

# Effective Computation of Generalized Spectral Sequences

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The  $n$ -**homology group** of  $C_*$  is defined as

$$H_n(C_*) := \frac{\text{Ker } d_n}{\text{Im } d_{n+1}}$$

and its rank  $\beta_n$  is called  $n$ -th **Betti number**.

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A **filtration** of the chain complex  $C_*$  is a sequence  $(F_p C_*)_{p \in \mathbb{Z}}$

$$\cdots \subseteq F_{p-1} C_* \subseteq F_p C_* \subseteq \cdots \subseteq C_*$$

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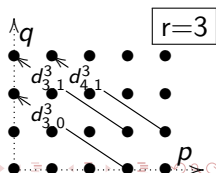
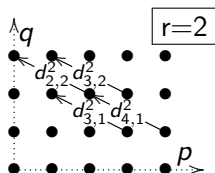
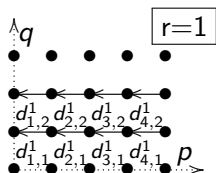
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# Generalized filtrations and spectral systems

The notion of spectral sequence of a filtered complex has been recently generalized by B. Matschke for a filtration indexed over a *poset*  $I$ , i.e. a collection of sub-chain complexes  $\{F_i C_*\}_{i \in I}$  with  $F_i C_* \subseteq F_j C_*$  if  $i \leq j$ , as a set of groups, for all  $z \leq s \leq p \leq b$  in  $I$  and for each degree  $n$ :

$$S_n[z, s, p, b] = \frac{F_p C_n \cap d_n^{-1}(F_z C_{n-1}) + F_s C_n}{d_{n+1}(F_b C_{n+1}) + F_s C_n}$$

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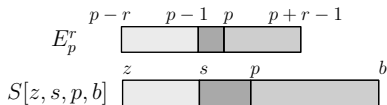
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Example:  $\mathbb{Z}$ -filtration  $(F_p)_{p \in \mathbb{Z}}$ , indices  $z \leq s \leq p \leq b$  in  $\mathbb{Z}$ :



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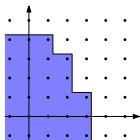
Consider  $\mathbb{Z}^m$ , seen as the poset  $(\mathbb{Z}^m, \leq)$  with the coordinate-wise order relation:  $P = (p_1, \dots, p_m) \leq Q = (q_1, \dots, q_m)$  if and only if  $p_i \leq q_i$ , for all  $1 \leq i \leq m$ .



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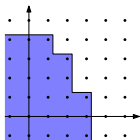
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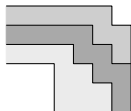
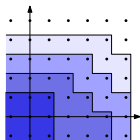
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We denote  $D(\mathbb{Z}^m)$  the collection of all downsets of  $\mathbb{Z}^m$ , which is a poset with respect to the inclusion  $\subseteq$ .



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## Theorem (Serre, 1951)

Let  $G \hookrightarrow E \rightarrow B$  be a **fibration** and suppose the base  $B$  is 1-reduced. There is a spectral sequence converging to  $H_*(E)$  whose second page is given by  $E_{p,q}^2 = H_p(B; H_q(G))$ .

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Consider a **tower of fibrations**

$$\begin{array}{ccccc} E & \longrightarrow & N & \longrightarrow & B \\ \uparrow & & \uparrow & & \\ G & & M & & \end{array}$$

and suppose the base  $B$  is 1-reduced. There exists a  $D(\mathbb{Z}^2)$ -spectral system converging to  $H_*(E)$  whose second page is given by

$$S_n^*(P; 2) = H_{p_2}(B; H_{p_1}(M; H_{n-p_1-p_2}(G))), \quad P = (p_1, p_2) \in \mathbb{Z}^2.$$

# Multidimensional persistence and persistence of $f$ -filtrations

Multidimensional filtrations (or  $\mathbb{Z}^m$ -filtrations) of simplicial complexes:

$$\begin{array}{ccccccc} K_{1N'} & \hookrightarrow & K_{2N'} & \hookrightarrow & \dots & \hookrightarrow & K_{NN'} \\ \uparrow & & \uparrow & & & & \uparrow \\ \dots & & \dots & & & & \dots \\ \uparrow & & \uparrow & & & & \uparrow \\ K_{12} & \hookrightarrow & K_{22} & \hookrightarrow & \dots & \hookrightarrow & K_{N2} \\ \uparrow & & \uparrow & & & & \uparrow \\ K_{11} & \hookrightarrow & K_{21} & \hookrightarrow & \dots & \hookrightarrow & K_{N1} \end{array}$$

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Associated invariant: **rank invariant**

$$\beta_n^{P,Q} := \dim_{\mathbb{F}} \operatorname{Im}(H_n(K_P) \rightarrow H_n(K_Q)), \quad P, Q \in \mathbb{Z}^m, \quad P \leq Q.$$

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Similarly, for an  $I$ -filtration  $(F_i)_{i \in I}$ , we define **rank invariant** the collection of integers

$$\beta_n(v, w) := \dim_{\mathbb{F}} \operatorname{Im}(H_n(F_v) \rightarrow H_n(F_w)), \quad v, w \in I, \quad v \leq w.$$





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The result is a *basis-divisors* description of the group, that is:

- a list of combinations  $(c_1, \dots, c_{\alpha+k})$
- a list of torsion coefficients  $(b_1, \dots, b_k, 0, \dots, 0)$ .



