On Continuity of the Roots of a Parametric Zero Dimensional Multivariate Polynomial Ideal

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The roots of an unary polynomial are continuous in $\mathbb{C}^m$.

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Given parameters $\bar{A}{=}A_1,\ldots,A_m$ and variables $\bar{X}{=}X_1,\ldots,X_n$,

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the roots of $F \subset \mathbb{Q}[\bar{A},\bar{X}]$ are continuous in $\mathbb{C}^m$???

Ex. Let V be the variety of $\{X_1X_2+AX_2-1, X_1^2+AX_2-1\}$.

$$V=\begin{cases}\{(\frac{-A\pm\sqrt{4+A^2}}{2},\frac{-A\pm\sqrt{4+A^2}}{2}),(0,\frac{1}{A})\} & (A\neq 0)\\ \{(\pm 1,\pm 1)\} & (A=0)\end{cases}.$$

- We can not treat $(0,\frac{1}{A})$ in the case $A=0$, and
- $\#(V)=\begin{cases}3 & (A\neq 0)\\ 2 & (A=0)\end{cases}$ counting the multiplicities. □

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In this talk, each root is counted with multiplicity.

To treat the continuity of the roots of $F \subset \mathbb{Q}[\bar{A}, \bar{X}]$ in $\mathcal{S} \subset \mathbb{C}^m$, we have to assume that

$$\forall \bar{a} \in \mathcal{S} \quad \#(V_{\mathbb{C}}(F(\bar{a}))) \text{ has the same cardinality},$$

where $V_{\mathbb{C}}(F(\bar{a}))$ is the variety of $F(\bar{a}) = \{f(\bar{a}, \bar{X}) : f \in F\}$ in $\mathbb{C}$.

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For that, we use a Comprehensive Gröbner System (CGS)

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which is an ideal tool for handling parametric ideals.

By the result, we improve a quantifier elimination method.

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Definition

Def. 1. Let $\mathcal{P} \subset \mathbb{C}^m$ and $\mathcal{S}_1, \ldots, \mathcal{S}_t \subset \mathcal{P}$ and $S = \{\mathcal{S}_1, \ldots, \mathcal{S}_t\}$.

S is a partition of $\mathcal{P} \overset{\text{def}}{\Leftrightarrow}$ the properties **1** and **2** are satisfied:

1. $\cup_{i=1}^{t} \mathcal{S}_i = \mathcal{P}$ and $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$ $(i \neq j)$.
2. $\forall i \; \exists P, Q \subset \mathbb{Q}[\bar{A}] \; [\mathcal{S}_i = V_{\mathbb{C}}(P) \backslash V_{\mathbb{C}}(Q)]$. □

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Fix a term order on the set of terms of $\bar{X}$.

HC_g denotes the head coefficient of $g \in \mathbb{Q}[\bar{A}, \bar{X}]$.

Rem. $\mathrm{HC}_g \in \mathbb{Q}[\bar{A}]$ for $g \in \mathbb{Q}[\bar{A}, \bar{X}]$. □

Definition

Def. 2. Let $\mathcal{S}_1,\ldots,\mathcal{S}_t \subset \mathcal{P} \subset \mathbb{C}^m$ and $F, G_1,\ldots,G_t \subset \mathbb{Q}[\bar{A},\bar{X}]$.

$\{(\mathcal{S}_1,G_1),\ldots,(\mathcal{S}_t,G_t)\}$ is a CGS of $\langle F\rangle$ on $\mathcal{P} \overset{\text{def}}{\Leftrightarrow}$ for each $\bar{a} \in \mathcal{S}_i$

1. $G_i(\bar{a})$ is a Gröbner Basis (GB) of $\langle F(\bar{a})\rangle$,
2. $\forall g \in G_i\ (\mathrm{HC}_g(\bar{a}) \neq 0)$, $\{\mathcal{S}_1,\ldots,\mathcal{S}_t\}$ is a partition of $\mathcal{P}$. □

Ex. Let $f_1 = AX_1 + X_2^2 - 1$, $f_2 = X_2^3 - 1$. Then we obtain that

$$\{(V_{\mathbb{C}}(0)\backslash V_{\mathbb{C}}(A),\{f_1,f_2\}),(V_{\mathbb{C}}(A),\{X_2-1\})\}$$

is a CGS of $\langle f_1, f_2\rangle$ w.r.t. $X_1 \succ_{\mathrm{lex}} X_2$ on $\mathbb{C}$. □

Well-known Fact

$\dim(L)$ denotes the dimension of a linear space L.

