Bivariate Dimension Polynomials of Non-Reflexive Prime Difference-Differential Ideals. The Case of One Translation

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Let $K$ be a difference-differential field, $\text{Char } K = 0$, with basic set of derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$ and a single endomorphism $\sigma$ (any two mappings of the set $\Delta \cup \{\sigma\}$ commute). We will often use prefix $\Delta-\sigma$- instead of "difference-differential".

Let $T$ be the free commutative semigroup generated by the set $\Delta \cup \{\sigma\}$.

If $\tau = \delta_1^{k_1} \ldots \delta_m^{k_m} \sigma^l \in T \quad (k_1, \ldots, k_m, l \in \mathbb{N})$, then the numbers $\text{ord}_\Delta \tau = \sum_{i=1}^{m} k_i$ and $\text{ord}_\sigma \tau = l$ are called the orders of $\tau$ with respect to $\Delta$ and $\sigma$, respectively.

If $r, s \in \mathbb{N}$, we set $T(r, s) = \{\tau \in T \mid \text{ord}_\Delta \tau \leq r, \text{ord}_\sigma \tau \leq s\}$.

Furthermore, $\Theta$ will denote the subsemigroup of $T$ generated by $\Delta$, so every element $\tau \in T$ can be written as $\tau = \theta \sigma^l$ where $\theta \in \Theta$, $l \in \mathbb{N}$. If $r \in \mathbb{N}$, we set $\Theta(r) = \{\theta \in \Theta \mid \text{ord}_\Delta \theta \leq r\}$. 
Theorem 1 (L., 2000)

With the above notation, let \( L = K\langle \eta_1, \ldots, \eta_n \rangle \) be a \( \Delta-\sigma \)-field extension of \( K \) generated by a finite set \( \eta = \{ \eta_1, \ldots, \eta_n \} \). (As a field, \( L = K(\{ \tau \eta_j | \tau \in T, 1 \leq j \leq n \}) \).) Then there exists a polynomial \( \phi_{\eta|K}(t_1, t_2) \in \mathbb{Q}[t_1, t_2] \) such that

(i) \( \phi_{\eta|K}(r, s) = \text{tr. deg}_K K(\{ \tau \eta_j | \tau \in T(r, s), 1 \leq j \leq n \}) \) for all sufficiently large \( (r, s) \in \mathbb{N}^2 \). (It means that there exist \( r_0, s_0 \in \mathbb{N} \) such that the equality holds for all \( (r, s) \in \mathbb{N}^2 \) with \( r \geq r_0, s \geq s_0 \).)

(ii) \( \text{deg}_{t_1} \phi_{\eta|K} \leq m, \text{deg}_{t_2} \phi_{\eta|K} \leq 1 \) and \( \phi_{\eta|K} \) can be written as

\[
\phi_{\eta|K}(t_1, t_2) = \left( \sum_{i=0}^{m} a_i \binom{t_1 + i}{i} \right) t_2 + \sum_{i=0}^{m} b_i \binom{t_1 + i}{i}
\]

where \( a_i, b_i \in \mathbb{Z} \) (1 \( \leq i \leq m \)).
(iii) If $\phi_{\eta|K}(t_1, t_2) = \left( \sum_{i=0}^{m} a_i \binom{t_1 + i}{i} \right) t_2 + \sum_{i=0}^{m} b_i \binom{t_1 + i}{i}$ and

$$
\phi^{(1)}(t_1) = \sum_{i=0}^{m} a_i \binom{t_1 + i}{i}, \quad \phi^{(2)}(t_1) = \sum_{i=0}^{m} b_i \binom{t_1 + i}{i},
$$

then $a_m, \deg_{t_1} \phi_{\eta|K}, \deg_{t_2} \phi_{\eta|K}$ (which is 0 or 1), $d = \deg \phi^{(1)}$, $a_d$ (if $\phi^{(1)} = 0$, we set $\deg \phi^{(1)} = -1, a_d = 0$), and the coefficient of the monomial with the highest degree in $t_1$ do not depend on the choice of the system of $\Delta$-$\sigma$-generators $\eta$ of $L/K$.

