sim \mathcal{N}(\mu_i, \sigma_i^2) \text{ independent} \qquad g(\xi) = \mathbb{P}[Q \leqslant \xi]$$

$$g(\xi) = \frac{1}{\prod_{i=1}^{d} \sqrt{2\pi}\sigma_i} \int_{\sum_{i=1}^{d} x_i^2 \leqslant \xi} \exp\left(-\sum_{i=1}^{d} \frac{(x_i - \mu_i)^2}{2\sigma_i^2} \right) dx_1 \cdots dx_d$$

Up to renormalization, assume $\xi = 1$

Parameters: $m_i = \frac{\mu_i^2}{2\sigma_i^2}, \qquad p_i = \frac{1}{2\sigma_i^2}, \qquad p_1 \geqslant p_2 \geqslant \cdots \geqslant p_d$

Applications to: test statistics, spectral analysis and optimal control, space collision risk evaluation, radioastronomie, quantitative finance, SKAT (Sequence Kernel Association Test) in bioinformatics, . . .

State of the Art and Contributions

Existing methods:

- Numerical inversion of the characteristic function, e.g. [Imhof-1961, Davies-1973]
- Divergent series using saddlepoint methods, e.g. [Daniels-1954, Kuonen-1999]
- Convergent power series expansions, e.g. [Ruben-1962, Johnson-Kotz-1970]
- Various holonomic approaches, e.g., [Serra-Arzelier-Joldes-et-al-2016, Oaku-2018]

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Previous Contributions

- case $d = 2$ applied to orbital collision probability in the short-term encounter model, with explicit truncation bounds
- Recurrence unrolling rounding error analysis using Mezzarobba's method
- Extension to the case $d = 3$ (instantaneous probability of collision)

Roadmap of the approach and new contributions:

D-finite CDF → positive series → certified relative precision

closed-form Laplace transform $\Rightarrow$ D-finite power series $\Rightarrow$ truncation and roundoff error control

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Roadmap of the approach and new contributions:

D-finite CDF → positive series → positive recurrence (no cancellation) → certified relative precision

closed-form Laplace transform $\Rightarrow$ D-finite power series $\Rightarrow$ truncation and roundoff error control

State of the Art and Contributions

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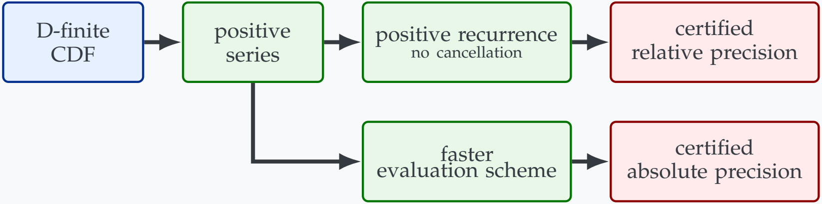

Algorithm CdfRec:

Extending the Recurrence to Arbitrary Dimension

Recurrence Through Closed-Form Laplace Transform – One Dimension

$$Q = X^2, \quad X \sim \mathcal{N}(\mu, \sigma^2) \quad p = \frac{1}{2\sigma^2}, \quad m = \frac{\mu^2}{2\sigma^2}$$

$$\begin{aligned} g(\xi) &= \mathbb{P}[Q \leqslant \xi] \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\sqrt{\xi}}^{\sqrt{\xi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \mathrm{d}x \end{aligned}$$

P-recursive $(g_n)_{n \geqslant 0}$

$$g(\xi) = \sum_{n \geqslant 0} g_n \xi^{n+1/2}$$

Recurrence Through Closed-Form Laplace Transform – One Dimension

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Laplace Transform $\mathcal{L}_g(\lambda) = \int_0^{+\infty} e^{-\lambda\xi} g(\xi) d\xi$

$$\widehat{g}(\lambda) = \frac{\mathcal{L}_g(\lambda^{-1})}{\lambda^{3/2}} = \frac{\sqrt{p}\, \exp\left(-\frac{m}{1+p\lambda}\right)}{\sqrt{1+p\lambda}}$$