Rem. 3. Let $\mathcal{G}$ be a CGS. $\forall(\mathcal{S}, G) \in \mathcal{G}$ satisfies that for $\bar{a} \in \mathcal{S}$

- the set of leading terms of $G(\bar{a})$ is invariant, so
- $\dim(\mathbb{C}[\bar{X}]/\langle G(\bar{a})\rangle)$ is invariant, so
- $\dim(\mathbb{C}[\bar{X}]/\langle G(\bar{a})\rangle)$ is finite $\Rightarrow \#(V_{\mathbb{C}}(G(\bar{a})))$ is invariant, so
- $\langle G(\bar{a})\rangle$ is zero-dimensional $\Rightarrow \#(V_{\mathbb{C}}(G(\bar{a})))$ is invariant.

We discuss the continuity of roots of G in $\mathcal{S} \subset \mathbb{C}^m$ such that

$$\langle G(\bar{a})\rangle \text{ is zero-dimensional for } \bar{a} \in \mathcal{S}. \qquad \square$$

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Definition

Def. 4. For $\bar{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and $\bar{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n$,

$$d(\bar{a}, \bar{b}) \stackrel{\text{def}}{=} \max(|a_i - b_i| : i). \qquad \square$$

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□

$\mathfrak{S}_l$ denotes the symmetric group of degree l.

Def. 5. For $\alpha = (\alpha_1, \ldots, \alpha_l), \beta = (\beta_1, \ldots, \beta_l) \in (\mathbb{C}^n)^l$, we define

$$\alpha \sim \beta \stackrel{\text{def}}{\Leftrightarrow} \exists \sigma \in \mathfrak{S}_l \forall i \in \{1, \ldots, l\} \, [\alpha_i = \beta_{\sigma(i)}],$$

an l-size multiset $\alpha_M \stackrel{\text{def}}{=} \{\beta \in (\mathbb{C}^n)^l : \alpha \sim \beta\}$, $(\mathbb{C}^n)_M \stackrel{\text{def}}{=} \{\alpha_M : \alpha \in \mathbb{C}^n\}$.

□

Ex. For $a \neq b$, $(a,a,b)_M = (a,b,a)_M$, but $(a,a,b)_M \neq (a,b,b)_M$.

□

Definition

Def. 6. For $\alpha_M = (\alpha_1, \ldots, \alpha_l)_M, \;\; \beta_M = (\beta_1, \ldots, \beta_l)_M \in (\mathbb{C}^n)_M$,

$$D(\alpha_M, \beta_M) \stackrel{\text{def}}{=} \min(\max(d(\alpha_i, \beta_{\sigma(i)}) : i) : \sigma \in \mathfrak{S}_l). \qquad \square$$

Ex. $D((1,3,4)_M, (2,3,5)_M) = 1.$ $\square$

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Def. 7. Let $\mathcal{S} \subset \mathbb{C}^m$, and $I \lhd \mathbb{Q}[\bar{A}, \bar{X}]$ be an ideal such that

$$\exists l \in \mathbb{N} \forall \bar{a} \in \mathcal{S}\ [l = \dim(\mathbb{C}[\bar{X}]/I(\bar{a}))].$$

Let $\theta(\bar{a})$ be the l-size multiset of the roots of $I(\bar{a})$ for $\bar{a} \in \mathcal{S}$.

