Exact Algorithms for Semidefinite Programs with Degenerate Feasible Set

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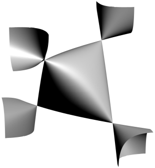
Problem statement



Input $A_0, \ldots, A_n$ symmetric matrices with entries in $\mathbb{Q}$ of size $m \times m$.

$\ell \in \mathbb{Q}[x_1, \ldots, x_n]$ of degree 1

Spectrahedron $\mathscr{S}(A) = \{\boldsymbol{x} \in \mathbb{R}^n \mid A_0 + \boldsymbol{x}_1 A_1 + \cdots + \boldsymbol{x}_n A_n \succeq 0\}$.

$\ell^\star = \inf_{\boldsymbol{x} \in \mathscr{S}(A)} \ell(\boldsymbol{x}) \rightsquigarrow$ real algebraic number

Output Assuming that $\ell^\star$ is reached

rational parametrization of a minimizer $q(t) = 0,\ x_i = \frac{v_i(t)}{\partial q / \partial t},\ 1 \leqslant i \leqslant n$

State of the art

Approximate solutions

Computed numerically through interior point methods.

- under the assumption that $\mathscr{S}(A)$ has **non-empty interior**
- polynomial time at fixed precision
- super efficient solvers based on floating/double point arithmetics

BUT not so unfrequent reliability issues.

Exact solving of SDP

The feasible set $\mathscr{S}(A)$ is a semi-algebraic set

- General algorithms for semi-algebraic sets
- Dedicated exact algorithms for solving LMI

Henrion/Naldi/Safey El Din, SIOPT

  - polynomial time when n or m is fixed
  - **regularity assumptions $\rightsquigarrow$ genericity of the A_i's**

Motivations

- Lyapunov stability for $\dot{x} = \mathbf{M}x$ **see e.g. Henrion/Garulli'05**

  Find P such that $P + (-\mathbf{M}^T P - P\mathbf{M}) \succ 0$

- SOS polynomials (sums of squares): $f(u) = f_1(u)^2 + \cdots + f_t(u)^2$, with $u = (u_1, \ldots, u_n)$, is equivalent to a LMI of type

  $$f(u) = v(u)^T \cdot A \cdot v(u), \quad A \succeq 0$$

- More generally: $f^* = \inf f(u)$ iff $f^* = \sup \lambda : f - \lambda \geqslant 0$

  SOS relaxation : $f - \lambda = g_1^2 + \cdots + g_t^2$

Irrational certificates: Scheiderer's examples $\rightsquigarrow$ **hardness of solving !**

- Many examples of feasible sets with empty interiors

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Need of exact algorithms for solving SDP when $\mathscr{S}(A)$ is degenerate

Main results

Assumptions on the input

- $\ell^\star$ is reached
- genericity assumption on $\ell \in \mathbb{Q}[x_1, \ldots, x_n]$

- Dedicated algorithm for computing a minimizer solution to the SDP represented by a rational parametrization

no assumption on $\mathscr{S}(A)$

- arithmetic complexity polynomial in $\binom{n+m}{n}$
- preliminary implementation for small sized problems

Can handle degenerate cases

Useful for just deciding the emptiness or grabbing sample points in the solution of a feasible set (possibly degenerate)

Overview of the ingredients

Degenerate SDP

↓

Regularized situation

$A(x) + \epsilon B \succeq 0 \rightsquigarrow$ non-empty interior

↓

Reduction to polynomial system solving

Highly structured systems

↓

Solve using symbolic homotopy

↓

Take the limit $(\epsilon \to 0)$

Projections of semi-algebraic sets

First result

Let $\mathbf{R}$ be a real closed field and $S \subset \mathbf{R}^n$ be a closed semi-algebraic set. For generic $\ell \in \mathbb{Q}[x_1, \ldots, x_n]$ with $\deg(\ell) = 1$,

$\ell(S)$ is closed

generalizes a result in S./Schost ISSAC'03

We start with $\mathscr{S}(A)$ defined by $A_0 + x_1 + \cdots + x_n A_n \succeq 0$.

- For generic ℓ, $\ell(\mathscr{S}(A))$ is closed.

Take B symmetric with $B \succ 0$.

- $\mathscr{S}(A) \subset \mathscr{S}(A(x) + \epsilon B) \subset \mathbf{R}^n = \mathbb{R}\langle\epsilon\rangle^n$
- $\ell(\mathscr{S}(A(x) + \epsilon B))$ is closed.

