Bivariate Dimension Polynomials of Non-Reflexive Prime Difference-Differential Ideals. The Case of One Translation

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Let K be a difference-differential field, $\operatorname{Char} K = 0$, with basic set of derivations $\Delta = \{\delta_1, \dots, \delta_m\}$ and a single endomorphism σ (any two mappings of the set $\Delta \bigcup \{\sigma\}$ commute). We will often use prefix Δ-σ- instead of "difference-differential".

Let T be the free commutative semigroup generated by the set $\Delta \bigcup \{\sigma\}$.

$$\text{If } \tau = \delta_1^{k_1} \dots \delta_m^{k_m} \sigma^l \in T \quad (k_1, \dots, k_m, l \in \mathbb{N}), \text{ then}$$

the numbers $\operatorname{ord}_\Delta \tau = \sum_{i=1}^{m} k_i$ and $\operatorname{ord}_\sigma \tau = l$ are called the *orders* of τ with respect to Δ and σ, respectively.

If $r, s \in \mathbb{N}$, we set $T(r, s) = \{\tau \in T \mid \operatorname{ord}_\Delta \tau \le r, \operatorname{ord}_\sigma \tau \le s\}$.

Furthermore, Θ will denote the subsemigroup of T generated by Δ, so every element $\tau \in T$ can be written as $\tau = \theta\sigma^l$ where $\theta \in \Theta$, $l \in \mathbb{N}$. If $r \in \mathbb{N}$, we set $\Theta(r) = \{\theta \in \Theta \mid \operatorname{ord}_\Delta \theta \le r\}$.

Theorem 1 (L., 2000)

With the above notation, let $L = K\langle\eta_1, \dots, \eta_n\rangle$ *be a* Δ-σ-*field extension of* K *generated by a finite set* $\eta = \{\eta_1, \dots, \eta_n\}$. *(As a field,* $L = K(\{\tau\eta_j \mid \tau \in T, 1 \le j \le n\})$.*) Then there exists a polynomial* $\phi_{\eta|K}(t_1, t_2) \in \mathbb{Q}[t_1, t_2]$ *such that*

(i) $\phi_{\eta|K}(r, s) = \operatorname{tr.deg}_K K(\{\tau\eta_j | \tau \in T(r, s), 1 \le j \le n\})$ *for all sufficiently large* $(r, s) \in \mathbb{N}^2$. *(It means that there exist* $r_0, s_0 \in \mathbb{N}$ *such that the equality holds for all* $(r, s) \in \mathbb{N}^2$ *with* $r \ge r_0, s \ge s_0$.*)*

(ii) $\deg_{t_1} \phi_{\eta|K} \le m$, $\deg_{t_2} \phi_{\eta|K} \le 1$ *and* $\phi_{\eta|K}$ *can be written as*

$$\phi_{\eta|K}(t_1, t_2) = \left(\sum_{i=0}^{m} a_i \binom{t_1 + i}{i}\right) t_2 + \sum_{i=0}^{m} b_i \binom{t_1 + i}{i}$$

where $a_i, b_i \in \mathbb{Z}$ *(*$1 \le i \le m$*).*

(iii) If $\phi_{\eta|K}(t_1, t_2) = \left(\sum_{i=0}^{m} a_i \binom{t_1 + i}{i}\right) t_2 + \sum_{i=0}^{m} b_i \binom{t_1 + i}{i}$ and

$$\phi^{(1)}(t_1) = \sum_{i=0}^{m} a_i \binom{t_1 + i}{i}, \ \ \phi^{(2)}(t_1) = \sum_{i=0}^{m} b_i \binom{t_1 + i}{i},$$

then a_m, $\deg_{t_1} \phi_{\eta|K}$, $\deg_{t_2} \phi_{\eta|K}$ (which is 0 or 1), $d = \deg \phi^{(1)}$, a_d (if $\phi^{(1)} = 0$, we set $\deg \phi^{(1)} = -1$, $a_d = 0$), and the coefficient of the monomial with the highest degree in t_1 do not depend on the choice of the system of Δ-σ-generators η of L/K.