Then, I has continuous roots on $\mathcal{S} \stackrel{\text{def}}{\Leftrightarrow} \forall \bar{a} \in \mathcal{S}\ \forall \epsilon > 0\ \exists \delta > 0$

$$\forall \bar{b} \in \mathcal{S}\ [d(\bar{a}, \bar{b}) < \delta \Rightarrow D(\theta(\bar{a}), \theta(\bar{b})) < \epsilon]. \qquad \square$$

Main Theorem

Th. 8. Let $\mathcal{G}$ be a CGS of $I \lhd \mathbb{Q}[\bar{A},\bar{X}]$, and $(\mathcal{S},G) \in \mathcal{G}$ s.t.

$$\forall \bar{a} \in \mathcal{S}\ \langle G(\bar{a})\rangle \text{ is zero-dimensional}.$$

Then I has continuous roots on $\mathcal{S}$. □

Ex. Let $I = \langle X_1X_2+AX_2-1, X_1^2+AX_2-1\rangle$ with a parameter A,

$$\mathcal{S}_1 = V_{\mathbb{C}}(0)\setminus V_{\mathbb{C}}(A),\quad \mathcal{S}_2 = V_{\mathbb{C}}(A).$$

I has continuous roots on $\mathcal{S}_1$ and $\mathcal{S}_2$, since we get a CGS

$$\{(\mathcal{S}_1,\{g_1,g_2,g_3\}),(\mathcal{S}_2,\{X_1^2-1,X_2-X_1\})\},$$

where $g_1 = X_1^3+AX_1^2+X_1$, $g_2 = AX_2+X_1^2-1$, $g_3 = X_1X_2-X_1^2$. □

Main Theorem: Proof (Outline)

Taking an arbitrary $\bar{a} \in \mathcal{S}$, we introduce $l = \dim(\mathbb{C}[\bar{X}]/I(\bar{a}))$,

$$\begin{aligned} \alpha_i &\in V_{\mathbb{C}}(I(\bar{a})) \text{ s.t. } \alpha_i = (\alpha_i^{(1)}, \ldots, \alpha_i^{(n)}), \\ \theta &: \mathcal{S} \to (\mathbb{C}^n)_M \,;\, \bar{a} \mapsto (\alpha_1, \ldots, \alpha_l)_M, \\ \pi_j &: \mathcal{S} \to \mathbb{C}_M \,;\, \bar{a} \mapsto (\alpha_1^{(j)}, \ldots, \alpha_l^{(j)})_M. \end{aligned}$$

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We consider finite $B(\bar{a}) = \{\alpha_1^{(1)}, \ldots, \alpha_l^{(1)}\} \times \cdots \times \{\alpha_1^{(n)}, \ldots, \alpha_l^{(n)}\}$.

Main Theorem: Proof (Outline): $\alpha_i \neq \alpha_k$

Let $h(\bar{X}) = \sum_{j=1}^{n} c_j X_j \in \mathbb{Q}[\bar{X}]$ s.t. $\forall \alpha \neq \alpha' \in B(\bar{a})[h(\alpha) \neq h(\alpha')]$,

$$\epsilon_0 < \min(|h(\alpha) - h(\alpha')| : \alpha \neq \alpha' \in B(\bar{a})),$$

$\bar{b} \in \mathcal{S}$ and $\delta > 0$ s.t. $d(\bar{a}, \bar{b}) < \delta \to D(\pi_j(\bar{a}), \pi_j(\bar{b}) < \epsilon_0$.

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3 $\forall \beta_i \in V_{\mathbb{C}}(I(\bar{b}))\ \exists \alpha' \in B(\bar{a})\ [d(\alpha', \beta_i) < \epsilon_0]$. (∵ **1**)

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4 If α' is not a root of $I(\bar{a})$, we get a contradiction. ($\because$ a property of h and ϵ_0)

5 If the multiplicity μ_i of α_i satisfies the property such that

$$\mu_i \neq \#(V_{\mathbb{C}}(I(\bar{b})) \cap \{z \in \mathbb{C}^n : d(\alpha_i, z) < \epsilon_0\}),$$

we get a contradiction by a property of h and ϵ_0. □

Contents

ISSAC 2015

Main parts of any CGS-QE methods eliminate $\exists \bar{X} \in \mathbb{R}^n$ from

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where $F, H \subset \mathbb{Q}[\bar{A}, \bar{X}]$ and a quantifier free formula ϕ satisfy

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For that, we need the parametric saturation $\langle F \rangle : (\prod_{h \in H} h)^\infty$.