D-finite first-order ODE for $\widehat{g}$
explicit initial conditions

$$\widehat{g}(\lambda) = \sum_{n \geqslant 0} \Gamma\left(n + \frac{3}{2}\right) g_n \lambda^n$$

termwise inverse

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numerical cancellation

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termwise inverse

$$g(\xi) = \sum_{n \geqslant 0} g_n \xi^{n+1/2}$$

Recurrence Through Closed-Form Laplace Transform – Preconditioning

P-recursive $(c_n)_{n\geqslant 0}$, $\quad c_n \geqslant 0$

$$Q = X^2, \quad X \sim \mathcal{N}(\mu, \sigma^2) \quad p = \frac{1}{2\sigma^2}, \quad m = \frac{\mu^2}{2\sigma^2}$$

$$f(\xi) = e^{p\xi} g(\xi)$$

$g(\xi) = e^{-p\xi} f(\xi)$

$$f(\xi) = \sum_{n\geqslant 0} c_n \xi^{n+1/2}$$

Shifted Laplace Transform $\quad \mathcal{L}_f(\lambda) = \mathcal{L}_g(\lambda - p)$

termwise inverse

$$\widehat{f}(\lambda) = \frac{\mathcal{L}_f(\lambda^{-1})}{\lambda^{3/2}} = \frac{\sqrt{p}\, \exp(-m(1-p\lambda))}{1-p\lambda}$$

$$\widehat{f}(\lambda) = \sum_{n\geqslant 0} \Gamma\left(n + \frac{3}{2}\right) c_n \lambda^n$$

D-finite first-order ODE for $\widehat{f}$
explicit initial conditions

Differential Equation

$$\widehat{f}'(\lambda) = (pm + \underbrace{\frac{p}{1-p\lambda}}_{\geqslant 0})\widehat{f}(\lambda), \qquad \widehat{f}(0) = \sqrt{p}\, e^{-m}$$

Recurrence Through Closed-Form Laplace Transform – d Dimensions

P-recursive $(c_n)_{n\geqslant 0}$, $c_n \geqslant 0$

$$Q = \sum_{i=1}^{d} X_i^2, \quad X_i \sim \mathcal{N}(\mu_i, \sigma_i^2), \quad p_i = \frac{1}{2\sigma_i^2}, \quad m_i = \frac{\mu_i^2}{2\sigma_i^2}$$

$$f(\xi) = e^{p_1 \xi} g(\xi), \quad p_1 \geqslant p_2 \geqslant \ldots p_d, \quad \alpha_i = p_1 - p_i$$

$g(\xi) = e^{-p_1\xi} f(\xi)$

$$f(\xi) = \sum_{n\geqslant 0} c_n \xi^{n+d/2}$$

Shifted Laplace Transform

termwise inverse

$$\widehat{f}(\lambda) = \frac{\mathcal{L}_f(\lambda^{-1})}{\lambda^{d/2+1}} = \frac{\prod_{i=1}^{d} \sqrt{p_i} \exp\left[-\sum_{i=1}^{d} m_i \frac{1-p_1\lambda}{1-\alpha_i\lambda}\right]}{(1-p_1\lambda)\prod_{i=2}^{d}\sqrt{1-\alpha_i\lambda}}$$

$$\widehat{f}(\lambda) = \sum_{n\geqslant 0} \Gamma\left(n + \frac{d}{2} + 1\right) c_n \lambda^n$$

D-finite first-order ODE for $\widehat{f}$
explicit initial conditions

Differential Equation

$$\frac{\widehat{f}'(\lambda)}{\widehat{f}(\lambda)} = p_1 m_1 + \frac{p_1}{1-p_1\lambda} + \sum_{i=2}^{d}\left(\frac{\alpha_i}{2(1-\alpha_i\lambda)} + \frac{p_i m_i}{(1-\alpha_i\lambda)^2}\right),$$

$$\widehat{f}(0) = \prod_{i=1}^{d} \sqrt{p_i}\, e^{-m_i}$$

CdfRec: Evaluation of Positive Series Using Scalar Recurrence

Require: d, $(p_i)_{1\leqslant i\leqslant d}$, $(m_i)_{1\leqslant i\leqslant d}$, number of terms $\mathbf{N}$.