Furthermore, a_m is equal to the Δ-σ-transcendence degree of L/K (denoted by Δ-σ-$\mathrm{tr.\,deg}_K L$), that is, to the maximal number of elements $\xi_1, \dots, \xi_k \in L$ such that the set $\{\tau\xi_i \,|\, \tau \in T, 1 \leq i \leq k\}$ is algebraically independent over K.

$\phi_{\eta|K}(t_1, t_2)$ is called the **Δ-σ-dimension polynomial** of the extension L/K associated with the set of Δ-σ-generators η.

Let $R = K\{y_1, \dots, y_n\}$ be the ring of Δ-σ-polynomials in n Δ-σ-indeterminates over K.
As a ring, $R = K[\{\tau y_i \mid \tau \in T,\ 1 \leq i \leq n\}]$. The Δ-σ-structure on R is obtained by the extension of the action of elements of T on K by setting $\tau'(\tau y_i) = (\tau'\tau)y_i$ for any $\tau, \tau' \in T$, $1 \leq i \leq n$.)
Elements of the set $TY = \{\tau y_i \mid \tau \in T,\ 1 \leq i \leq n\}$ are called **terms**.

By a Δ-σ-ideal of R we mean an ideal P of this ring such that $\delta_i(P) \subseteq P$ ($1 \leq i \leq m$) and $\sigma(P) \subseteq P$. P is said to be a prime Δ-σ-ideal if it is prime in the usual sense.

A Δ-σ-ideal P is said to be *reflexive* if the inclusion $\sigma(a) \in P$ ($a \in R$) implies that $a \in P$. In this case the factor ring R/P has the natural structure of a Δ-σ-ring: $\tau(a + P) = \tau(a) + P$ for every $a \in R$, $\tau \in T$.

If P is a prime reflexive Δ-σ-ideal in the ring of Δ-σ-polynomials $R = K\{y_1, \dots, y_n\}$, then the quotient field $L = \mathrm{q.f.}(R/P)$ has a natural structure of a Δ-σ-field extension of K: $L = K\langle \eta_1, \dots, \eta_n \rangle$ where η_i is the canonical image of y_i in R/P ($1 \le i \le n$). Then the Δ-σ-dimension polynomial of the extension L/K is called the **Δ-σ-dimension polynomial** of P.

If $f \in R$, then $f(\eta)$ will denote the image of f under the natural homomorphism $R \to L$ ($\eta_i \mapsto y_i + P$ for $i = 1, \dots, n$).
If $F \subset R$, we set $F(\eta) = \{f(\eta) \mid f \in F\}$.

If P is a non-reflexive Δ-σ-ideal of R, then

$$P^* = \{f \in R \mid \sigma^k(f) \in P \text{ for some } k \in \mathbb{N}\}$$

is the smallest reflexive Δ-σ-ideal of R containing P. It is called the **reflexive closure** of P. If P is prime, so is P^*.

The original proof of Theorem 1 was based on the properties of dimension polynomials of Δ-σ-modules and modules of Kähler differentials associated with a field extension. The following generalization of the Ritt-Kolchin characteristic set method gives another proof of Theorem 1 and a method of computation of Δ-σ-dimension polynomials.

We consider two orderings $<_\Delta$ and $<_\sigma$ on T and on the set of terms TY of $K\{y_1, \dots, y_n\}$ such that if $\tau = \delta_1^{k_1} \dots \delta_m^{k_m}\sigma^l$, $\tau' = \delta_1^{k'_1} \dots \delta_m^{k'_m}\sigma^{l'} \in T$, then

$\tau <_\Delta \tau'$ iff $(\text{ord}_\Delta \tau, k_1, \dots, k_m, l) <_P (\text{ord}_\Delta \tau', k'_1, \dots, k'_m, l')$ and

$\tau <_\sigma \tau'$ iff $(l, \text{ord}_\Delta \tau, k_1, \dots, k_m) <_P (l, \text{ord}_\Delta \tau, k'_1, \dots, k'_m)$.

Furthermore, $\tau y_i <_\Delta (<_\sigma) \tau' y_j$ iff $\tau <_\Delta (<_\sigma) \tau'$ or $\tau = \tau'$, $i < j$.