Furthermore, $a_m$ is equal to the $\Delta$-$\sigma$-transcendence degree of $L/K$ (denoted by $\Delta$-$\sigma$-tr. $\deg_K L$), that is, to the maximal number of elements $\xi_1, \ldots, \xi_k \in L$ such that the set

$$\{ \tau \xi_i \mid \tau \in T, 1 \leq i \leq k \}$$

is algebraically independent over $K$.

$\phi_{\eta|K}(t_1, t_2)$ is called the $\Delta$-$\sigma$-dimension polynomial of the extension $L/K$ associated with the set of $\Delta$-$\sigma$-generators $\eta$. 
Let $R = K\{y_1, \ldots, y_n\}$ be the ring of $\Delta$-$\sigma$-polynomials in $n$ $\Delta$-$\sigma$-indeterminates over $K$.

As a ring, $R = K[\{\tau y_i \mid \tau \in T, 1 \leq i \leq n\}]$. The $\Delta$-$\sigma$-structure on $R$ is obtained by the extension of the action of elements of $T$ on $K$ by setting $\tau'(\tau y_i) = (\tau' \tau)y_i$ for any $\tau, \tau' \in T, 1 \leq i \leq n$.

Elements of the set $TY = \{\tau y_i \mid \tau \in T, 1 \leq i \leq n\}$ are called terms.

By a $\Delta$-$\sigma$-ideal of $R$ we mean an ideal $P$ of this ring such that $\delta_i(P) \subseteq P$ ($1 \leq i \leq m$) and $\sigma(P) \subseteq P$. $P$ is said to be a prime $\Delta$-$\sigma$-ideal if it is prime in the usual sense.

A $\Delta$-$\sigma$-ideal $P$ is said to be reflexive if the inclusion $\sigma(a) \in P$ ($a \in R$) implies that $a \in P$. In this case the factor ring $R/P$ has the natural structure of a $\Delta$-$\sigma$-ring: $\tau(a + P) = \tau(a) + P$ for every $a \in R$, $\tau \in T$. 
If $P$ is a prime reflexive $\Delta$-$\sigma$-ideal in the ring of $\Delta$-$\sigma$-polynomials $R = K\{y_1, \ldots, y_n\}$, then the quotient field $L = q.f.(R/P)$ has a natural structure of a $\Delta$-$\sigma$-field extension of $K$: $L = K\langle \eta_1, \ldots, \eta_n \rangle$ where $\eta_i$ is the canonical image of $y_i$ in $R/P$ ($1 \leq i \leq n$). Then the $\Delta$-$\sigma$-dimension polynomial of the extension $L/K$ is called the $\Delta$-$\sigma$-dimension polynomial of $P$.

If $f \in R$, then $f(\eta)$ will denote the image of $f$ under the natural homomorphism $R \rightarrow L$ ($\eta_i \mapsto y_i + P$ for $i = 1, \ldots, n$).

If $F \subset R$, we set $F(\eta) = \{f(\eta) \mid f \in F\}$.

If $P$ is a non-reflexive $\Delta$-$\sigma$-ideal of $R$, then

$$P^* = \{f \in R \mid \sigma^k(f) \in P \text{ for some } k \in \mathbb{N}\}$$

is the smallest reflexive $\Delta$-$\sigma$-ideal of $R$ containing $P$. It is called the reflexive closure of $P$. If $P$ is prime, so is $P^*$. 
The original proof of Theorem 1 was based on the properties of dimension polynomials of $\Delta$-$\sigma$-modules and modules of Kähler differentials associated with a field extension. The following generalization of the Ritt-Kolchin characteristic set method gives another proof of Theorem 1 and a method of computation of $\Delta$-$\sigma$-dimension polynomials.

We consider two orderings $<_{\Delta}$ and $<_{\sigma}$ on $T$ and on the set of terms $TY$ of $K\{y_1, \ldots, y_n\}$ such that if $\tau = \delta_1^{k_1} \ldots \delta_m^{k_m} \sigma^l$, $\tau' = \delta_1^{k'_1} \ldots \delta_m^{k'_m} \sigma'^l \in T$, then

$\tau <_{\Delta} \tau'$ iff $(\text{ord}_{\Delta} \tau, k_1, \ldots, k_m, l) <_P (\text{ord}_{\Delta} \tau', k'_1, \ldots, k'_m, l')$ and

$\tau <_{\sigma} \tau'$ iff $(l, \text{ord}_{\Delta} \tau, k_1, \ldots, k_m) <_P (l, \text{ord}_{\Delta} \tau', k'_1, \ldots, k'_m)$.