Ensure: $s_N \approx g(1)$.

$\alpha_i = p_1 - p_i, \quad 1 \leqslant i \leqslant d;$

$\varphi(\lambda) = p_1 m_1 + \dfrac{p_1}{1 - p_1\lambda} + \displaystyle\sum_{i=2}^{d} \left(\dfrac{\alpha_i/2}{1-\alpha_i\lambda} + \dfrac{p_i m_i}{(1-\alpha_i\lambda)^2} \right);$

$P(\lambda) \leftarrow \mathrm{numerator}(\varphi), \quad Q(\lambda) \leftarrow \mathrm{denominator}(\varphi);$

$c_0 = \dfrac{\left(\prod_{i=1}^{d} \sqrt{p_i}\right) \exp\left(-\sum_{i=1}^{d} m_i\right)}{\Gamma\left(\frac{d}{2}+1\right)}; \quad s \leftarrow c_0;$

for $n = 1$ to $\mathbf{N} - \mathbf{1}$ **do**

  $c_n \leftarrow 0; \quad \mathrm{fac} \leftarrow 1;$

  for $i = 1$ to $\min(2d, n)$ **do**

    $\mathrm{fac} \leftarrow \left(\frac{d}{2} + n + 1 - i\right) \mathrm{fac};$

    $c_n \leftarrow c_n + (-1)^{i-1} \dfrac{P_{i-1} + Q_i(n-i)}{\mathrm{fac}} c_{n-i};$

  end for

  $c_n \leftarrow c_n / n;$

  $s \leftarrow s + c_n;$

end for

return $s_N = \exp(-p_1)s.$

construct rational ODE — initial coefficient — unroll recurrence — positive summation — preconditioning

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for $n = 1$ to $\mathbf{N - 1}$ **do**

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construct rational ODE

initial coefficient

unroll recurrence

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choose N from truncation bound

roundoff error bound

construct rational ODE — initial coefficient — unroll recurrence

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return $s_N = \exp(-p_1)s.$

choose N from truncation bound

roundoff error bound

mixed signs in the recurrence!!

construct rational ODE

initial coefficient

unroll recurrence

positive summation

preconditioning

Algorithm CdfPosRec:

A Positive Recurrence for Reducing the Roundoff Error

Coupled Recurrence With Positive Coefficients

$$\widehat{f}' = \left(p_1 m_1 + \frac{p_1}{1 - p_1\lambda} + \sum_{i=2}^{d} \left(\frac{\alpha_i/2}{1 - \alpha_i\lambda} + \frac{p_i m_i}{(1 - \alpha_i\lambda)^2} \right) \right) \widehat{f}$$

$$\gamma_n := \Gamma(n + \tfrac{d}{2} + 1)$$

$$p_1 m_1 \widehat{f} = \sum_{n \geqslant 0} p_1 m_1 c_n \gamma_n \lambda^n$$

$$\frac{p_1 \widehat{f}}{1 - p_1\lambda} = \sum_{n \geqslant 0} p_1 a_{1,n} \gamma_n \lambda^n$$

$$\frac{\alpha_i \widehat{f}}{2(1 - \alpha_i\lambda)} = \sum_{n \geqslant 0} \frac{\alpha_i}{2} a_{i,n} \gamma_n \lambda^n$$

$$\frac{p_i m_i \widehat{f}}{(1 - \alpha_i\lambda)^2} = \sum_{n \geqslant 0} p_i m_i b_{i,n} \gamma_n \lambda^n$$

Coupled Recurrence With Positive Coefficients

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$$\frac{p_i m_i \widehat{f}}{(1 - \alpha_i \lambda)^2} = \sum_{n \geqslant 0} p_i m_i b_{i,n} \gamma_n \lambda^n$$

$$n(n + d/2) c_n = p_1 m_1 c_{n-1} + p_1 a_{1,n-1} + \sum_{i=2}^{d} \left(\frac{\alpha_i}{2} a_{i,n-1} + p_i m_i b_{i,n-1} \right)$$