($<_P$ denotes the product order on the set $\mathbb{N}^{m+2}$:
$a = (a_1, \dots, a_{m+2}) \leq_P a' = (a'_1, \dots, a'_{m+2})$ iff $a_i \leq a'_i$ for $i = 1, \dots, m+2$; $a <_P a'$ iff $a \leq_P a'$ and $a \neq a'$.)

If $u = \tau y_k \in TY$, we set $\operatorname{ord}_\Delta u = \operatorname{ord}_\Delta \tau$ and $\operatorname{ord}_\sigma u = \operatorname{ord}_\sigma \tau$.

A term $\tau' y_i$ is said to be a **transform** of a term τy_j if $i = j$ and $\tau \mid \tau'$ (that is, $\tau' = \tau\tau''$ for some $\tau'' \in T$).

If $A \in K\{y_1, \dots, y_n\} \setminus K$, then the highest terms of A with respect to $<_\Delta$ and $<_\sigma$ are called the **Δ-leader** and **σ-leader** of A, respectively. They are denoted, respectively, by u_A and v_A.

If A is written as a polynomial in v_A,

$$A = I_d v_A^d + I_{d-1} v_A^{d-1} + \cdots + I_0$$

($I_d, I_{d-1}, \dots, I_0$ do not contain v_A), then I_d is called the **initial** of A; it is denoted by I_A.

$\partial A/\partial v_A = dI_d v_A^{d-1} + (d-1)I_{d-1} v_A^{d-2} + \cdots + I_1$ is called a **separant** of A; it is denoted by S_A.

If $A, B \in K\{y_1, \ldots, y_n\}$, we say that A has lower rank than B and write $\operatorname{rk} A < \operatorname{rk} B$ if either $A \in K$, $B \notin K$, or $(v_A, \deg_{v_A} A, \operatorname{ord}_\Delta u_A) <_{lex} (v_B, \deg_{v_B} B, \operatorname{ord}_\Delta u_B)$ where v_A and v_B are compared with respect to $<_\sigma$. If the two vectors are equal (or $A, B \in K$), we say that A and B are of the same rank and write $\operatorname{rk} A = \operatorname{rk} B$.

If $A, B \in K\{y_1, \ldots, y_n\}$, then B is said to be **reduced** with respect to A if
(i) B does not contain terms τv_A such that $\operatorname{ord}_\Delta \tau > 0$ and $\operatorname{ord}_\Delta(\tau u_A) \leq ord_\Delta u_B$.

(ii) If B contains a term τv_A where $\operatorname{ord}_\Delta \tau = 0$, then either $\operatorname{ord}_\Delta u_B < \operatorname{ord}_\Delta u_A$ or $\operatorname{ord}_\Delta u_A \leq \operatorname{ord}_\Delta u_B$ and $\deg_{\tau v_A} B < \deg_{v_A} A$.

If $B \in K\{y_1, \ldots, y_n\}$, then B is said to be reduced with respect to a set $\mathcal{A} \subseteq K\{y_1, \ldots, y_n\}$ if B is reduced with respect to every element of $\mathcal{A}$.
A set of Δ-σ-polynomials $\mathcal{A}$ in $K\{y_1, \ldots, y_n\}$ is called **autoreduced** if $\mathcal{A} \bigcap K = \emptyset$ and every element of $\mathcal{A}$ is reduced with respect to any other element of this set.

Proposition 1

Every autoreduced set of Δ-σ-polynomials in the ring $K\{y_1, \ldots, y_n\}$ is finite.

In what follows we always list elements of an autoreduced set in the order of increasing rank.

Proposition 2

Let $\mathcal{A} = \{A_1, \ldots, A_d\}$ be an autoreduced set in $K\{y_1, \ldots, y_s\}$ and let I_k and S_k denote the initial and separant of A_k, respectively. Let

$I(\mathcal{A}) = \{X \in K\{y_1, \ldots, y_n\} \mid X = 1$ *or* X *is a product of finitely many elements of the form* $\sigma^i(I_k)$ *and* $\sigma^j(S_k)$ *where* $i, j \in \mathbb{N}\}$.

