Constructive Arithmetics in Ore Localizations with Enough Commutativity

Johannes Hoffmann, Viktor Levandovskyy[1]

RWTH Aachen University, Germany

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Localization of commutative domains

Definition

A subset S of a ring R is called **multiplicative set** if

- $0 \notin S$,
- $1 \in S$ and
- S is **multiplicatively closed**, that is, $\forall s, t \in S : s \cdot t \in S$.

Notation: $[S] :=$ the smallest multiplicative superset of S.

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Theorem (Classical)

Let S be a multiplicative set in a commutative domain R. Then

$$S^{-1}R := \left\{ \frac{r}{s} \mid r \in R, s \in S \right\} = \{ s^{-1}r \mid (s, r) \in S \times R \}$$

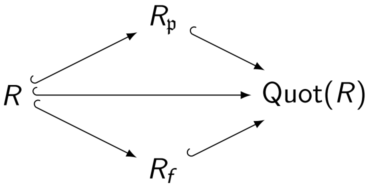
is a commutative domain (with the usual addition and multiplication of fractions).

Commutative examples

Classical localizations

Let R be a commutative domain.

- $\mathrm{Quot}(R) := \left\{ \frac{p}{q} \mid p, q \in R, q \neq 0 \right\} = (R \setminus \{0\})^{-1} R$
- $R_{\mathfrak{p}} := \left\{ \frac{p}{q} \mid p, q \in R, q \notin \mathfrak{p} \right\} = (R \setminus \mathfrak{p})^{-1} R$, $\mathfrak{p}$ prime ideal of R
- $R_f := \left\{ \frac{p}{f^k} \mid p \in R, k \in \mathbb{N}_0 \right\} = [f]^{-1} R$, $f \in R \setminus \{0\}$



The hierarchy of Ore localizations: localization of...

commutative domains

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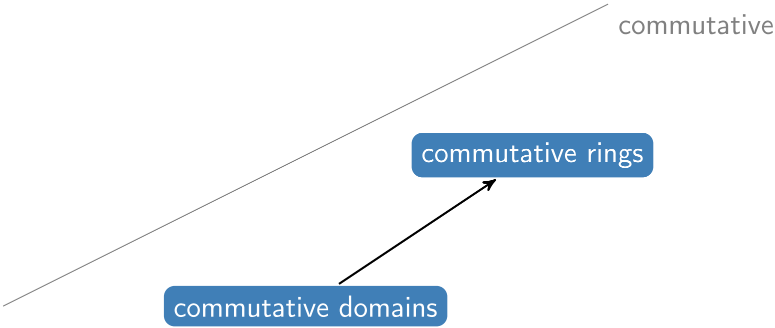

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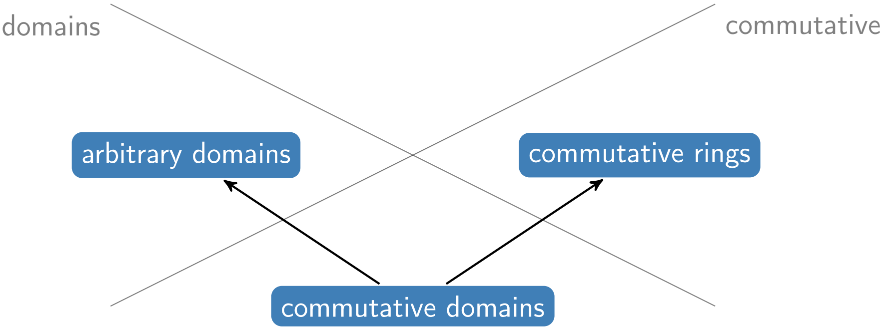

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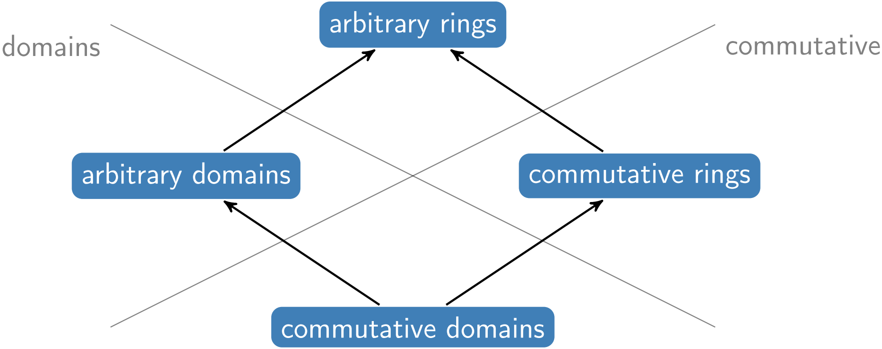

Left Ore sets, left denominator sets

Definition

Let S be a subset of a ring R.