Coupled Recurrence With Positive Coefficients

$$\widehat{f}' = \left(p_1 m_1 + \frac{p_1}{1-p_1\lambda} + \sum_{i=2}^{d}\left(\frac{\alpha_i/2}{1-\alpha_i\lambda} + \frac{p_i m_i}{(1-\alpha_i\lambda)^2}\right)\right)\widehat{f}$$

$$\gamma_n := \Gamma(n + \tfrac{d}{2} + 1)$$

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$$a_{1,n} = \frac{p_1}{n+d/2} a_{1,n-1} + c_n$$

$$a_{i,n} = \frac{\alpha_i}{n+d/2} a_{i,n-1} + c_n$$

$$b_{i,n} = \frac{\alpha_i}{n+d/2} b_{i,n-1} + a_{i,n}$$

CdfPosRec: Evaluation Using Coupled Recurrence With Positive Coefficients

Require: d, $(p_i)_{1 \leqslant i \leqslant d}$, $(m_i)_{1 \leqslant i \leqslant d}$, number of terms $\mathbf{N}$.

Ensure: $s_N \approx g(1)$.

$\alpha_i = p_1 - p_i, \quad 1 \leqslant i \leqslant d;$

$c_0 = \dfrac{\left(\prod_{i=1}^{d} \sqrt{p_i}\right) \exp\left(-\sum_{i=1}^{d} m_i\right)}{\Gamma\left(\frac{d}{2}+1\right)}; \qquad a_{i,0} \leftarrow c_0, \quad b_{i,0} \leftarrow c_0;$

$s \leftarrow c_0;$

for $n = 1$ to $\mathbf{N-1}$ **do**

$\quad c_n \leftarrow \dfrac{p_1 m_1 c_{n-1} + p_1 a_{1,n-1} + \sum_{i=2}^{d}\left(\frac{\alpha_i}{2} a_{i,n-1} + p_i m_i b_{i,n-1}\right)}{n\left(n+\frac{d}{2}\right)};$

$\quad a_{1,n} \leftarrow \dfrac{p_1}{n+\frac{d}{2}} a_{1,n-1} + c_n;$

$\quad$ **for** $i = 2$ to d **do**

$\quad\quad a_{i,n} \leftarrow \dfrac{\alpha_i}{n+\frac{d}{2}} a_{i,n-1} + c_n; \qquad b_{i,n} \leftarrow \dfrac{\alpha_i}{n+\frac{d}{2}} b_{i,n-1} + a_{i,n};$

$\quad$ **end for**

$\quad s \leftarrow s + c_n;$

end for

return $s_N = \exp(-p_1)s.$

initial terms — positive recurrence unrolling

positive summation — preconditioning

Complexity of CdfPosRec with Target Relative Accuracy

Truncation Error Bounds [Arzelier-Bréhard-Joldes-2026]

Let $r_N := \left| g(1) - \exp(-p_1) \sum_{n=0}^{N-1} c_n \right|$ be the truncation error bound:

(i) *(absolute error)* $r_N \leqslant 2^{-t}$ for $N = O\left(p_1 + t\right)$

(ii) *(relative error)* $r_N \leqslant 2^{-t} g(1)$ for $N = O\left(p_1 + \sum_i m_i + \sum_i \ln\left(\frac{p_1}{p_i}\right) + t\right)$

Complexity of Algorithm CdfPosRec [Arzelier-Bréhard-Joldes-2026]

CdfPosRec computes $g(1)$ with relative accuracy 2^{-t} in

$$\tilde{O}\left(d \underbrace{\left(p_1 + \sum_i m_i + \sum_i \ln\left(\frac{p_1}{p_i}\right) + t\right)}_{N \text{ terms}} \underbrace{t}_{\substack{\text{working prec. up to} \\ \text{log factors in } p_1,\, d,\, m_i,\, t}} \right) \quad \text{bit operations}$$

Numerical Experiment: CdfRec vs. CdfPosRec

- Comparing `cdf_rec` and `cdf_pos_rec`, implemented in Julia (here in `binary64`)
- `iso` = central isotropic ($\mu_i = 0$, σ_i equal), `aniso` = central anisotropic ($\mu_i = 0$), `noncent` = noncentral anisotropic, `hard` = large N, `das` = examples from [Das-2005]