Furthermore, $\tau y_i <_{\Delta} (<_\sigma) \tau' y_j$ iff $\tau <_{\Delta} (<_\sigma) \tau'$ or $\tau = \tau'$, $i < j$.

($<_P$ denotes the product order on the set $\mathbb{N}^{m+2}$: $a = (a_1, \ldots, a_{m+2}) <_P a' = (a'_1, \ldots, a'_{m+2})$ iff $a_i \leq a'_i$ for $i = 1, \ldots, m+2$; $a <_P a'$ iff $a <_P a'$ and $a \neq a'$.)
If \( u = \tau y_k \in TY \), we set \( \text{ord}_\Delta u = \text{ord}_\Delta \tau \) and \( \text{ord}_\sigma u = \text{ord}_\sigma \tau \).

A term \( \tau' y_i \) is said to be a **transform** of a term \( \tau y_j \) if \( i = j \) and \( \tau \mid \tau' \) (that is, \( \tau' = \tau \tau'' \) for some \( \tau'' \in T \)).

If \( A \in K\{y_1, \ldots, y_n\} \setminus K \), then the highest terms of \( A \) with respect to \( <_\Delta \) and \( <_\sigma \) are called the **\( \Delta\)**-leader and **\( \sigma\)**-leader of \( A \), respectively. They are denoted, respectively, by \( u_A \) and \( v_A \).

If \( A \) is written as a polynomial in \( v_A \),

\[
A = l_d v_A^d + l_{d-1} v_A^{d-1} + \cdots + l_0
\]

\((l_d, l_{d-1}, \ldots, l_0 \text{ do not contain } v_A)\), then \( l_d \) is called the **initial** of \( A \); it is denoted by \( l_A \).

\[
\frac{\partial A}{\partial v_A} = dl_d v_A^{d-1} + (d - 1)l_{d-1} v_A^{d-2} + \cdots + l_1
\]

is called a **separant** of \( A \); it is denoted by \( S_A \).
If \( A, B \in K\{y_1, \ldots, y_n\} \), we say that \( A \) has lower rank than \( B \) and write \( \text{rk} \ A < \text{rk} \ B \) if either \( A \in K \), \( B \notin K \), or 
\[
(v_A, \deg_{v_A} A, \text{ord}_\Delta u_A) <_{\text{lex}} (v_B, \deg_{v_B} B, \text{ord}_\Delta u_B)
\]
where \( v_A \) and \( v_B \) are compared with respect to \( <_\sigma \). If the two vectors are equal (or \( A, B \in K \)), we say that \( A \) and \( B \) are of the same rank and write \( \text{rk} \ A = \text{rk} \ B \).

If \( A, B \in K\{y_1, \ldots, y_n\} \), then \( B \) is said to be reduced with respect to \( A \) if
(i) \( B \) does not contain terms \( \tau v_A \) such that \( \text{ord}_\Delta \tau > 0 \) and \( \text{ord}_\Delta (\tau u_A) \leq \text{ord}_\Delta u_B \).
(ii) If \( B \) contains a term \( \tau v_A \) where \( \text{ord}_\Delta \tau = 0 \), then either \( \text{ord}_\Delta u_B < \text{ord}_\Delta u_A \) or \( \text{ord}_\Delta u_A \leq \text{ord}_\Delta u_B \) and \( \deg_{\tau v_A} B < \deg_{v_A} A \).
If $B \in K\{y_1, \ldots, y_n\}$, then $B$ is said to be reduced with respect to a set $A \subseteq K\{y_1, \ldots, y_n\}$ if $B$ is reduced with respect to every element of $A$.

A set of $\Delta$-$\sigma$-polynomials $A$ in $K\{y_1, \ldots, y_n\}$ is called **autoreduced** if $A \cap K = \emptyset$ and every element of $A$ is reduced with respect to any other element of this set.

**Proposition 1**

Every autoreduced set of $\Delta$-$\sigma$-polynomials in the ring $K\{y_1, \ldots, y_n\}$ is finite.

