Additive Decompositions in Primitive Extensions

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Outline

- Additive decomposition problem
- Previous results
- Additive decompositions in primitive extensions
  - Hermite reduction
  - Polynomial reduction
- Applications

Terminologies

Let F be a field of characteristic zero.

- A derivation on F is a map $' : F \to F$ s.t. for all $a, b \in F$,

$$(a+b)' = a' + b' \quad \text{and} \quad (ab)' = ab' + a'b.$$

- $(F, \, ')$ is a differential field.
- $C_F = \{a \in F \mid a' = 0\}$ is the subfield of constants.
- A differential field (E, D) is a differential extension of F if

$$F \subseteq E \quad \text{and} \quad D\mid_F = \, '.$$

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Example. Set $\; ' = d/dx$.

$$\mathbb{C}(x), \; \mathbb{C}(x, \log(x)), \; \mathbb{C}(x, e^x), \; \mathbb{C}(x, \sqrt{x}), \; \ldots$$

are differential fields.

Additive decomposition problem

Notation. $F' := \{f' \mid f \in F\}$.

Problem. Given $f \in F$, find $g, r \in F$ s.t.

$$f = g' + r$$

with the properties that

- $f \in F' \iff r = 0$,
- r is minimal in some sense.

Previous results

- Rational functions in $\mathbb{C}(x)$ (Ostrogradsky 1845, Hermite 1872)
- Rational functions in $\mathbb{C}(x_1, \ldots, x_n)$ (Bostan, Lairez and Salvy 2013)
- Hyperexponential functions over $\mathbb{C}(x)$ (Bostan, Chen, Chyzak, Li and Xin 2013)
- Algebraic functions over $\mathbb{C}(x)$ (Chen, Kauers, Koutschan 2016)
- Fuchsian D-finite functions over $\mathbb{C}(x)$ (Chen, van Hoeij, Kauers, Koutschan 2017)
- D-finite functions over $\mathbb{C}(x)$ (van der Hoeven 2017, 2018, Bostan, Chyzak, Lairez and Salvy 2018)

Primitive towers

Definition. Let $(F, ') \subset (E, ')$. $t \in E$ is a primitive monomial if $t' \in F$, t is transcendental over F and $C_{F(t)} = C_F$.

Examples.

- $\log(x)$ and $\arctan(x)$ are primitive monomials over $\mathbb{C}(x)$,
- $\mathrm{Li}(x) := \int \frac{dx}{\log(x)}$ is a primitive monomial over $\mathbb{C}(x, \log(x))$.

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A primitive tower is

$$\begin{array}{ccccccc} F_0 & \subset & F_1 & \subset & \cdots & \subset & F_n \\ \parallel & & \parallel & & & & \parallel \\ \mathbb{C}(x) & & F_0(t_1) & & & & F_{n-1}(t_n) \end{array}$$

where t_i is a primitive monomial over F_{i-1} for all $1 \le i \le n$.

Hermite reduction

Definition. Given a primitive tower $F_0 \subset \cdots \subset F_n$,

- $p \in F_{n-1}[t_n]$ is t_n-normal if $\gcd(p, p') \in F_{n-1}$;
- $f \in F_n$ is t_n-simple if f is proper and $\mathrm{den}(f)$ is t_n-normal.

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Lemma. For $f \in F_n$, there exist $g, h \in F_n$ and $p \in F_{n-1}[t_n]$ s.t.

$$f = g' + h + p.$$

where h is t_n-simple. Moreover,

$$f \in F_n' \implies h = 0.$$

Polynomial reduction

Problem P. For $p \in F_{n-1}[t_n]$, find $g, q \in F_{n-1}[t_n]$ s.t.

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Main idea. For $a \in F_{n-1}$ and $d \in \mathbb{N}$,

$$a\, t_n^d = g' + q \quad \text{with} \;\; \deg_{t_n}(q) < d.$$

$$\Updownarrow$$

$$a - c\, t_n' \in F_{n-1}' \quad \text{for some } c \in \mathbb{C}.$$

Hermitian parts

By Hermite reduction, for $f \in F_i$, $\exists!$ t_i-simple $h \in F_i$ s.t.

$$f = g' + h + p,$$

where $g \in F_i$ and $p \in F_{i-1}[t_i]$ for $1 \leq i \leq n$.

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Definition. Call h the Hermitian part of f, denoted by $\mathrm{hp}_{t_i}(f)$.

If $a - c\, t_n' \in F_{n-1}'$ and $\mathrm{hp}_{t_{n-1}}(t_n') \neq 0$, then

$$c = \frac{\mathrm{hp}_{t_{n-1}}(a)}{\mathrm{hp}_{t_{n-1}}(t_n')}.$$

Straight towers

Definition. A primitive tower $F_{-1} \subset F_0 \subset \cdots \subset F_n$ with $F_{-1} = \mathbb{C}$ and $F_0 = \mathbb{C}(t_0)$ is straight if $\mathrm{hp}_{t_{i-1}}(t_i') \neq 0$ for all $1 \leq i \leq n$.