But the computation of $\langle F \rangle : (\prod_{h \in H} h)^\infty$ is heavy.

We need some device for its efficient computation.

To appear in MCS (Saturation)

Def. 9. Let $I \lhd \mathbb{R}[\bar{X}]$ be zero-dimensional, and $h \in \mathbb{R}[\bar{X}]$, and

m_p be a map $\mathbb{R}[\bar{X}]/I \to \mathbb{R}[\bar{X}]/I;\, a \mapsto ap\ (p \in \mathbb{R}[\bar{X}]/I)$,

$\{v_1, \ldots, v_l\}$ be a basis of $\mathbb{R}[\bar{X}]/I$.

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Lem. 10. Let χ_p^I be the characteristic polynomials of M_p^I s.t.

$$\chi_p^I(Y) = Y^l + C_{p,1}Y^{l-1} + \cdots C_{p,r}Y^{r-1} \in \mathbb{R}[Y]$$

for $p \in \mathbb{R}[\bar{X}]$. Then, for $h \in \mathbb{R}[\bar{X}]$, $C_{h,r} = C_{1,r} \prod_{\alpha \in V_{\mathbb{C}}(I)} h(\alpha)$.

If $C_{1,r} \neq 0$, $C_{h,r}/C_{1,r} \neq 0 \Leftrightarrow I = I : h^\infty$. □

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$N_{1,i}, N_{h,i}$ be the numerator of $C_{1,i}, C_{h,i}/C_{1,i}$, respectively.

We construct $\mathcal{S}_{d-r} = \mathcal{S} \cap (V_{\mathbb{C}}(0) \setminus V_{\mathbb{C}}(N_{h,d-r}))$ and

$$\mathcal{S}_i = \mathcal{S} \cap (V_{\mathbb{C}}(N_{1,d-r}, \ldots, N_{1,i+1}) \setminus V_{\mathbb{C}}(N_{h,i})).$$

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We construct $\mathcal{S}_{d-r} = \mathcal{S} \cap (V_{\mathbb{C}}(0) \backslash V_{\mathbb{C}}(N_{h,d-r}))$ and

$$\mathcal{S}_i = \mathcal{S} \cap (V_{\mathbb{C}}(N_{1,d-r}, \ldots, N_{1,i+1}) \backslash V_{\mathbb{C}}(N_{h,i})).$$

Lem. 10 implies $\forall \bar{a} \in \cup_{i=1}^{d-r} \mathcal{S}_i \cap \mathbb{R}^m \; [I(\bar{a}) = I(\bar{a}) : h(\bar{a}, \bar{X})^{\infty}]$. □

To appear in MCS (Parametric Saturation)

Pro. 11. Let $h \in \mathbb{Q}[\bar{A},\bar{X}]$, $\mathcal{G}$ be a CGS of a parametric ideal I,

$(\mathcal{S},G) \in \mathcal{G}$ s.t. $\langle G(\bar{a})\rangle$ is zero-dimensional for $\bar{a} \in \mathcal{S}$,

$\chi_p^{\langle G\rangle}(Y) = Y^l + \sum_{i=1}^{d-r} C_{p,i} Y^{l-i} \mathbb{Q}(\bar{A})[Y]$ for $p \in \mathbb{Q}[\bar{A},\bar{X}]$,

$N_{1,i}, N_{h,i}$ be the numerator of $C_{1,i}, C_{h,i}/C_{1,i}$, respectively.