Then for any Δ-σ-polynomial B, there exist $B_0 \in K\{y_1, \ldots, y_n\}$ and $J \in I(\mathcal{A})$ such that B_0 is reduced with respect to $\mathcal{A}$ and $JB \equiv B_0 \mod [\mathcal{A}]$ (that is, $JB - B_0 \in [\mathcal{A}]$).

The Δ-σ-polynomial B_0 is called the *remainder* of B with respect to $\mathcal{A}$. We also say that *B reduces to B_0 modulo $\mathcal{A}$*.

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If $\mathcal{A} = \{A_1, \ldots, A_p\}$, $\mathcal{B} = \{B_1, \ldots, B_q\}$ are two autoreduced sets, we say that $\mathcal{A}$ has lower rank than $\mathcal{B}$ if one of the following two cases holds:
(1) There exists $k \in \mathbb{N}$ such that $k \leq \min\{p, q\}$, $\operatorname{rk} A_i = \operatorname{rk} B_i$ for $i = 1, \ldots, k-1$ and $\operatorname{rk} A_k < \operatorname{rk} B_k$.

(2) $p > q$ and $\operatorname{rk} A_i = \operatorname{rk} B_i$ for $i = 1, \ldots, q$.

If $p = q$ and $\operatorname{rk} A_i = \operatorname{rk} B_i$ for $i = 1, \ldots, p$, then $\operatorname{rk} \mathcal{A} = \operatorname{rk} \mathcal{B}$.

Proposition 3

In every nonempty family of autoreduced sets of Δ-σ-polynomials there exists an autoreduced set of lowest rank. In particular, every ideal I of $K\{y_1, \ldots, y_s\}$ contains an autoreduced set of lowest rank called a **characteristic** *set of I. If $\mathcal{A}$ is a characteristic set of a Δ-σ-ideal I, then an element $B \in I$ is reduced with respect to $\mathcal{A}$ if and only if $B = 0$.*

Now we need some results about dimension polynomials of subsets of $\mathbb{N}^{m+1}$ (m is a positive integer) treated as a Cartesian product $\mathbb{N}^m \times \mathbb{N}$ (we single out the last coordinate).

If $a = (a_1, \dots, a_{m+1}) \in \mathbb{N}^{m+1}$, we set $\operatorname{ord}_1 a = \sum_{i=1}^{m} a_i$ and $\operatorname{ord}_2 a = a_{m+1}$. Furthermore, we treat $\mathbb{N}^{m+1}$ as a partially ordered set with respect to the product order $\leq_P$.

If $A \subseteq \mathbb{N}^{m+1}$, then V_A will denote the set of all elements $v \in \mathbb{N}^{m+1}$ such that there is no $a \in A$ with $a \leq_P v$. Thus, $v = (v_1, \dots, v_{m+1}) \in V_A$ if and only if for any element $(a_1, \dots, a_{m+1}) \in A$, there exists $i \in \mathbb{N}, 1 \leq i \leq m+1$, such that $a_i > v_i$.

Furthermore, for any $r, s \in \mathbb{N}$, we set

$$A(r,s) = \{x = (x_1, \dots, x_{m+1}) \in A \mid \operatorname{ord}_1 x \leq r, \operatorname{ord}_2 x \leq s\}.$$

Theorem 2

Let $A \subseteq \mathbb{N}^{m+1}$*. Then there exists a polynomial* $\omega_A(t_1, t_2) \in \mathbb{Q}[t_1, t_2]$ *such that*

(i) $\omega_A(r, s) = \text{Card } V_A(r, s)$ *for all sufficiently large* $(r, s) \in \mathbb{N}^2$.

(ii) $\deg_{t_1} \omega_A \leq m$ *and* $\deg_{t_2} \omega_A \leq 1$ *(hence* $\deg \omega_A \leq m + 1$*).*

(iii) $\deg \omega_A = m + 1$ *if and only if* $A = \emptyset$*. In this case* $\omega_A(t_1, t_2) = \binom{t_1+m}{m}(t_2 + 1)$.

(iv) $\omega_A = 0$ *if and only if* $(0, \dots, 0) \in A$.