- S satisfies the **left Ore condition** in R if

$$\forall\ s \in S, r \in R \qquad \exists\ \tilde{s} \in S, \tilde{r} \in R : \qquad \tilde{s}r = \tilde{r}s.$$

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- S is **left reversible** in R if

$$\forall\ s \in S, r \in R : \qquad rs = 0 \quad \Rightarrow \quad \exists \tilde{s} \in S : \tilde{s}r = 0.$$

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Consequences of the left Ore condition on S in R

- Finite collections of elements from S have common left multiples in S.
- Any right fraction rs^{-1} can be rewritten as a left fraction $\tilde{s}^{-1}\tilde{r}$.

Construction of the left Ore localization

Theorem (Ore, 1931)

- *The following is an equivalence relation on $S \times R$:*

$$(s_1, r_1) \sim (s_2, r_2) \;\Leftrightarrow\; \exists\; \tilde{s} \in S, \tilde{r} \in R :\; \tilde{s}s_2 = \tilde{r}s_1 \text{ and } \tilde{s}r_2 = \tilde{r}r_1$$

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- $S^{-1}R := (S \times R/\sim, +, \cdot)$ *is a ring via*

$$+ : S^{-1}R \times S^{-1}R \to S^{-1}R,\; (s_1, r_1) + (s_2, r_2) := (\tilde{s}s_1, \tilde{s}r_1 + \tilde{r}r_2),$$

where $\tilde{s} \in S$ *and* $\tilde{r} \in R$ *satisfy* $\tilde{s}s_1 = \tilde{r}s_2$*, and*

$$\cdot : S^{-1}R \times S^{-1}R \to S^{-1}R,\; (s_1, r_1) \cdot (s_2, r_2) := (\tilde{s}s_1, \tilde{r}r_2),$$

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Partial classification of Ore localizations

Definition

Let K be a field and R a K-algebra, S a left denominator set in R.

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Monoidal localization: S is generated as a monoid by countably many elements

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Geometric localization: Let $n \in \mathbb{N}$, $K[\underline{x}] := K[x_1, \ldots, x_n]$, J an ideal in $K[\underline{x}]$, $\mathfrak{p}$ a prime ideal in $K[\underline{x}]/J$ and $S = (K[\underline{x}]/J) \setminus \mathfrak{p}$

Example: $K[x]_{\mathfrak{p}}$

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Rational localization: $T \subseteq R$ is a K-subalgebra, $S = T \setminus \{0\}$

Special case: R is generated by a set X of variables and T is generated by a subset of $X \Rightarrow S^{-1}R$ is essential rational

Example: $(K[x] \setminus \{0\})^{-1}K[x, y] \cong K(x)[y]$

Previously on ISSAC'17

Setup: a left Ore set S in a (not necessarily commutative) domain R.

Goal: provide algorithms for basic arithmetic in $S^{-1}R$.

Restrictions: R is a G-algebra, S belongs to one of the types above[a].

Key problem: intersection of left ideal with submonoid

Result: library `olga.lib` for SINGULAR:PLURAL

Johannes Hoffmann and Viktor Levandovskyy.
A Constructive Approach to Arithmetics in Ore Localizations.
In *Proc. ISSAC'17*, pages 197–204. ACM Press, 2017.

[a] Note that further computability restrictions apply.

Addressing the key problem

The intersection problem

Let S be a left denominator set in a ring R and I a left ideal in R. The **intersection problem** is to decide whether $I \cap S = \emptyset$ and to compute an element contained in this intersection when the answer is negative.

Addressing the key problem

The intersection problem

Let S be a left denominator set in a ring R and I a left ideal in R. The **intersection problem** is to decide whether $I \cap S = \emptyset$ and to compute an element contained in this intersection when the answer is negative.

Recent result (Posur, 2018)

The intersection problem is a main ingredient for solving linear systems over localizations of commutative rings.

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Follow-up question

What can we do in the commutative setting?

Content

The intersection problem in polynomial algebras

Definition

Let K be a field, $n \in \mathbb{N}$, $K[\underline{x}] := K[x_1, \ldots, x_n]$ and J an ideal in $K[\underline{x}]$. Then we consider the **polynomial algebra** $R := K[\underline{x}]/J$.

The intersection problem in polynomial algebras

Definition

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What can we do?