We construct $\mathcal{S}_{d-r} = \mathcal{S} \cap (V_{\mathbb{C}}(0) \backslash V_{\mathbb{C}}(N_{h,d-r}))$ and

$$\mathcal{S}_i = \mathcal{S} \cap (V_{\mathbb{C}}(N_{1,d-r},\ldots,N_{1,i+1}) \backslash V_{\mathbb{C}}(N_{h,i})).$$

Lem. 10 implies $\forall \bar{a} \in \cup_{i=1}^{d-r} \mathcal{S}_i \cap \mathbb{R}^m\ [I(\bar{a}) = I(\bar{a}) : h(\bar{a},\bar{X})^\infty]$. □

Rem. We need not to compute $I : h^\infty$ on $\cup_{i=1}^{d-r} \mathcal{S}_i \cap \mathbb{R}^m$.

Improvement

Pro. 12. Let $h \in \mathbb{Q}[\bar{A},\bar{X}]$, $\mathcal{G}$ be a CGS of a parametric ideal I,

$(\mathcal{S},G) \in \mathcal{G}$ s.t. $\langle G(\bar{a})\rangle$ is zero-dimensional for $\bar{a} \in \mathcal{S}$,

$\chi_p^{\langle G\rangle}(Y) = Y^l + \sum_{i=1}^{d-r} C_{p,i} Y^{l-i} \mathbb{Q}(\bar{A})[Y]$ for $p \in \mathbb{Q}[\bar{A},\bar{X}]$,

$N_{h,d-r}$ the numerator of $C_{h,d-r}/C_{1,d-r}$, $\mathcal{T} = V_{\mathbb{C}}(0) \setminus V_{\mathbb{C}}(N_{h,d-r})$.

Improvement

Pro. 12. Let $h \in \mathbb{Q}[\bar{A},\bar{X}]$, $\mathcal{G}$ be a CGS of a parametric ideal I,

$(\mathcal{S},G) \in \mathcal{G}$ s.t. $\langle G(\bar{a})\rangle$ is zero-dimensional for $\bar{a} \in \mathcal{S}$,

$\chi_p^{\langle G\rangle}(Y) = Y^l + \sum_{i=1}^{d-r} C_{p,i} Y^{l-i} \mathbb{Q}(\bar{A})[Y]$ for $p \in \mathbb{Q}[\bar{A},\bar{X}]$,

$N_{h,d-r}$ the numerator of $C_{h,d-r}/C_{1,d-r}$, $\mathcal{T} = V_{\mathbb{C}}(0) \setminus V_{\mathbb{C}}(N_{h,d-r})$.

Assuming that $\forall \bar{a} \in \mathcal{S} \cap \mathbb{R}^m$ is non-isolated, we obtain that

$$\forall \bar{a} \in \mathcal{S} \cap \mathcal{T} \cap \mathbb{R}^m [I(\bar{a}) = I(\bar{a}) : h(\bar{a},\bar{X})^\infty]$$

by **Th. 8**. □

Improvement

Pro. 12. Let $h \in \mathbb{Q}[\bar{A},\bar{X}]$, $\mathcal{G}$ be a CGS of a parametric ideal I,

$(\mathcal{S},G) \in \mathcal{G}$ s.t. $\langle G(\bar{a})\rangle$ is zero-dimensional for $\bar{a} \in \mathcal{S}$,

$\chi_p^{\langle G\rangle}(Y) = Y^l + \sum_{i=1}^{d-r} C_{p,i} Y^{l-i} \mathbb{Q}(\bar{A})[Y]$ for $p \in \mathbb{Q}[\bar{A},\bar{X}]$,

$N_{h,d-r}$ the numerator of $C_{h,d-r}/C_{1,d-r}$, $\mathcal{T} = V_{\mathbb{C}}(0) \setminus V_{\mathbb{C}}(N_{h,d-r})$.

Assuming that $\forall \bar{a} \in \mathcal{S} \cap \mathbb{R}^m$ is non-isolated, we obtain that

$$\forall \bar{a} \in \mathcal{S} \cap \mathcal{T} \cap \mathbb{R}^m [I(\bar{a}) = I(\bar{a}) : h(\bar{a},\bar{X})^{\infty}]$$

by **Th. 8**. □

Rem. We need not to compute $I : h^{\infty}$ on $\mathcal{S} \cap \mathcal{T} \cap \mathbb{R}^m$.

Improvement

Let $f = X^2 + AX + B$.