$\omega_A(t_1, t_2)$ is called the dimension polynomial of the set $A \subseteq \mathbb{N}^{m+1}$ associated with the orders ord_1 and ord_2.

The proof of the theorem and a closed-form formula for $\omega_A(t_1, t_2)$ can be found in [Kondrateva, M. V., Levin, A. B., Mikhalev, A. V., Pankratev, E. V. *Differential and Difference Dimension Polynomials*. Kluwer Acad. Publ., 1999.]

Let K be a Δ-σ-field, $R = K\{y_1, \dots, y_n\}$, and P a prime Δ-σ-ideal in R. Let P^* denote the reflexive closure of P (P^* is also a prime) and for every $r, s \in \mathbb{N}$, let $R_{rs} = K[\{\tau y_i \,|\, \tau \in T(r,s), 1 \leq i \leq n\}]$. (It is a polynomial ring over K in indeterminates τy_i such that $\operatorname{ord}_\Delta \tau \leq r$ and $\operatorname{ord}_\sigma \tau \leq s$.)

Let $P_{rs} = P \bigcap R_{rs}$, $P^*_{rs} = P^* \bigcap R_{rs}$, and let L, L^*, L_{rs} and L^*_{rs} denote the quotient fields of the integral domains R/P, R/P^*, R_{rs}/P_{rs} and R_{rs}/P^*_{rs}, respectively.

If η_i denotes the canonical image of y_i in R/P^*, then L^* is a Δ-σ-field extension of K, $L^* = K\langle \eta_1, \dots, \eta_n \rangle$, and $L^*_{rs} \cong K(\{\tau\eta_i \,|\, \tau \in T(r,s),\, 1 \leq i \leq n\})$.

Theorem 3

With the above notation, let $\mathcal{A} = \{A_1, \dots, A_p\}$ be a characteristic set of P^* and for any $r, s \in \mathbb{N}$, let

$U'_{rs} = \{u \in TY \mid \operatorname{ord}_\Delta u \le r, \operatorname{ord}_\sigma u \le s$ *and u is not a transform of any v_{A_i}*$\}$ *and*

$U''_{r,s} = \{u \in TY \mid \operatorname{ord}_\Delta u \le r, \operatorname{ord}_\sigma u \le s$ *and there exist $A \in \mathcal{A}$ such that $u = \tau v_A$ and* $\operatorname{ord}_\Delta(\tau u_A) > r\}$.

Then $U'_{rs}(\eta) \bigcup U''_{rs}(\eta)$ is a transcendence basis of L^*_{rs} over K.

By Theorem 2, there exists $\phi^{(1)}(t_1, t_2) \in \mathbb{Q}[t_1, t_2]$ such that $\phi^{(1)}(r, s) = \operatorname{Card} U'_{rs}$ for all sufficiently large $(r, s) \in \mathbb{N}^2$, $\deg_{t_1} \phi^{(1)} \le m$, and $\deg_{t_2} \phi^{(1)} \le 1$. Furthermore, $\operatorname{Card} U''_{r,s}$ is expressed by a polynomial $\phi^{(2)}(t_1, t_2)$ which is an alternating sum of bivariate polynomials of the form $\binom{t_1 + m + a}{m}(t_2 + b)$ $(a, b \in \mathbb{Z})$.

It shows that there exists a polynomial $\phi_{P^*}(t_1, t_2) \in \mathbb{Q}[t_1, t_2]$ such that
(i) $\phi_{P^*}(r, s) = \operatorname{tr.deg}_K L^*_{rs}$ for all sufficiently large $(r, s) \in \mathbb{N}^2$.
(ii) $\phi_{P^*}(t_1, t_2)$ is linear with respect to t_2 and $\deg_{t_1} \phi_{P^*} \le m$; it is of the form

$$\phi_{P^*}(t_1, t_2) = \phi^{(1)}_{P^*}(t_1)t_2 + \phi^{(2)}_{P^*}(t_1)$$

where $\phi^{(1)}_{P^*}(t_1)$ and $\phi^{(2)}_{P^*}(t_1)$ are polynomials in one variable with rational coefficients that can be written as

$$\phi^{(1)}_{P^*}(t_1) = \sum_{i=0}^{m} a_i \binom{t_1 + i}{i} \text{ and } \phi^{(2)}_{P^*}(t_1) = \sum_{i=0}^{m} b_i \binom{t_1 + i}{i}$$

with $a_i,\, b_i \in \mathbb{Z}$ $(1 \le i \le m)$.

