Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

FWF Der Wissenschaftsfonds.

Algorithmic Arithmetics with DD-Finite Functions

Implementation and Issues

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Outline

Notation

Throughout this talk we consider:

- K: a **computable** field
- $K[[x]]$: ring of formal power series over K.
- Given a field F:

$$V_F(f) = \langle f, f', f'', \ldots \rangle_F.$$

D-finite functions

Definition

Let $f \in K[[x]]$. We say that f is *D-finite* (or *holonomic*) if there exist $d \in \mathbb{N}$ and polynomials $p_0(x), \ldots, p_d(x)$ such that:

$$p_d(x) f^{(d)}(x) + \ldots + p_0(x) f(x) = 0.$$

We say that d is the *order* of f.

Non-D-finite examples

There are power series that are not D-finite:

- Double exponential: $f(x) = e^{e^x}$.
- Tangent: $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

- Gamma function: $f(x) = \Gamma(x+1)$.
- Partition Generating Function: $f(x) = \sum_{n\geq 0} p(n)x^n$.

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$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0.$$

Examples

The set is bigger than the D-finite functions:

$$
\begin{array}{ccc}
f \text{ is D-finite} & \Rightarrow & f \text{ is DD-finite}, \\
f(x) = e^{e^x} & \Rightarrow & f'(x) - e^x f(x) = 0, \\
f(x) = \tan(x) & \Rightarrow & \cos(x)^2 f''(x) - 2f(x) = 0, \\
f(x) = e^{\int_0^x J_n(t)dt} & \Rightarrow & f'(x) - J_n(x)f(x) = 0
\end{array}
$$

Differentially Definable Functions

Definition

Let $f \in K[[x]]$. We say that f is *DD-finite* if there exist $d \in \mathbb{N}$ and D-finite elements $r_0(x), \dots, r_d(x)$ such that:

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Differentially Definable Functions

Definition

Let $f \in K[[x]]$ and $R \subset K[[x]]$ a ring. We say that f is differentially definable over R if there exist $d \in \mathbb{N}$ and elements in R $r_0(x), \ldots, r_d(x)$ such that:

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0.$$

$\mathrm{D}(R)$: differentially definable functions over R.

Characterization Theorem

The following are equivalent:

$$f(x) \in \mathrm{D}(R).$$

There are elements $r_0(x), \dots, r_d(x) \in R$ and $g(x) \in \mathrm{D}(R)$ such:

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Let F be the *field of fractions* of R:

$$\dim(V_F(f)) < \infty$$

Closure properties

$f(x), g(x) \in \mathrm{D}(R)$ of order d_1, d_2.
F the field of fractions of R.
$a(x)$ algebraic over F of degree p.

Property	**Is in** $\mathrm{D}(R)$	**Order bound**
Addition	$(f+g)$	$d_1 + d_2$
Product	(fg)	$d_1 d_2$
Differentiation	f'	d_1
Integration	$\int f$	$d_1 + 1$
Be Algebraic	$a(x)$	p

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$\longrightarrow$ Proof by direct formula
$\longrightarrow$ Proof by linear algebra

Vector spaces

Let $R \subset K[[x]]$, F its field of fractions and $V_F(f)$ the F-vector space spanned by f and its derivatives.

The Characterization theorem provides

$$f(x) \in \mathrm{D}(R) \quad \Leftrightarrow \quad \dim(V_F(f)) < \infty$$

The ansatz method

Specifications

Input: A power series $h(x)$ ($f(x)+g(x)$, $f(x)g(x)$ or $a(x)$)
Output: An operator $\mathcal{A} \in R[\partial]$ such that $\mathcal{A}h = 0$

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1. Compute $W \subset K[[x]]$ such that $\dim(W) < \infty$ and $V_F(h) \subset W$.

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Method

4. Set up the ansatz:

$$\alpha_0 h(x) + \ldots + \alpha_n h^{(n)} = 0.$$

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6. Return $\mathcal{A} = \alpha_n \partial^n + \ldots + \alpha_1 \partial + \alpha_0$.

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Derivation matrices

Let V be an F-vector space with derivation ∂ and Φ be n generators of V.

Derivation matrix

$M \in F^{n\times n}$ is a derivation matrix w.r.t Φ if

$$\partial \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = M \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} + \begin{pmatrix} \alpha_1' \\ \vdots \\ \alpha_n' \end{pmatrix}$$

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Example: a derivation matrix in $V_F(f)$ is the companion matrix $\mathcal{C}_f$.

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Let $f, g \in \mathrm{D}(R)$ of orders d_1 and d_2 respectively. Consider $h = f + g$

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Computing the space W_+ and the generators Φ_+

The proof of the closure property **addition** shows:

$$W_+ = V(f) \oplus V(g),$$

hence the generators of W_+ are the union of the generators of $V(f)$ and $V(g)$:

$$\Phi_+ = \{f, f', \ldots, f^{(d_1-1)}, \\ g, g', \ldots, g^{(d_2-1)}\}$$

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Computing the derivation matrix M_+ w.r.t Φ_+

$$M_+ = \mathcal{C}_f \oplus \mathcal{C}_g.$$

Computing the initial vector v_0 w.r.t Φ_+

As we have $h = f + g$, the initial vector is:

$$v_0 = e_{d_1,1} \oplus e_{d_2,1},$$

Product

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Computing the space W_* and the generators Φ_*

The proof of the closure property **product** shows:

$$W = V(f) \otimes V(g).$$

Hence the generators of W_* are the tensor product of the generators of $V(f)$ and $V(g)$:

$$\Phi = \{ \begin{array}{cccc} fg, & f'g, & \ldots, & f^{(d_1-1)}g, \\ fg', & f'g', & \ldots, & f^{(d_1-1)}g', \\ \vdots, & \vdots, & \ddots, & \vdots, \\ fg^{(d_2-1)}, & f'g^{(d_2-1)}, & \ldots, & f^{(d_1-1)}g^{(d_2-1)} \end{array} \}$$

Product

Let $f, g \in \mathrm{D}(R)$ of orders d_1 and d_2 respectively. Consider $h = fg$

Computing the derivation matrix M_* w.r.t Φ_*

$$M_* = \mathcal{C}_f \otimes \mathcal{I}_{d_2} + \mathcal{I}_{d_1} \otimes \mathcal{C}_g.$$

Computing the initial vector v_0 w.r.t Φ_*

As we have $h = fg$, the initial vector is:

$$v_0 = e_{d_1,1} \otimes e_{d_2,1} = (1, 0, 0, 0 \ldots),$$

Coefficient growth

In the case $R = \mathrm{D}(K[x])$, computing closure properties means computing D-finite closure properties on the coefficient level.

- Each sum possibly increases the order of the equation.
- Each product possibly increases the order of the equation.
- Each derivative possibly increases the order of the equation.

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In practice: **huge coefficient growth**

Lazy computations

Solution: skip computations until the end.

1. The coefficients of the original equations are converted to new variables.

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Conclusions

Achievements

- Extended the framework of D-finite to a wider class of computable functions
- Implemented closure properties for DD-finite
- Code available for SAGE

Conclusions

Future work

- Improve performance of the current code
- Study analytic properties of DD-finite functions
- Study combinatorial properties of DD-finite functions
- Study the analogue of DD-finite functions in sequences
- Generalize to other types of operators (*q-holonomic*).
- Multivariate case

Thank you!

Contact webpage:

- https://www.dk-compmath.jku.at/people/antonio
- https://www.risc.jku.at/home/ajpastor

SAGE code:

- http://git.risc.jku.at/gitweb/?p=ajpastor/diff_defined_functions.git