Ex. Consider $\exists X[f = 0 \wedge X > 0]$. Let $I = \langle f \rangle$. Computing

$$M_1^I = \begin{pmatrix} 2 & -A \\ -A & A^2 - 2B \end{pmatrix}, \quad M_X^I = \begin{pmatrix} -A & A^2 - 2B \\ A^2 - 2B & -A^3 + 3AB \end{pmatrix},$$

we obtain theirs characteristic polynomials χ_1^I, χ_X^I of the form

$$\begin{aligned} \chi_1^I(Y) &= Y^2 + (-A^2 + 2B - 2) + A^2 - 4B, \\ \chi_X^I(Y) &= Y^2 + (A^3 + A - 3AB) + B(A^2 - 4B). \end{aligned}$$

For $(a,b) \in V_{\mathbb{C}}(0) \backslash V_{\mathbb{C}}(B) \cap \mathbb{R}^2$, $I(a,b) = I(a,b) : X^\infty$. $\square$

Benchmark Data

Our group develops a CGS-QE package on Maple.

- ‘new’ denotes our new package worked on Maple.
- ‘old’ denotes our old package worked on Maple.
- ‘syn’ denotes SyNRAC (Fujitsu Lab.) worked on Maple.
- ‘rc’ denotes RegularChains worked on Maple.
- ‘red’ denotes Reduce worked on Mathematica.
- ‘rl’ denotes RedLog worked on Reduce.
- ‘qep’ denotes QEPCAD.

Benchmark Data

Input $\#(*)$ denotes the number of $*$.

I1	$\#(\exists) = 4,$	$\#(=) = 6,$	$\#(\neq) = 7,$	$\#(>) = 1.$
I2	$\#(\exists) = 2,$	$\#(=) = 1,$	$\#(\neq) = 1,$	$\#(>) = 0.$
I5	$\#(\exists) = 2,$	$\#(=) = 1,$	$\#(\neq) = 1,$	$\#(>) = 0.$
I8	$\#(\exists) = 1,$	$\#(=) = 1,$	$\#(\neq) = 3,$	$\#(>) = 0.$
I9	$\#(\exists) = 16,$	$\#(=) = 19,$	$\#(\neq) = 3,$	$\#(>) = 1.$

Benchmark Data

Input $\#(*)$ denotes the number of $*$.

I1	$\#(\exists) = 4,$	$\#(=) = 6,$	$\#(\neq) = 7,$	$\#(>) = 1.$
I2	$\#(\exists) = 2,$	$\#(=) = 1,$	$\#(\neq) = 1,$	$\#(>) = 0.$
I5	$\#(\exists) = 2,$	$\#(=) = 1,$	$\#(\neq) = 1,$	$\#(>) = 0.$
I8	$\#(\exists) = 1,$	$\#(=) = 1,$	$\#(\neq) = 3,$	$\#(>) = 0.$
I9	$\#(\exists) = 16,$	$\#(=) = 19,$	$\#(\neq) = 3,$	$\#(>) = 1.$

Time Computation time is written in second.
"N" means the computation doesn't terminate within 1h.

	new	old	syn	rc	red	rl	qep
I1	48	N	N	N	N	N	N
I2	2	N	N	N	N	N	1883
I5	1	N	482	N	N	N	N
I8	8	N	N	N	239	N	118
I9	27	N	N	53	N	N	N

Contents

Conclusion

- We obtained the following main result at **Th. 8**

  Let $\mathcal{G}$ be a CGS of $I \lhd \mathbb{Q}[\bar{A},\bar{X}]$, and $(\mathcal{S},G) \in \mathcal{G}$ s.t.

  $$\forall \bar{a} \in \mathcal{S}\ \langle G(\bar{a})\rangle \text{ is zero-dimensional.}$$

  Then I has continuous roots on $\mathcal{S}$. □

- We applied the main result to a QE method using CGS.

	new	old	syn	rc	red	rl	qep
I1	48	N	N	N	N	N	N
I2	2	N	N	N	N	N	1883
I5	1	N	482	N	N	N	N
I8	8	N	N	N	239	N	118
I9	27	N	N	53	N	N	N

Thank you for your kind attention !!