Theorem 4

Let K be a Δ-σ-field, $R = K\{y_1, \dots, y_n\}$ and P a prime non-reflexive Δ-σ-ideal in R. For every $r, s \in \mathbb{N}$, let $R_{rs} = K[\{\tau y_i \mid \tau \in T(r,s), 1 \le i \le n\}]$, $P_{rs} = P \bigcap R_{rs}$, and $L_{rs} = \text{q.f.}(R_{rs}/P_{rs})$. Then there exists a bivariate polynomial $\psi_P(t_1, t_2) \in \mathbb{Q}[t_1, t_2]$ such that

(i) $\psi_P(r, s) = \text{tr.deg}_K L_{rs}$ *for all sufficiently large $(r, s) \in \mathbb{N}^2$.*

(ii) *The polynomial $\psi_P(t_1, t_2)$ is linear with respect to t_2 and* $\deg_{t_1} \psi_P \le m$*, so it can be written as*

$$\psi_P(t_1, t_2) = \psi_P^{(1)}(t_1)t_2 + \psi_P^{(2)}(t_1)$$

where $\psi_P^{(1)}(t_1)$ and $\psi_P^{(2)}(t_1)$ are polynomials in one variable with rational coefficients of degree at most m.

In the case of a non-reflexive prime difference polynomial ideal P (when $\Delta = \emptyset$), this result was proved in [E. Hrushovski. The Elementary Theory of the Frobenius Automorphisms. *arXiv:math*/0406514v1, 2004, 1–135. The updated version (2012): http://www.ma.huji.ac.il/ ehud/FROB.pdf]

and

[M. Wibmer. Algebraic Difference Equations. Lecture Notes (2013). https://www.math.upenn.edu/wibmer/ AlgebraicDifferenceEquations.pdf]

We will outline a proof based on the properties of characteristic sets. It will also give a method of computation of dimension polynomials associated with a non-reflexive prime Δ-σ-polynomial ideal.

We start with the case $\Delta = \emptyset$ and use prefix σ- instead of Δ-σ-.

Let $\mathcal{A} = \{A_1, \dots, A_p\}$ be a characteristic set of P^* (the reflexive closure of P), let v_j denote the σ-leader of A_j ($1 \le j \le p$), and let $\eta_i = y_i + P \in K\{y_1, \dots, y_n\}/P$ ($1 \le i \le n$).

Let $L = \text{q.f.}(K\{y_1, \dots, y_n\}/P) = K(\{\sigma^k \eta_i \,|\, k \in \mathbb{N},\, 1 \le i \le n\})$ and $L_s = K(\{\sigma^k \eta_i \,|\, 0 \le k \le s,\, 1 \le i \le n\})$.

For every $j = 1, \dots, p$, let s_j be the smallest nonnegative integer such that $\sigma^{s_j}(A_j) \in P$. Furthermore, let

$$V = \{v \in TY \,|\, v \neq \sigma^i v_j \text{ for any } i \in \mathbb{N},\, 1 \le j \le p\},$$

$V_r = \{v \in V \,|\, \text{ord}_\sigma v \le r\}$ ($r \in \mathbb{N}$), $V(\eta) = \{v(\eta) \,|\, v \in V\}$,
$W = \{\sigma^k v_j \,|\, 1 \le j \le p,\, 0 \le k \le s_j - 1\}$, and
$W(\eta) = \{u(\eta) \,|\, u \in W\}$.

It is easy to see that the set $V(\eta)$ is algebraically independent over K: if $f(v_1(\eta), \dots, v_k(\eta)) = 0$ for some polynomial f and $v_1, \dots, v_k \in V$, then $f(v_1, \dots, v_k) \in P \subseteq P^*$ and $f(v_1, \dots, v_k)$ is reduced with respect to the characteristic set $\mathcal{A}$, hence $f = 0$.