Let I be an ideal of R and S a multiplicative set in R. Then we can solve the intersection problem $I \cap S$ in either of the following cases:

- $S^{-1}R$ is monoidal and S is finitely generated.
- $S^{-1}R$ is geometric.
- $S^{-1}R$ is essential rational.

We can also decide whether a multiplicative submonoid of R contains 0.

Toolbox

Setting

- K is a field, $K[\underline{x}] := K[x_1, \ldots, x_n]$ and $K[\underline{y}] := K[y_1, \ldots, y_m]$,
- $I = {}_{K[\underline{x}]}\langle h_1, \ldots, h_k \rangle$ and $J = {}_{K[\underline{y}]}\langle g_1, \ldots, g_l \rangle$ are ideals,
- $f_1, \ldots, f_n \in K[\underline{y}]$.

Consider the homomorphism

$$\varphi : K[\underline{x}]/I \to K[\underline{y}]/J, \quad x_i \mapsto f_i.$$

Computing the kernel of a polynomial algebra homomorphism

1. Let $H := {}_{K[\underline{x},\underline{y}]}\langle h_1, \ldots, h_k, g_1, \ldots, g_l, x_1 - f_1, \ldots, x_n - f_n \rangle$.
2. Compute $\ker(\varphi) = H \cap K[\underline{x}]$ by eliminating $y_1, \ldots, y_m$.

Observation

Computing kernels is equivalent to computing preimages of ideals.

Polynomial algebras I: essential rational

Setting

- K a field, $K[\underline{x}] := K[x_1, \ldots, x_n]$
- J an ideal in $K[\underline{x}]$, I an ideal in $R := K[\underline{x}]/J$
- $r \in \{1, \ldots, n\}$, $K[\underline{t}] := K[t_1, \ldots, t_r]$
- $\hat{S} = K[x_1 + J, \ldots, x_r + J] \subseteq R$, $S = \hat{S} \setminus \{0\}$

Essential rational intersection in polynomial algebras

1. Let $\varphi : K[\underline{t}] \to R,\ t_i \mapsto x_i$.
2. Compute the preimage $\varphi^{-1}(I) =: {}_{K[\underline{t}]}\langle w_1, \ldots, w_k \rangle$.
3. If $\varphi(w_i) \neq 0$ for some i return $\varphi(w_i)$.
4. Otherwise return 0.

Polynomial algebras II: geometric

Setting

- K a field, $K[\underline{x}] := K[x_1, \ldots, x_n]$
- J an ideal in $K[\underline{x}]$, I an ideal in $R := K[\underline{x}]/J$
- $\mathfrak{p}$ a prime ideal in R, $S = R \setminus \mathfrak{p}$

Geometric intersection in polynomial algebras

1. Let $\pi : K[\underline{x}] \twoheadrightarrow R$.
2. Compute the preimage $\mathfrak{q} := \pi^{-1}(\mathfrak{p})$.
3. If $\mathrm{NF}(h|\mathfrak{q}) \neq 0$ for some generator h of I return $h + J$.
4. Otherwise return 0.

Polynomial algebras III: finitely generated monoidal

Setting

- K a field, $K[\underline{x}] := K[x_1, \ldots, x_n]$
- J an ideal in $K[\underline{x}]$, I an ideal in $R := K[\underline{x}]/J$
- $f_1, \ldots, f_k \in K[\underline{x}]$, $K[\underline{t}] = K[t_1, \ldots, t_k]$, $S = [f_1 + J, \ldots, f_k + J]$

Decide whether $0 \in S$

1. Let $\psi : K[\underline{t}] \to R, \; t_i \mapsto f_i + J$.
2. Compute $\ker(\psi) \subseteq K[\underline{t}]$.
3. Compute the saturation $M := \ker(\psi) : \langle t_1 \cdot \ldots \cdot t_m \rangle^{\infty}$.
4. If $1 \in M$ return "yes", otherwise return "no".

Polynomial algebras III: finitely generated monoidal

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Finitely generated monoidal intersection

1. Let $\psi : K[\underline{t}] \to R, \ t_i \mapsto f_i + J$.
2. Compute the preimage $L := \psi^{-1}(I) \subseteq K[\underline{t}]$.
3. If $\psi(L) = \{0\}$ return 0.
4. Compute $\ker(\psi) \subseteq K[\underline{t}]$.
5. Let $\varphi : K[\underline{t}] \to K[\underline{t}, \underline{q}^{\pm 1}] := K[\underline{t}, q_1, q_1^{-1}, \ldots q_k, q_k^{-1}], \ t_i \mapsto q_i t_i$.
6. Compute a reduced GB of the monomial ideal $W := {}_{K[\underline{t},\underline{q}^{\pm 1}]}\langle L \rangle \cap K[\underline{t}]$.
7. If $\mathrm{NF}(w | \ker(\psi)) \neq 0$ for some monomial gen. w of W return $w + J$.
8. Otherwise return 0.