Case	Type	d	p_{ref}	N	T_{posrec} (s)	T_{rec} (s)	Dig. posrec	Dig. rec
S1	iso	10	0.3555	30	1.01e-5	5.78e-5	15.9	15.9
S2	iso	30	0.4908	50	3.41e-5	1.71e-4	15.9	15.5
S3	aniso	10	0.9974	70	2.38e-5	1.61e-4	14.9	10.3
S4	aniso	10	0.8746	100	3.33e-5	2.38e-4	14.5	4.5
S5	aniso	10	0.5960	150	4.99e-5	3.67e-4	14.2	0.0
S6	aniso	10	0.2553	330	1.18e-4	8.68e-4	15.8	0.0
S7	noncent	10	0.5799	150	5.29e-5	3.85e-4	14.7	0.0
S8	noncent	10	0.5288	150	5.33e-5	3.82e-4	14.8	0.1
S9	noncent	10	0.4352	150	5.21e-5	3.85e-4	14.0	0.4
S10	noncent	10	0.2957	150	5.34e-5	3.84e-4	14.1	1.0
S11	aniso	30	0.9606	200	1.46e-4	1.36e-3	14.0	0.0
S12	aniso	30	0.5599	340	2.49e-4	2.45e-3	14.2	0.0
S13	noncent	30	0.5068	340	2.50e-4	2.44e-3	13.7	0.0
S14	noncent	30	0.3383	340	2.48e-4	2.49e-3	14.6	0.0
S15	hard	10	4.451e-04	**7,910**	2.93e-3	2.30e-2	12.8	0.0
S16	hard	30	2.162e-13	**7,900**	6.03e-3	7.44e-2	12.8	0.0
S17	hard	10	4.101e-06	**183,500**	9.94e-2	6.21e-1	10.7	0.0
S18	hard	30	4.515e-23	**183,500**	1.96e-1	1.80e+0	10.7	0.0
D1	das	3	0.4936	30	9.04e-6	2.90e-5	15.9	15.0
D5	das	8	0.5913	60	1.99e-5	1.18e-4	15.2	11.4
D9	das	10	0.5848	80	2.87e-5	1.93e-4	14.4	10.0
D11	das	34	0.5736	210	1.79e-4	1.56e-3	15.5	0.0

Algorithm CdfFast:

An Alternative Approach for Taming High Dimensions

Overview of Algorithm CdfFast – Taming High Dimensions

$$\widehat{f}(\lambda) = \frac{\prod_i \sqrt{p_i} e^{-m_i}}{1 - p_1\lambda} \exp \underbrace{\left[p_1 m_1 \lambda + \sum_{i=2}^{d} \left(\frac{p_i m_i \lambda}{1 - \alpha_i \lambda} - \frac{1}{2} \ln(1 - \alpha_i \lambda) \right) \right]}_{\psi(\lambda)\text{: analytic over } \overline{B}(0, \frac{1}{p_1})}$$

1. Compute $\psi(\lambda)$ up to degree N in $\widetilde{O}(\max(N, d))$ FP operations

$\rightsquigarrow$ fast transposed multipoint evaluation to compute sums $\displaystyle\sum_{i=1}^{d} \frac{w_i}{1 - \alpha_i \lambda} \,\%\, \lambda^N$

2. Compute $\exp(\psi(\lambda))$ up to degree N in $\widetilde{O}(N)$ FP operations

$\rightsquigarrow$ use a FFT-based exponentiation scheme over a disk of radius $\dfrac{1-\varepsilon}{p_1}$

3. Compute $\widehat{f}(\lambda)$ and eventually $g(1)$ in $\widetilde{O}(N)$ FP operations

A Fast Numerical Transposed Multipoint Evaluation Scheme

Transposed Multipoint Evaluation Problem

$$\sum_{i=1}^{d} \frac{w_i}{1-\alpha_i z} \,\%\, z^N = \sum_{n=0}^{N-1} \underbrace{\left(\sum_{i=1}^{d} w_i \alpha_i^n \right)}_{u_n} z^n$$