Furthermore, every element of the field L is algebraic over its subfield $K\left(V(\eta) \bigcup W(\eta)\right)$. Let $\{w_1, \dots, w_q\}$ be a maximal subset of W such that the set $\{w_1(\eta), \dots, w_q(\eta)\}$ is algebraically independent over $K\left(V(\eta)\right)$. Then $V(\eta) \bigcup \{w_1(\eta), \dots, w_q(\eta)\}$ is a transcendence basis of L/K.

Since the set W is finite, there exists $r_0 \in \mathbb{N}$ such that
(i) $w_1, \dots, w_q \in R_{r_0} = K[\{\sigma^k y_i \,|\, 0 \leq k \leq r_0,\, 1 \leq i \leq n\}]$;
(ii) $r_0 \geq \max\{\operatorname{ord}_\sigma v_j + s_j \,|\, 1 \leq j \leq p\}$;
(iii) Every element of $W(\eta)$ is algebraic over the field $K\left(V_{r_0}(\eta) \bigcup \{w_1(\eta), \dots, w_q(\eta)\}\right)$.

Let $r \geq r_0$. $R_r = K[\{\sigma^k y_i \mid 1 \leq i \leq n,\ 0 \leq k \leq r\}]$, and $P_r = P \bigcap R_r$. Let L_r denote the quotient field of the integral domain R_r/P_r and $\zeta_i^{(r)} = y_i + P_r \in R_r/P_r \subseteq L_r$ ($1 \leq i \leq n$). Furthermore, let $\zeta^{(r)} = \{\zeta_1^{(r)}, \dots, \zeta_n^{(r)}\}$, and $V_r(\zeta^{(r)}) = \{v(\zeta^{(r)}) \mid v \in V_r\}$. Then one can show that

$$B_r = V_r(\zeta^{(r)}) \bigcup \{w_1(\zeta^{(r)}), \dots, w_q(\zeta^{(r)})\}$$

is a transcendence basis of L_r over K.

This completes the proof of the theorem in the case $\Delta = \emptyset$ and also shows that $\psi_P(t) = \phi_{P^*}(t) + q$ where q is a constant. As a consequence of this result we obtain that any strictly ascending chain of prime σ-ideals between P and P^* has length at most q and that $K\{y_1, \dots, y_n\}$ satisfies the ascending chain condition for prime (not necessarily reflexive) σ-ideals.

In order to complete the proof Theorem 4 in the case $\operatorname{Card}\Delta = m > 0$, we treat L_{rs} as the subfield $K(\{\theta\sigma^j\xi_i \,|\, \theta \in \Theta(r), 0 \le j \le s, 1 \le i \le n\})$ of the differential (Δ-) overfield $K\langle\{\sigma^j\xi_i \,|\, 0 \le j \le s, 1 \le i \le n\}\rangle_\Delta$ of K.
(Here ξ_i is the canonical image of y_i in the factor ring $K\{\sigma^j y_i, 1 \le j \le s, 1 \le i \le n\}_\Delta / P \bigcap K\{\sigma^j y_i, 1 \le j \le s, 1 \le i \le n\}_\Delta$.)

By the Kolchin's theorem on differential dimension polynomial, for any $s \in \mathbb{N}$, there exists a polynomial

$$\chi_s(t) = \sum_{i=0}^{m} a_i(s)\binom{t+i}{i}$$

($a_i(s) \in \mathbb{Z}$) such that $\chi_s(r) = \operatorname{tr.deg}_K L_{rs}$ for all sufficiently large $r \in \mathbb{N}$.

On the other hand, the first part of the proof (with the use of the finite set of σ-indeterminates $\{\Theta(r)y_i \mid \theta \in \Theta(r),\ 1 \le i \le n\}$ instead of $\{y_1, \dots, y_n\}$) shows that

$$\operatorname{tr.deg}_K L_{rs} = \operatorname{Card} V_{rs} + \lambda(r)$$

where

$V_{rs} = \{u = \tau y_i \in TY \mid \tau \in T(r,s)$ and $u \neq \tau' v_j$ for any $\tau' \in T,\ 1 \le j \le p\}$.