Content

The closure problem

Definition

Let S be a left Ore set in a ring R and M a left R-module. The **left Ore localization** of M at S is $S^{-1}M := S^{-1}R \otimes_R M$. The homomorphism

$$\varepsilon = \varepsilon_{S,R,M} : M \to S^{-1}M, \quad m \mapsto (1, m) := (1, 1) \otimes m$$

is called the **localization map** of M with respect to S.

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Given a left R-submodule P of M, find the **local closure** or ***S*-closure**

$$P^S := \varepsilon^{-1}(S^{-1}P).$$

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Lemma

$$P^S = \{m \in M \mid \exists s \in S : sm \in P\} =: \mathrm{LSat}_S^M(P).$$

Example: symbolic powers of prime ideals

Observation

Powers of prime ideals need not be primary.

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Powers of prime ideals need not be primary.

Definition

Let $\mathfrak{p}$ be a prime ideal in a commutative ring R.
The ideal

$$\mathfrak{p}^{(k)} := \left\{ f \in R \mid \exists s \in R \setminus \mathfrak{p} : sf \in \mathfrak{p}^k \right\}$$

is called the ***k*-th symbolic power** of $\mathfrak{p}$.

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Powers of prime ideals need not be primary.

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Let $\mathfrak{p}$ be a prime ideal in a commutative ring R and $S = R \setminus \mathfrak{p}$.
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Lemma

The ideal $\mathfrak{p}^{(k)}$ is the smallest $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^k$.

Commutative decomposition closure: preparation

Lemma

Let S be a multiplicative set in a commutative ring R and $\mathfrak{q}$ a primary ideal of R. Then

$$\mathfrak{q}^S = \begin{cases} R, & \text{if } \mathfrak{q} \cap S \neq \emptyset, \\ \mathfrak{q}, & \text{if } \mathfrak{q} \cap S = \emptyset. \end{cases}$$

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Lemma

Let S be a left Ore set in a ring R, M a left R-module and $P_1, \ldots, P_k$ some left R-submodules of M. Then

$$\left(\bigcap_{j=1}^{k} P_j \right)^S = \bigcap_{j=1}^{k} P_j^S .$$

Commutative decomposition closure: algorithm

Let S be a multiplicative set in a commutative ring R and I an ideal in R.

Commutative decomposition closure

1. Obtain a decomposition of I into primary ideals: $I = \bigcap_{j=1}^{k} \mathfrak{q}_j$.
2. For each j set $\tilde{\mathfrak{q}}_j := \begin{cases} \mathfrak{q}_j, & \text{if } \mathfrak{q}_j \cap S = \emptyset, \\ R, & \text{if } \mathfrak{q}_j \cap S \neq \emptyset. \end{cases}$
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Observation

We need to solve several intersection problems with primary ideals.

G-algebras (PBW algebras, algebras of solvable type)

Definition

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The K-algebra

$$A := K$$

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For a field K, $n \in \mathbb{N}$ and $1 \leq i < j \leq n$ consider the constants $c_{i,j} \in K \setminus \{0\}$ and polynomials $d_{i,j} \in K[x_1, \ldots, x_n]$. The K-algebra

$$A := K\langle x_1, \ldots, x_n \mid \{x_j x_i = c_{i,j} x_i x_j + d_{i,j} : 1 \leq i < j \leq n\}\rangle$$

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- there exists a monomial total well-ordering $<$ on $K[x_1, \ldots, x_n]$ such that for any $1 \leq i < j \leq n$ either $d_{i,j} = 0$ or the leading monomial of $d_{i,j}$ with respect to $<$ is smaller than $x_i x_j$, and

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- $\{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}_0\}$ is a K-basis of A.

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Remark

- G-algebras are Noetherian domains.
- There exists a Gröbner basis theory for G-algebras plus implementation (most extensive in SINGULAR:PLURAL).

Examples of G-algebras

- Weyl algebras ($K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \mid \forall i : \partial_i x_i = x_i \partial_i + 1\rangle$)
- Shift algebras ($K\langle x_1, \dots, x_n, s_1, \dots, s_n \mid \forall i : s_i x_i = (x_i + 1) s_i\rangle$)
- q-Weyl algebras ($K\langle \underline{x}, \underline{\partial} \mid \forall i \exists q_i \in K^* : \partial_i x_i = q_i x_i \partial_i + 1\rangle$)
- q-Shift algebras ($K\langle \underline{x}, \underline{s} \mid \forall i \exists q_i \in K^* : s_i x_i = q_i x_i s_i\rangle$)
- Integration algebras ($K\langle \underline{x}, \underline{I} \mid \forall i : I_i x_i = x_i I_i + I_i^2\rangle$)
- Universal enveloping algebras of finite-dimensional Lie algebras
- Many quantum groups
- Tensor products over K of G-algebras
- ...