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Define a t_n-straight polynomial $q \in F_{n-1}[t_n]$:

- q is t_0-straight if $q = 0$,
- q is t_n-straight if $\mathrm{lc}_{t_n}(q) = u + v$ s.t.
  - $u \in F_{n-1}$ is t_{n-1}-simple,
  - $u \neq c\,\mathrm{hp}_{t_{n-1}}(t_n')$ for any nonzero $c \in \mathbb{C}$,
  - $v \in F_{n-2}[t_{n-1}]$ is t_{n-1}-straight.

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Prop. Let $q \in F_{n-1}[t_n]$ be t_n-straight. Then $q \in F_n' \iff q = 0$.

Flat towers

Definition. A primitive tower

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is flat if $t_i' \in F_0$ for all $1 \leq i \leq n$.

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Notation. For $1 \leq i \leq n$ and $p \in F_{i-1}[t_i, \ldots, t_n]$,

- $\mathrm{hm}_i(p)$ is the head monomial of p w.r.t $\prec_{\mathrm{plex}}$ $(t_i \prec \ldots \prec t_n)$.
- $\mathrm{hc}_i(p)$ is the head coefficient of p.

Flat polynomials

Definition. A polynomial $q \in F_{n-1}[t_n]$ is t_n-flat if:

- $\exists\, q_i \in F_{i-1}[t_i, \ldots, t_n]$ s.t. $q = \sum_{i=1}^{n} q_i$,
- $\mathrm{hc}_i(q_i)$ is t_{i-1}-simple for $1 \le i \le n$,
- $q_1 = 0$ or $\mathrm{hc}_0(q_1) \notin \mathrm{span}_{\mathbb{C}}\{t_1', \ldots, t_m'\}$ where

$$m = \begin{cases} n & \text{if } \mathrm{hm}_0(q_1) = 1, \\ s & \text{if } \mathrm{hm}_0(q_1) = t_s^{e_s} \cdots t_n^{e_n} \text{ with } e_s > 0 \end{cases}$$

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Prop. Let $q \in F_{n-1}[t_n]$ be t_n-flat. Then $q \in F_n' \iff q = 0$.

The main result

Theorem. Given a straight (flat) tower $F_0 \subset \cdots \subset F_n$ and $f \in F_n$, there are $g \in F_n$ and $q \in F_{n-1}[t_n]$ s.t.

$$f = \underbrace{g'}_{\text{integrable}} + \underbrace{\mathrm{hp}_{t_n}(f) + q}_{\text{non-integrable}},$$

where q is t_n-straight (t_n-flat).

Moreover,

- $f \in F_n' \iff \mathrm{hp}_{t_n}(f) = q = 0$,
- if $f = \tilde{g}' + \tilde{h} + \tilde{q}$ for t_n-proper $\tilde{h}$ and $\tilde{q} \in F_{n-1}[t_n]$, then

$$\mathrm{den}(\mathrm{hp}_{t_n}(f)) \mid \mathrm{den}(\tilde{h}) \quad \text{and} \quad \begin{cases} \deg_{t_n}(q) \le \deg_{t_n}(\tilde{q}) & \text{(straight)} \\ q \preceq_{\mathrm{plex}} \tilde{q} & \text{(flat).} \end{cases}$$

Examples

1. Straight:

$$f_1 = \frac{1}{\log(x)\mathrm{Li}(x)} + \left(\log(x) + \frac{1}{\log(x)}\right)\mathrm{Li}(x) - \frac{x}{\log(x)} \in \mathbb{C}(x, \log(x), \mathrm{Li}(x))$$

2. Flat:

$$f_2 = \left(\frac{\arctan(x)}{x^2+1}\right)^3 - \frac{\log(x)\arctan(x)^2}{x} + \log(x)^2 \in \mathbb{C}(x, \log(x), \arctan(x))$$

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Application I: elementary integrability

Given a straight (flat) tower $F_0 \subset \cdots \subset F_n$ and $f \in F_n$, we have

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If t_i is logarithmic over F_{i-1} for all $1 \leq i \leq n$, then

f is elementary integrable over F_n

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$$r \in \mathrm{span}_{\mathbb{C}}\{a'/a \mid a \in F_n\}.$$

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Example.

$$f_1 = (\cdots)' + \frac{1}{\log(x)\mathrm{Li}(x)} = (\cdots)' + \frac{\mathrm{Li}(x)'}{\mathrm{Li}(x)} = (\cdots)' + (\log \circ \mathrm{Li}(x))'$$

Application II: creative telescoping

$(F, \{D_x, D_y\})$: a differential field with $D_x D_y = D_y D_x$.

Problem. Given $f \in F$, find nonzero $L := \sum_{i=0}^{d} \ell_i D_x^i$ with $D_y(\ell_i) = 0$ and g in an elementary extension E of F s.t.

$$L(x, D_x)(f) = D_y(g)$$

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$$f = \underbrace{t + 1 - \frac{2y}{(x^2 + y^2)t^2}}_{\text{NOT } D\text{-finite}}$$

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$$\Downarrow$$

$$L = xD_x - 1 \quad \text{and} \quad g = \frac{-2x^2}{t^2(x^2+y^2)} - \frac{1}{t} - yt + y.$$

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