In what follows we always list elements of an autoreduced set in the order of increasing rank.
Proposition 2

Let $A = \{A_1, \ldots, A_d\}$ be an autoreduced set in $K\{y_1, \ldots, y_s\}$ and let $I_k$ and $S_k$ denote the initial and separant of $A_k$, respectively. Let

$$I(A) = \{X \in K\{y_1, \ldots, y_n\} \mid X = 1 \text{ or } X \text{ is a product of finitely many elements of the form } \sigma^i(I_k) \text{ and } \sigma^j(S_k) \text{ where } i, j \in \mathbb{N}\}.$$

Then for any $\Delta$-polynomial $B$, there exist $B_0 \in K\{y_1, \ldots, y_n\}$ and $J \in I(A)$ such that $B_0$ is reduced with respect to $A$ and $JB \equiv B_0 \mod [A]$ (that is, $JB - B_0 \in [A]$).

The $\Delta$-polynomial $B_0$ is called the remainder of $B$ with respect to $A$. We also say that $B$ reduces to $B_0$ modulo $A$. 
If $\mathcal{A} = \{A_1, \ldots, A_p\}$, $\mathcal{B} = \{B_1, \ldots, B_q\}$ are two autoreduced sets, we say that $\mathcal{A}$ has lower rank than $\mathcal{B}$ if one of the following two cases holds:

1. There exists $k \in \mathbb{N}$ such that $k \leq \min\{p, q\}$, $\text{rk} A_i = \text{rk} B_i$ for $i = 1, \ldots, k - 1$ and $\text{rk} A_k < \text{rk} B_k$.
2. $p > q$ and $\text{rk} A_i = \text{rk} B_i$ for $i = 1, \ldots, q$.

If $p = q$ and $\text{rk} A_i = \text{rk} B_i$ for $i = 1, \ldots, p$, then $\text{rk} \mathcal{A} = \text{rk} \mathcal{B}$.

**Proposition 3**

In every nonempty family of autoreduced sets of $\Delta$-\(\sigma\)-polynomials there exists an autoreduced set of lowest rank. In particular, every ideal $I$ of $K\{y_1, \ldots, y_s\}$ contains an autoreduced set of lowest rank called a **characteristic** set of $I$. If $\mathcal{A}$ is a characteristic set of a $\Delta$-\(\sigma\)$-ideal I, then an element $B \in I$ is reduced with respect to $\mathcal{A}$ if and only if $B = 0$. 
Now we need some results about dimension polynomials of subsets of $\mathbb{N}^{m+1}$ ($m$ is a positive integer) treated as a Cartesian product $\mathbb{N}^m \times \mathbb{N}$ (we single out the last coordinate).

If $a = (a_1, \ldots, a_{m+1}) \in \mathbb{N}^{m+1}$, we set $\text{ord}_1 a = \sum_{i=1}^{m} a_i$ and $\text{ord}_2 a = a_{m+1}$. Furthermore, we treat $\mathbb{N}^{m+1}$ as a partially ordered set with respect to the product order $\leq_P$.

If $A \subseteq \mathbb{N}^{m+1}$, then $V_A$ will denote the set of all elements $v \in \mathbb{N}^{m+1}$ such that there is no $a \in A$ with $a \leq_P v$. Thus, $v = (v_1, \ldots, v_{m+1}) \in V_A$ if and only if for any element $(a_1, \ldots, a_{m+1}) \in A$, there exists $i \in \mathbb{N}$, $1 \leq i \leq m + 1$, such that $a_i > v_i$.

Furthermore, for any $r, s \in \mathbb{N}$, we set

$$A(r, s) = \{x = (x_1, \ldots, x_{m+1}) \in A \mid \text{ord}_1 x \leq r, \text{ord}_2 x \leq s\}.$$
**Theorem 2**

Let \( A \subseteq \mathbb{N}^{m+1} \). Then there exists a polynomial 
\( \omega_A(t_1, t_2) \in \mathbb{Q}[t_1, t_2] \) such that

(i) \( \omega_A(r, s) = \text{Card } V_A(r, s) \) for all sufficiently large \((r, s) \in \mathbb{N}^2\).

(ii) \( \deg_{t_1} \omega_A \leq m \) and \( \deg_{t_2} \omega_A \leq 1 \) (hence \( \deg \omega_A \leq m + 1 \)).