Main Theorem : Proof (Outline)

Taking an arbitrary $\bar{a} \in \mathcal{S}$, we introduce $l = \dim(\mathbb{C}[\bar{X}]/I(\bar{a}))$,

$$\alpha_i \in V_{\mathbb{C}}(I(\bar{a})) \text{ s.t. } \alpha_i = (\alpha_i^{(1)}, \ldots, \alpha_i^{(n)}),$$

$$\theta : \mathcal{S} \to (\mathbb{C}^n)_M \,;\, \bar{a} \mapsto (\alpha_1, \ldots, \alpha_l)_M,$$

$$\pi_j : \mathcal{S} \to \mathbb{C}_M \,;\, \bar{a} \mapsto (\alpha_1^{(j)}, \ldots, \alpha_l^{(j)})_M.$$

1 Each π_j is continuous at $\bar{a}$.
($\because$ $G(\bar{a})$ is a GB of zero-dimensional $I(\bar{a})$)

2 θ is continuous at $\bar{a}$ in the case $\alpha_1 = \cdots = \alpha_l$. ($\because$ **1**)

We consider finite $B(\bar{a}) = \{\alpha_1^{(1)}, \ldots, \alpha_l^{(1)}\} \times \cdots \times \{\alpha_1^{(n)}, \ldots, \alpha_l^{(n)}\}$.

Main Theorem : Proof (Outline): $\alpha_i \neq \alpha_k$

Let $h(\bar{X}) = \sum_{j=1}^{n} c_j X_j \in \mathbb{Q}[\bar{X}]$ s.t. $\forall \alpha \neq \alpha' \in B(\bar{a})[h(\alpha) \neq h(\alpha')]$,

$$\epsilon_0 < \min(|h(\alpha) - h(\alpha')| : \alpha \neq \alpha' \in B(\bar{a})),$$

$\bar{b} \in \mathcal{S}$ s.t. s.t. $\exists \delta > 0\, [d(\bar{a}, \bar{b}) < \delta \to (D(\pi_j(\bar{a}), \pi_j(\bar{b})) < \epsilon_0]$.

3 $\forall \beta_i \in V_{\mathbb{C}}(I(\bar{b}))\, \exists \alpha' \in B(\bar{a})\, [d(\alpha', \beta_i) < \epsilon_0]$. (∵ **1**)

4 If α' is not a root of $I(\bar{a})$, we get a contradiction.
(∵ a property of h and ϵ_0)

5 If the multiplicity μ_i of α_i satisfies the property such that

$$\mu_i \neq \#(V_{\mathbb{C}}(I(\bar{b})) \cap \{z \in \mathbb{C}^n : d(\alpha_i, z) < \epsilon_0\}),$$

we get a contradiction by a property of h and ϵ_0. □

Main Theorem : Proof (Outline) : Step 1

Let ϕ_p be $\mathbb{C}[\bar{X}]/\langle G(\bar{a})\rangle \to \mathbb{C}[\bar{X}]/\langle G(\bar{a})\rangle$; $f \mapsto fp$, ($p \in \mathbb{Q}[\bar{X}]$)

and $\chi_p \in \mathbb{C}[Y]$ be its characteristic polynomial. Then

$p(\alpha_i)$'s are the roots of χ_p ($\because$ ϕ_p is a multiplication map)

$\Rightarrow$ π_j and θ_h are continuous, where ($\because$ $\chi_{X_j}, \chi_h \in (\bar{A})[Y]$)

$$\theta_h : \mathcal{S} \to \mathbb{C}_M \; ; \; \bar{a} \to (h(\alpha_1), \ldots, h(\alpha_l))_M .$$

Main Theorem : Proof (Outline) : Step 2

We consider the case $\alpha = \alpha_1 = \cdots = \alpha_l$.