Vandermonde matrix V_α: $\quad u = V_\alpha^T \cdot w \quad \leftrightsquigarrow \quad y = V_\alpha \cdot p$

Multipoint Evaluation Problem

$$p(z) = \sum_{n=0}^{N-1} p_n z^n \quad \rightsquigarrow \quad y_i = p(\alpha_i) = \sum_{n=0}^{N-1} p_n \alpha_i^n, \quad 1 \leqslant i \leqslant d$$

Tellegen's transposition principle

For a linear map $A : \mathbb{K}^n \to \mathbb{K}^m$, if $A \cdot v$ can be computed by a SLP of size L, then $A^T \cdot w$ can be computed by a SLP of size $L - n + m$.

Moroz' Fast Numerical Multipoint Evaluation Algorithm (assume $N = d$)

Multipoint Evaluation Problem

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- Symbolic algorithms in $\widetilde{O}(d)$ arithmetic operations, e.g. [Bostan-Lecerf-Schost-2003]
- in FP arithmetic $\rightsquigarrow$ eval. error $= O(2^d)$ $\rightsquigarrow$ need extra d bits

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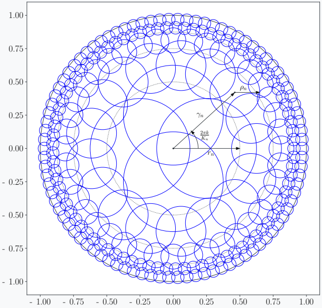

Credit: Guillaume Moroz

0. Compute the t-hyperbolic covering of the unit disk
   $\rightsquigarrow$ radii tend to zero close to the boundary

1. Shift and truncate p to degree t on each small disk

2. On each small disk, evaluate all $p(\alpha_i)$ for α_i in the disk (with error $O(2^t)$)

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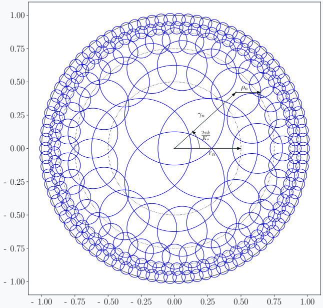

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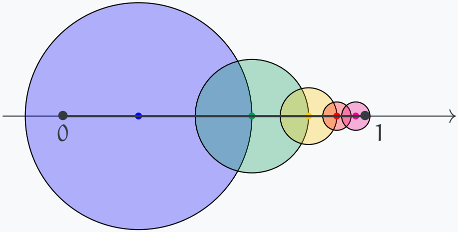
$$\text{Error} \leqslant \|p\|_1 2^{-t} \text{ in } \widetilde{O}\big(d(t + \ln \|p\|_1)\big) \text{ bit op.}$$

Transposing Numerical Fast Multipoint Evaluation

0. Compute the t-hyperbolic covering of $[0, 1]$

$\rightsquigarrow$ Disks $D_k := \overline{B}(\gamma_k, \rho_k)$; $\quad \alpha_i \mapsto \beta_i = \frac{\alpha_i - \gamma_k}{\rho_k}$ with k such that $\alpha_i \in D_k$



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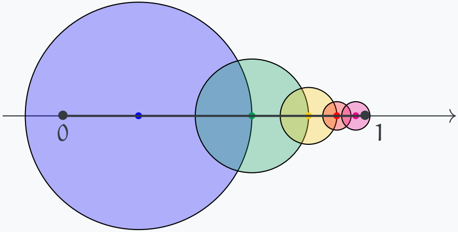

2^T. Compute $U_k := \sum_{i | \alpha_i \in D_k} \frac{w_i}{1 - \beta_i z} \% z^t$ using a transposed multipoint evaluation algorithm (with error $O(2^t)$)

1^T. (Shift and Truncate)^T = Pad and Transposed Shift

$$\mathrm{shift}^T_{\gamma_k, \rho_k} : U(z) \mapsto \frac{1}{1 - \gamma_k z} U\left(\frac{\rho_k z}{1 - \gamma_k z}\right) \quad \Rightarrow \quad \frac{1}{1 - \beta_i z} \mapsto \frac{1}{1 - \alpha_i z}$$