We shall use B and ϵ to regularize the problem.

Let $A_\epsilon(x) = A(x) + \epsilon B$

From semi-algebraic to algebraic formulation

Define:

For $A_\epsilon(x)$: $\mathcal{D}_r = \{x \in \mathbb{C}\langle\epsilon\rangle^n \mid rank(A_\epsilon(x)) \leqslant r\}$

For $A_\epsilon(x)$, and $\mathscr{S}(A_\epsilon(x)) \neq \emptyset$: $r(A_\epsilon) = \min\{rank\, A_\epsilon(x) \mid x \in \mathscr{S}(A_\epsilon(x))\}$

So one has nested sequences

$$\mathcal{D}_0 \subset \cdots \subset \mathcal{D}_{m-1}$$
$$\mathcal{D}_0 \cap \mathbb{R}\langle\epsilon\rangle^n \subset \cdots \subset \mathcal{D}_{m-1} \cap \mathbb{R}\langle\epsilon\rangle^n$$

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Smallest Rank Property Henrion-Naldi-S. SIOPT 2015

Let C be a conn. comp. of $\mathcal{D}_{r(A_\epsilon)} \cap \mathbb{R}^n$ s.t. $C \cap \mathscr{S}(A_\epsilon(x)) \neq \emptyset$. **Then** $C \subset \mathscr{S}(A_\epsilon(x))$. **In particular** $C \subset \mathcal{D}_{r(A_\epsilon)} \setminus \mathcal{D}_{r(A_\epsilon)-1}$.

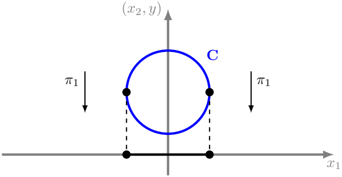
Critical points and incidence varieties

$$A_\epsilon(x) = A(x) + \epsilon B$$

1st step *Lifting of the determinantal variety*:

$$A_\epsilon(x)\, Y(y) = A_\epsilon(x) \begin{bmatrix} \mathbf{y}_{1,1} & \cdots & \mathbf{y}_{1,m-r} \\ \vdots & & \vdots \\ \mathbf{y}_{m,1} & \cdots & \mathbf{y}_{m,m-r} \end{bmatrix} = 0.$$

$$U\, Y(y) = I_{m-r}$$



If B and ℓ are generic, the lifted algebraic set $\mathcal{V}_r$ is **smooth** and **equidimensional**

2nd step *Compute critical points* of the map $(x, y) \mapsto \ell(x)$ on $\mathcal{V}_r$:

When ℓ is generic, there are **finitely many** critical points.

Symbolic homotopy

We use Lagrange polynomial systems to encode critical points.

$$A_\epsilon(x)Y(y) = 0, UY(y) = Id \quad \to \quad \mathbf{F}(x, y) = 0$$

$\downarrow$ z is a vector of new variables

$$\mathbf{F}(x, y) = 0, \quad z.jac(\mathbf{F}, \ell) = 0$$

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- System defining incidence variety $\to$ bi-linear in (x, y)
- Lagrange system $\to$ globally tri-linear in (x, y, z)
  but all equations are bi-linear
- Multi-homogeneous structure not handled efficiently by the litterature
  except **S. /Schost, JSC '18 (Symbolic homotopy) $\to$ 0-dim case**

Complexity quadratic in the Multi-homogeneous Bézout bound

Complexity

Using symbolic homotopy

- Here we have introduced one parameter ϵ to regularize the problem
- Handling infinitesimal parameters in real geometry

  Basu/Pollack/Roy, Rouillier/Roy/S.

  Efficient use of lifting techniques combined with degree bounds
  Schost 03

Classical strategy

- Instantiate ϵ to a randomly chosen value
- Solve the zero-dimensional system
- Lift ϵ and compute the limit.

All steps run in time polynomial in the
multi-homogeneous bound associated to the system
This is $\binom{n+m}{m}$!

Conclusions / Perspectives

- First dedicated algorithm for solving SDP with degenerate feasible sets
- Reasonable overhead w.r.t. the regular situation
- Geometric results that may be used in further/more general algorithms
- **What is missing:**
  - What happens when the linear form to optimize is not generic?
  - Implementation of symbolic homotopy algorithms
  - Potential impact on numerical algorithms?

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Thank you ...

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Thank you ... and many thanks to Éric for giving this talk