(iii) \( \deg \omega_A = m + 1 \) if and only if \( A = \emptyset \). In this case 
\( \omega_A(t_1, t_2) = (t_1 + m)(t_2 + 1). \)

(iv) \( \omega_A = 0 \) if and only if \((0, \ldots, 0) \in A. \)

\( \omega_A(t_1, t_2) \) is called the dimension polynomial of the set \( A \subseteq \mathbb{N}^{m+1} \) associated with the orders \( \text{ord}_1 \) and \( \text{ord}_2 \).

The proof of the theorem and a closed-form formula for 
\( \omega_A(t_1, t_2) \) can be found in [Kondrateva, M. V., Levin, A. B., Mikhalev, A. V., Pankratev, E. V. *Differential and Difference Dimension Polynomials*. Kluwer Acad. Publ., 1999.]
Let $K$ be a $\Delta-\sigma$-field, $R = K\{y_1, \ldots, y_n\}$, and $P$ a prime $\Delta-\sigma$-ideal in $R$. Let $P^*$ denote the reflexive closure of $P$ ($P^*$ is also a prime) and for every $r, s \in \mathbb{N}$, let $R_{rs} = K[\{\tau y_i \mid \tau \in T(r, s), 1 \leq i \leq n\}]$. (It is a polynomial ring over $K$ in indeterminates $\tau y_i$ such that $\text{ord}_\Delta \tau \leq r$ and $\text{ord}_\sigma \tau \leq s$.)

Let $P_{rs} = P \cap R_{rs}$, $P^*_{rs} = P^* \cap R_{rs}$, and let $L, L^*$, $L_{rs}$ and $L^*_{rs}$ denote the quotient fields of the integral domains $R/P$, $R/P^*$, $R_{rs}/P_{rs}$ and $R_{rs}/P^*_{rs}$, respectively.

If $\eta_i$ denotes the canonical image of $y_i$ in $R/P^*$, then $L^*$ is a $\Delta-\sigma$-field extension of $K$, $L^* = K\langle \eta_1, \ldots, \eta_n \rangle$, and $L^*_{rs} \cong K(\{\tau \eta_i \mid \tau \in T(r, s), 1 \leq i \leq n\})$. 
Theorem 3

With the above notation, let $\mathcal{A} = \{A_1, \ldots, A_p\}$ be a characteristic set of $P^*$ and for any $r, s \in \mathbb{N}$, let

$U'_{rs} = \{u \in TY \mid \ord_{\Delta} u \leq r, \ord_{\sigma} u \leq s \text{ and } u \text{ is not a transform of any } v_{A_i}\}$ and

$U''_{r,s} = \{u \in TY \mid \ord_{\Delta} u \leq r, \ord_{\sigma} u \leq s \text{ and there exist } A \in \mathcal{A} \text{ such that } u = \tau v_A \text{ and } \ord_{\Delta}(\tau u_A) > r\}.$

Then $U'_{rs}(\eta) \cup U''_{rs}(\eta)$ is a transcendence basis of $L^*_rs$ over $K$.

By Theorem 2, there exists $\phi^{(1)}(t_1, t_2) \in \mathbb{Q}[t_1, t_2]$ such that $\phi^{(1)}(r, s) = \text{Card } U'_{rs}$ for all sufficiently large $(r, s) \in \mathbb{N}^2$, $\deg_{t_1} \phi^{(1)} \leq m$, and $\deg_{t_2} \phi^{(1)} \leq 1$. Furthermore, $\text{Card } U''_{r,s}$ is expressed by a polynomial $\phi^{(2)}(t_1, t_2)$ which is an alternating sum of bivariate polynomials of the form $\binom{t_1 + m + a}{m}(t_2 + b)$ ($a, b \in \mathbb{Z}$).
It shows that there exists a polynomial $\phi_{P^*}(t_1, t_2) \in \mathbb{Q}[t_1, t_2]$ such that

(i) $\phi_{P^*}(r, s) = \text{tr. deg}_K L_{rs}^*$ for all sufficiently large $(r, s) \in \mathbb{N}^2$.