To compute the differential map

$d : S_2 \equiv S[z_2, s_2, p_2, b_2] \rightarrow S_1 \equiv S[z_1, s_1, p_1, b_1]$  applied to an element  $a = [x]$  given by a list of coordinates  $(a_1, \dots, a_r)$ :

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- We compute the coefficients of  $d(y)$  with respect to the set of generators of  $S_1$ .
- We reduce them considering the corresponding divisors.



If a  $I$ -filtered chain complex  $C_*$  is not of finite type, we use the effective homology method and we consider a pair of *reductions*  $C_* \Leftarrow \hat{C}_* \Rightarrow D_*$  from the initial chain complex  $C_*$  to another one  $D_*$  of finite type (also filtered over  $I$ ).

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## Theorem

Let  $\rho = (f, g, h) : C_* \Rightarrow D_*$  be a reduction between the  $I$ -filtered chain complexes  $(C_*, F)$  and  $(D_*, F')$ , and suppose that  $f$  and  $g$  are compatible with the filtrations. Then, given four indices  $z \leq s \leq p \leq b$  in  $I$ , the map  $f$  induces an isomorphism  $f^{z,s,p,b} : S[z, s, p, b]_n \rightarrow S'[z, s, p, b]_n$  whenever the homotopy  $h : (C_*, F) \rightarrow (C_{*+1}, F)$  satisfies the conditions

$$h(F_z) \subseteq F_s \quad \text{and} \quad h(F_p) \subseteq F_b.$$

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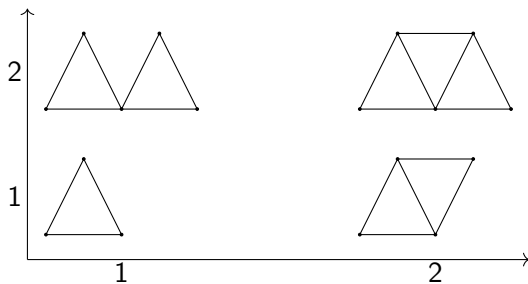
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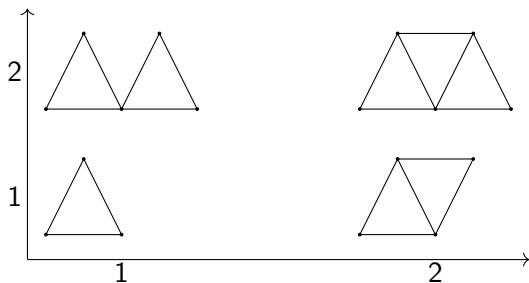
## Corollary

Under the same hypotheses, the generalized spectral sequences associated with the  $I$ -filtrations of  $C_*$  and  $C_*^c$  are isomorphic.

# Example



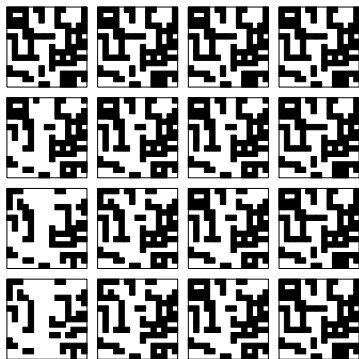
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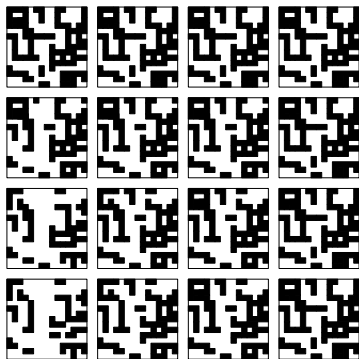
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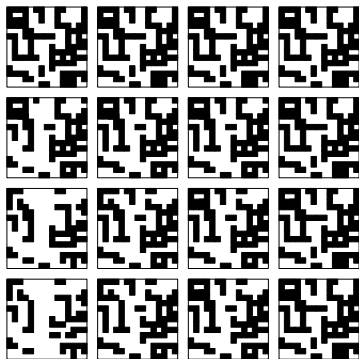
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Associated simplicial complex: 203 vertices, 408 edges and 208 triangles.

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Reduced chain complex: 21 vertices, 23 edges and 5 triangles.



# Generalized Serre spectral sequence: example

First stages of the Postnikov tower for computing the homotopy groups of the sphere  $S^3$ , given by the following tower of fibrations:

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```

```
> (gen-spsq-group K '((-1 -1)) '((-1 -1)) '((12 12)) '((12 12)) 6)
Generalized spectral sequence S[((-1 -1)),((-1 -1)),((12 12)),
((12 12))]._{6}
```

```
Component Z/6Z
```

Thank you!