Recent result (Heinle, L., 2012-2017)

G-algebras are finite factorization domains. Factorization in G-algebras is algorithmic and implemented in `ncfactor.lib` in SINGULAR:PLURAL.

Central saturation

Definition

Let R be a ring, $q \in Z(R)$, $k \in \mathbb{N}$ and I a left R-submodule of R^k.

- The **central quotient** of I by q is
$$I : q := \left\{ f \in R^k \mid qf = fq \in I \right\}.$$
- The **central saturation** of I by q is
$$I : q^\infty := \bigcup_{j \in \mathbb{N}_0} (I : q^j) = \left\{ f \in R^k \mid \exists n \in \mathbb{N}_0 : q^n f \in I \right\}.$$
- The **central saturation index** of I by q is the smallest $n \in \mathbb{N}_0 \cup \{\infty\}$ such that $(I : q^n) = (I : q^\infty)$, denoted by $\operatorname{Satindex}(I, q)$.

Central monoidal closure

Task

Let A be a G-algebra and I a left A-submodule of A^k.
Let $g_1, \ldots, g_k \in Z(A)$, then $S := [g_1, \ldots, g_k]$ is a left Ore set in A.
Goal: compute I^S.

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Solution

1. Set $g := \prod_{j=1}^{k} g_j$, then $T := [g]$ is a left Ore set in A.
2. Compute $I : g^\infty = I^S$ via central saturation.

Central essential rational closure: problem

Task

Let K be a field and consider a G-algebra

$$A := K\langle \underbrace{x_1, \ldots, x_n}_{\text{central}}, y_1, \ldots, y_m \mid \text{Relations} \rangle$$

such that $\underline{x}$ generates a sub-G-algebra $B \subseteq Z(A)$ of A.
Then $S := B \setminus \{0\}$ is a left Ore set in A and B.
Let I be a left R-submodule of A^r.
Goal: compute I^S.

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Example

For intuition: think of $\mathcal{D}[\underline{s}] = K[s_1, \ldots, s_n] \otimes_K A_{\frac{m}{2}}(K)$, where $A_{\frac{m}{2}}(K)$ is a Weyl algebra generated by $x_1, \ldots, x_{\frac{m}{2}}, \partial_1, \ldots, \partial_{\frac{m}{2}}$.

Central essential rational closure: algorithm

For an ordering $\leq$ on A, denote by $\leq^{\mathrm{POT}}$ the position-over-term ordering on A^k based on $\leq$.
Let $\varepsilon : A^r \to S^{-1}A^r,\ m \mapsto (1, m)$.

Algorithm

1. Let $\leq = (\leq_1, \leq_2)$ be an antiblock-ordering on A and $\preceq = \leq^{\mathrm{POT}}$.
2. Compute a Gröbner basis G of I with respect to $\preceq$.
3. Let $h := \sqrt{\prod_{g \in G} \mathrm{lc}_{\preceq_2}(\varepsilon(g))} \in K[\underline{x}] \setminus \{0\}$, where $\preceq_2 = {\leq_2}^{\mathrm{POT}}$.
4. Compute $k := \mathrm{Satindex}(I : h)$.
5. Return $(I : h^k)$.

Algorithmic conclusion

We can solve the intersection problem $I \cap S$...

- ISSAC'17: in G-algebras R, where $S^{-1}R$ is...
  - finitely generated monoidal with commuting generators (if $I \cap S \neq \emptyset$),
  - geometric in a Weyl algebra, or
  - essential rational (if elimination is possible).
- ISSAC'18: in polynomial algebras $R = K[\underline{x}]/J$, where $S^{-1}R$ is
  - finitely generated monoidal with commuting generators,
  - geometric, or
  - essential rational.

We can solve the closure problem I^S...

- in commutative rings, if I is effectively decomposable and we can solve the intersection problem $\mathfrak{q} \cap S$ for every involved primary ideal
- in G-algebras, where $S^{-1}R$ is...
  - finitely generated monoidal with central generators, or
  - central essential rational.

The latest version of SINGULAR is available at:
http://www.singular.uni-kl.de

The latest version of olga.lib is available at:
http://www.math.rwth-aachen.de/~Johannes.Hoffmann/singular.html