(ii) $\phi_{P^*}(t_1, t_2)$ is linear with respect to $t_2$ and $\deg_{t_1} \phi_{P^*} \leq m$; it is of the form

$$\phi_{P^*}(t_1, t_2) = \phi^{(1)}_{P^*}(t_1)t_2 + \phi^{(2)}_{P^*}(t_1)$$

where $\phi^{(1)}_{P^*}(t_1)$ and $\phi^{(2)}_{P^*}(t_1)$ are polynomials in one variable with rational coefficients that can be written as

$$\phi^{(1)}_{P^*}(t_1) = \sum_{i=0}^{m} a_i \binom{t_1 + i}{i} \quad \text{and} \quad \phi^{(2)}_{P^*}(t_1) = \sum_{i=0}^{m} b_i \binom{t_1 + i}{i}$$

with $a_i, b_i \in \mathbb{Z} \ (1 \leq i \leq m)$. 
Theorem 4

Let $K$ be a $\Delta$-$\sigma$-field, $R = K\{y_1, \ldots, y_n\}$ and $P$ a prime non-reflexive $\Delta$-$\sigma$-ideal in $R$. For every $r, s \in \mathbb{N}$, let $R_{rs} = K[\{\tau y_i \mid \tau \in T(r, s), 1 \leq i \leq n\}]$, $P_{rs} = P \cap R_{rs}$, and $L_{rs} = q. f. (R_{rs}/P_{rs})$. Then there exists a bivariate polynomial $\psi_P(t_1, t_2) \in \mathbb{Q}[t_1, t_2]$ such that

(i) $\psi_P(r, s) = \text{tr. deg}_K L_{rs}$ for all sufficiently large $(r, s) \in \mathbb{N}^2$.

(ii) The polynomial $\psi_P(t_1, t_2)$ is linear with respect to $t_2$ and $\deg_{t_1} \psi_P \leq m$, so it can be written as

$$\psi_P(t_1, t_2) = \psi_P^{(1)}(t_1)t_2 + \psi_P^{(2)}(t_1)$$

where $\psi_P^{(1)}(t_1)$ and $\psi_P^{(2)}(t_1)$ are polynomials in one variable with rational coefficients of degree at most $m$. 

In the case of a non-reflexive prime difference polynomial ideal \( P \) (when \( \Delta = \emptyset \)), this result was proved in


and


We will outline a proof based on the properties of characteristic sets. It will also give a method of computation of dimension polynomials associated with a non-reflexive prime \( \Delta-\sigma \)-polynomial ideal.
We start with the case $\Delta = \emptyset$ and use prefix $\sigma$- instead of $\Delta\sigma$-.
Let $A = \{A_1, \ldots, A_p\}$ be a characteristic set of $P^*$ (the reflexive closure of $P$), let $v_j$ denote the $\sigma$-leader of $A_j$ ($1 \leq j \leq p$), and let $\eta_i = y_i + P \in K\{y_1, \ldots, y_n\}/P$ ($1 \leq i \leq n$).
Let $L = \text{q.f.}(K\{y_1, \ldots, y_n\}/P) = K(\{\sigma^k\eta_i \mid k \in \mathbb{N}, 1 \leq i \leq n\})$
and $L_s = K(\{\sigma^k\eta_i \mid 0 \leq k \leq s, 1 \leq i \leq n\})$.
For every $j = 1, \ldots, p$, let $s_j$ be the smallest nonnegative integer such that $\sigma^{s_j}(A_j) \in P$. Furthermore, let

$$ V = \{v \in TY \mid v \neq \sigma^i v_j \text{ for any } i \in \mathbb{N}, 1 \leq j \leq p\}, $$
$$ V_r = \{v \in V \mid \text{ord}_\sigma v \leq r\} \ (r \in \mathbb{N}), \quad V(\eta) = \{v(\eta) \mid v \in V\}, $$
$$ W = \{\sigma^k v_j \mid 1 \leq j \leq p, 0 \leq k \leq s_j - 1\}, $$
and $$ W(\eta) = \{u(\eta) \mid u \in W\}. $$
It is easy to see that the set \( V(\eta) \) is algebraically independent over \( K \): if \( f(v_1(\eta), \ldots, v_k(\eta)) = 0 \) for some polynomial \( f \) and \( v_1, \ldots, v_k \in V \), then \( f(v_1, \ldots, v_k) \in P \subseteq P^* \) and \( f(v_1, \ldots, v_k) \) is reduced with respect to the characteristic set \( \mathcal{A} \), hence \( f = 0 \).