(v_j denotes the σ-leader of the element A_j of a characteristic set $\mathcal{A} = \{A_1, \dots, A_p\}$ of the reflexive closure P^* of P.)

Since the set W in the first part of the proof is finite and depends only on the σ-orders of terms of A_j, $1 \le j \le p$, the number of elements of the corresponding set in the general case depends only on r; we have denoted it by $\lambda(r)$.

By Theorem 2, there exist $r_0, s_0 \in \mathbb{N}$ and a bivariate numerical polynomial $\omega(t_1, t_2)$ such that $\omega(r, s) = \mathrm{Card}\, V_{rs}$ for all $r \geq r_0,\ s \geq s_0$, $\deg_{t_1} \omega \leq m$ and $\deg_{t_2} \omega \leq 1$. Thus,

$$\mathrm{tr.deg}_K L_{rs} = \omega(r, s) + \lambda(r)$$

for all $r \geq r_0,\ s \geq s_0$. At the same time, we have seen that

$$\mathrm{tr.deg}_K L_{rs_0} = \chi_{s_0}(r) = \sum_{i=0}^{m} a_i(s_0)\binom{r+i}{i}$$

for all sufficiently large $r \in \mathbb{N}$ ($a_i(s_0) \in \mathbb{Z}$). It follows that $\lambda(r)$ is a polynomial of r for all sufficiently large $r \in \mathbb{N}$, say, for all $r \geq r_1$. Therefore, for any $s \geq s_0,\ r \geq \max\{r_0, r_1\}$, $\mathrm{tr.deg}_K L_{rs} = \omega(r, s) + \lambda(r)$ is expressed as a bivariate numerical polynomial in r and s.

Example

Let K be a Δ-σ-field with two basic derivations, $\Delta = \{\delta_1, \delta_2\}$, and one basic endomorphism σ. Let $K\{y\}$ be the ring of Δ-σ-polynomials in one Δ-σ-indeterminate y and P a linear (and therefore prime) Δ-σ-ideal of $K\{y\}$ generated by the Δ-σ-polynomial $A = \sigma^2 y + \sigma\delta_1^2 y + \sigma\delta_2^2 y$ (that is, $P = [A]$). Then $P^* = [B]$, where $B = \sigma y + \delta_1^2 y + \delta_2^2 y$, and $\{B\}$ is a characteristic set of the Δ-σ-ideal P^*. With the notation of the proof of Theorem 1, we have $U'_{rs} = \{u \in TY \mid \operatorname{ord}_\Delta u \leq r, \operatorname{ord}_\sigma u \leq s$ and u is not a multiple of $\sigma y\}$ and $U''_{rs} = \{u \in TY \mid \operatorname{ord}_\Delta u \leq r, \operatorname{ord}_\sigma u \leq s$ and there is $\tau \in T$ such that $u = \tau(\sigma y)$ and $\operatorname{ord}_\Delta(\tau\delta_1^2) > r\}$.

Then $\text{Card } U'_{rs} = \text{Card}\{\delta_1^i \delta_2^j y \mid i + j \leq r\} = \binom{r+2}{2}$ and

$\text{Card } U''_{rs} = \text{Card}\{\sigma^i \delta_1^j \delta_2^k y \mid 1 \leq i \leq s,\ r-2 < j+k \leq r\} =$

$s\left(\binom{r+2}{2} - \binom{r+2-2}{2}\right) = (2r+1)s.$

Since $\sigma B \in P$, the proof of Theorem 4 shows that if $\psi_P(t_1, t_2)$ is the Δ-σ-dimension polynomial of P, then

$$\psi(r,s) = \text{Card } U'_{rs} + \text{Card } U''_{rs} + \text{Card}\{\sigma\delta_1^i \delta_2^j y \mid i + j \leq r-2\}$$

for all sufficiently large $(r,s) \in \mathbb{N}^2$. It follows that

$\psi_P(t_1, t_2) = (2t_1+1)t_2 + \binom{t_1+2}{2} + \binom{t_1}{2}$, that is

$$\psi_P(t_1, t_2) = (2t_1+1)t_2 + t_1^2 + t_1 + 1.$$

Thanks!