$\Rightarrow$ Sum the $V_k := \mathrm{shift}^T_{\gamma_k, \rho_k}\left(U_k\right)$

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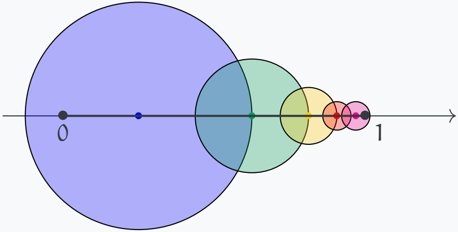

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Complexity of CdfFast vs. CdfPosRec with Target Absolute Accuracy

Complexity of Algorithm CdfPosRec [Arzelier-Bréhard-Joldes-2026]

CdfPosRec$(d, (p_i), (m_i))$ computes $g(1)$ with absolute accuracy 2^{-t} in

$$\tilde{O}\left(\left(\underbrace{p_1 + t}_{N \text{ terms}}\right) d \underbrace{t}_{\text{working prec. up to}}\right) \quad \text{bit operations}$$

log factors in p_1, d, m_i, t

Complexity of Algorithm CdfFast [Arzelier-Bréhard-Joldes-2026]

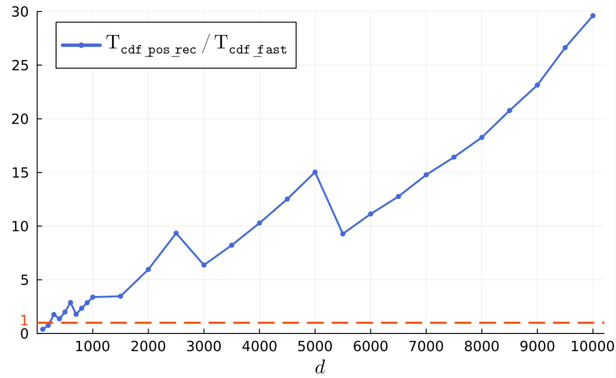
CdfFast$(d, (p_i), (m_i))$ computes $g(1)$ with absolute accuracy 2^{-t} in

$$\tilde{O}\left(\left(\underbrace{p_1 + t}_{N \text{ terms}} + d\right) \underbrace{t}_{\text{working prec. up to}}\right) \quad \text{bit operations}$$

log factors in p_1, d, m_i, t

Numerical Experiment in High Dimensions: CdfFast vs. CdfPosRec

- $\sigma_i \sim \mathcal{U}([1,2])$; $\mu_i \sim \mathcal{N}(0,1)$; rescale by $\left(\sum_i (\sigma_i^2 + \mu_i^2)\right)^{-\frac{1}{2}} \rightsquigarrow g(1) \approx \frac{1}{2}$
- Set $N = 2d$ and use 64-bit precision $\rightsquigarrow$ absolute error bounded by 10^{-10}



- Speedup of CdfFast over CdfPosRec reaches 30 for $d = 10,000$ (3.5 s vs. 104 s)
- Fast polynomial multiplication (FFT) and fast transposed multipoint evaluation (for base case t-blocks) not implemented yet

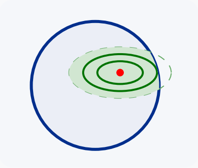
Conclusion and Future Work

Contributions:

- Generalization of [Serra-Arzelier-Joldes-et al-2016] to arbitrary dimension d
- Reduced evaluation error using a positive coupled recurrence
- Faster algorithm for high-dimensional cases
- Prototype implementations in Julia and Matlab

Future Research Directions: Handling χ^2 distributions (σ_i equal) more efficiently? Relative error bounds for CdfFast? Connections with Fast Multipole Methods (FMM)? …

Applications: ill-conditioned and high-dimensional instances (e.g., SKAT)



Quadratiques.jl

fast and reliable CDF evaluation

gitlab.laas.fr/arzelier/quadratic

Scan for repository

Thank you!