Furthermore, every element of the field \( L \) is algebraic over its subfield \( K(\,V(\eta) \cup W(\eta)) \). Let \( \{w_1, \ldots, w_q\} \) be a maximal subset of \( W \) such that the set \( \{w_1(\eta), \ldots, w_q(\eta)\} \) is algebraically independent over \( K(\,V(\eta)) \). Then \( V(\eta) \cup \{w_1(\eta), \ldots, w_q(\eta)\} \) is a transcendence basis of \( L/K \).

Since the set \( W \) is finite, there exists \( r_0 \in \mathbb{N} \) such that

(i) \( w_1, \ldots, w_q \in R_{r_0} = K[\{\sigma^k y_i \mid 0 \leq k \leq r_0, 1 \leq i \leq n\}] \);

(ii) \( r_0 \geq \max\{\text{ord}_\sigma v_j + s_j \mid 1 \leq j \leq p\} \);

(iii) Every element of \( W(\eta) \) is algebraic over the field \( K(\,V_{r_0}(\eta) \cup \{w_1(\eta), \ldots, w_q(\eta)\}) \).
Let $r \geq r_0$. $R_r = K[\{\sigma^k y_i \mid 1 \leq i \leq n, 0 \leq k \leq r\}]$, and $P_r = P \cap R_r$. Let $L_r$ denote the quotient field of the integral domain $R_r/P_r$ and $\zeta_i^{(r)} = y_i + P_r \in R_r/P_r \subseteq L_r \ (1 \leq i \leq n)$. Furthermore, let $\zeta^{(r)} = \{\zeta_1^{(r)}, \ldots, \zeta_n^{(r)}\}$, and $V_r(\zeta^{(r)}) = \{v(\zeta^{(r)}) \mid v \in V_r\}$. Then one can show that

$$B_r = V_r(\zeta^{(r)}) \bigcup \{w_1(\zeta^{(r)}), \ldots, w_q(\zeta^{(r)})\}$$

is a transcendence basis of $L_r$ over $K$.

This completes the proof of the theorem in the case $\Delta = \emptyset$ and also shows that $\psi_P(t) = \phi_{P^*}(t) + q$ where $q$ is a constant. As a consequence of this result we obtain that any strictly ascending chain of prime $\sigma$-ideals between $P$ and $P^*$ has length at most $q$ and that $K\{y_1, \ldots, y_n\}$ satisfies the ascending chain condition for prime (not necessarily reflexive) $\sigma$-ideals.
In order to complete the proof Theorem 4 in the case \( \text{Card} \, \Delta = m > 0 \), we treat \( L_{rs} \) as the subfield \( K(\{\theta\sigma^j\xi_i \mid \theta \in \Theta(r), 0 \leq j \leq s, 1 \leq i \leq n\}) \) of the differential \( (\Delta-) \) overfield \( K(\{\sigma^j\xi_i \mid 0 \leq j \leq s, 1 \leq i \leq n\})_\Delta \) of \( K \).

(Here \( \xi_i \) is the canonical image of \( y_i \) in the factor ring \( K\{\sigma^jy_i, 1 \leq j \leq s, 1 \leq i \leq n\}_\Delta / P \cap K\{\sigma^jy_i, 1 \leq j \leq s, 1 \leq i \leq n\}_\Delta \).

By the Kolchin’s theorem on differential dimension polynomial, for any \( s \in \mathbb{N} \), there exists a polynomial

\[
\chi_s(t) = \sum_{i=0}^{m} a_i(s) \binom{t + i}{i}
\]

\((a_i(s) \in \mathbb{Z})\) such that \( \chi_s(r) = \text{tr. deg}_K L_{rs} \) for all sufficiently large \( r \in \mathbb{N} \).
On the other hand, the first part of the proof (with the use of the finite set of \( \sigma \)-indeterminates \( \{ \Theta(r)y_i \mid \theta \in \Theta(r), 1 \leq i \leq n \} \) instead of \( \{y_1, \ldots, y_n\} \)) shows that

\[ \text{tr. deg}_K L_{rs} = \text{Card } V_{rs} + \lambda(r) \]

where

\[ V_{rs} = \{ u = \tau y_i \in TY \mid \tau \in T(r, s) \text{ and } u \neq \tau' v_j \text{ for any } \tau' \in T, 1 \leq j \leq p \}. \]

(\( v_j \) denotes the \( \sigma \)-leader of the element \( A_j \) of a characteristic set \( A = \{A_1, \ldots, A_p\} \) of the reflexive closure \( P^* \) of \( P \).)

Since the set \( W \) in the first part of the proof is finite and depends only on the \( \sigma \)-orders of terms of \( A_j, 1 \leq j \leq p \), the number of elements of the corresponding set in the general case depends only on \( r \); we have denoted it by \( \lambda(r) \).
By Theorem 2, there exist $r_0, s_0 \in \mathbb{N}$ and a bivariate numerical polynomial $\omega(t_1, t_2)$ such that $\omega(r, s) = \text{Card } V_{rs}$ for all $r \geq r_0$, $s \geq s_0$, $\deg_{t_1} \omega \leq m$ and $\deg_{t_2} \omega \leq 1$. Thus,

$$\text{tr. deg}_K L_{rs} = \omega(r, s) + \lambda(r)$$

for all $r \geq r_0$, $s \geq s_0$. At the same time, we have seen that

$$\text{tr. deg}_K L_{rs_0} = \chi_{s_0}(r) = \sum_{i=0}^{m} a_i(s_0) \binom{r+i}{i}$$

for all sufficiently large $r \in \mathbb{N}$ ($a_i(s_0) \in \mathbb{Z}$). It follows that $\lambda(r)$ is a polynomial of $r$ for all sufficiently large $r \in \mathbb{N}$, say, for all $r \geq r_1$. Therefore, for any $s \geq s_0$, $r \geq \max\{r_0, r_1\}$,

$$\text{tr. deg}_K L_{rs} = \omega(r, s) + \lambda(r)$$

is expressed as a bivariate numerical polynomial in $r$ and $s$. 
Example

Let $K$ be a $\Delta$-$\sigma$-field with two basic derivations, $\Delta = \{\delta_1, \delta_2\}$, and one basic endomorphism $\sigma$. Let $K\{y\}$ be the ring of $\Delta$-$\sigma$-polynomials in one $\Delta$-$\sigma$-indeterminate $y$ and $P$ a linear (and therefore prime) $\Delta$-$\sigma$-ideal of $K\{y\}$ generated by the $\Delta$-$\sigma$-polynomial $A = \sigma^2 y + \sigma \delta_1^2 y + \sigma \delta_2^2 y$ (that is, $P = [A]$). Then $P^* = [B]$, where $B = \sigma y + \delta_1^2 y + \delta_2^2 y$, and $\{B\}$ is a characteristic set of the $\Delta$-$\sigma$-ideal $P^*$. With the notation of the proof of Theorem 1, we have

$U'_rs = \{u \in TY \mid \ord_{\Delta} u \leq r, \ord_{\sigma} u \leq s$ and $u$ is not a multiple of $\sigma y\}$ and $U''_rs = \{u \in TY \mid \ord_{\Delta} u \leq r, \ord_{\sigma} u \leq s$ and there is $\tau \in T$ such that $u = \tau(\sigma y)$ and $\ord_{\Delta}(\tau \delta_1^2) > r\}$. 
Then \( \text{Card } U'_rs = \text{Card}\{\delta_i^j \delta_2^y \mid i + j \leq r\} = \binom{r+2}{2} \) and
\[
\text{Card } U''_rs = \text{Card}\{\sigma^i \delta_1^j \delta_2^k y \mid 1 \leq i \leq s, \, r - 2 < j + k \leq r\} = s \left( \binom{r+2}{2} - \binom{r+2-2}{2} \right) = (2r + 1)s.
\]
Since \( \sigma B \in P \), the proof of Theorem 4 shows that if \( \psi_P(t_1, t_2) \) is the \( \Delta-\sigma \)-dimension polynomial of \( P \), then
\[
\psi(r, s) = \text{Card } U'_rs + \text{Card } U''_rs + \text{Card}\{\sigma^i \delta_1^j y \mid i + j \leq r - 2\}
\]
for all sufficiently large \((r, s) \in \mathbb{N}^2\). It follows that
\[
\psi_P(t_1, t_2) = (2t_1 + 1)t_2 + \binom{t_1 + 2}{2} + \binom{t_1}{2}, \text{ that is}
\]
\[
\psi_P(t_1, t_2) = (2t_1 + 1)t_2 + t_1^2 + t_1 + 1.
\]
Thanks!