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall \bar{b} \in \mathcal{S}\ [d(\bar{a},\bar{b}) < \delta \to D(\pi_j(\bar{a}),\pi_j(\bar{b})) < \epsilon]$$

($\because$ π_j is continuous)

$$\Rightarrow \forall \epsilon{>}0\ \exists \delta{>}0\ \forall \bar{b}{\in}\mathcal{S}\ \forall \beta_j{\in}V_{\mathbb{C}}(I(\bar{b}))\ [d(\bar{a},\bar{b}){<}\delta \to d(\alpha,\beta_j){<}\epsilon]$$

$(\because\ \alpha = \alpha_1 = \cdots = \alpha_l)$

$$\Rightarrow \forall \epsilon > 0\ \exists \delta > 0\ \forall \bar{b} \in \mathcal{S}\ [d(\bar{a},\bar{b}) < \delta \to D(\theta(\bar{a}),\theta(\bar{b})) < \epsilon]$$

$(\because\ \theta(\bar{a}) = (\alpha,\ldots,\alpha)_M,\ \theta(\bar{b}) = (\beta_1,\ldots,\beta_l)_M)$

$\Rightarrow$ θ is continuous at $\bar{a}$.

Main Theorem : Proof (Outline) : Step 3

We consider the case $\alpha_i \neq \alpha_k$ for some i, k.

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall \bar{b} \in \mathcal{S}\ [d(\bar{a}, \bar{b}) < \delta \to D(\pi_j(\bar{a}), \pi_j(\bar{b})) < \epsilon]$$

($\because \pi_j$ is continuous)

$\Rightarrow$ For $\bar{b} \in \mathcal{S}$ s.t. $\exists \delta > 0\ d(\bar{a}, \bar{b}) < \delta \to D(\pi_j(\bar{a}), \pi_j(\bar{b})) < \epsilon_0$,

$$\forall \beta_i \in V_{\mathbb{C}}(I(\bar{b})) \exists \alpha' \in B(\bar{a})[d(\alpha', \beta_i) < \epsilon_0].$$

($\because \alpha' = (\alpha_1^{(i_1)}, \ldots, \alpha_n^{(i_n)})$ with $i_1, \ldots, i_n \in \{1, \ldots, l\}$)

Main Theorem : Proof (Outline) : Step 4

We consider an element $\bar{b} \in \mathcal{S}$, a positive number $\delta \in \mathbb{R}$ s.t.

$$d(\bar{a},\bar{b}) < \delta \to D(\pi_j(\bar{a}),\pi_j(\bar{b})), D(\theta_h(\bar{a}),\theta_h(\bar{b})) < \epsilon_0,$$

since π_j and θ_h are continuous at $\bar{a}$.

Let $O_\alpha(\epsilon_0) = \{z \in \mathbb{C}^n : |z - h(\alpha)| < \epsilon_0\}$ for $\alpha \in B(\bar{A})$. Then

$$\forall \alpha \neq \alpha' \in B(\bar{a}) \ [O_\alpha(\epsilon_0) \cap O_{\alpha'}(\epsilon_0) = \emptyset].$$

We can assume $\max(|c_1|,\ldots,|c_n|) < 1/n$. For $(\gamma,\gamma') \in \mathbb{C}^n$

$$|h(\gamma) - h(\gamma')| < \frac{1}{n}\sum_{j=1}^{n} |\gamma^{(j)} - \gamma^{(j)'}| < d(\gamma,\gamma').$$

Main Theorem : Proof (Outline) : Step 4

Assume that α' is not a root and α is a root of $I(\bar{a})$.

$\Rightarrow \quad |h(\alpha) - h(\alpha')| > 2\epsilon_0 \qquad (\because O_\alpha(\epsilon_0) \cap O_{\alpha'}(\epsilon_0) = \emptyset)$

$\Rightarrow \quad |h(\alpha) - h(\beta)| \geqslant |h(\alpha) - h(\alpha')| - |h(\alpha') - h(\beta)| > \epsilon_0$

$(\because |h(\alpha') - h(\beta)| < \epsilon_0)$

$\Rightarrow$ contradiction. $\qquad (\because D(\theta_h(\bar{a}), \theta_h(\bar{b})) < \epsilon_0)$

Let $\mathcal{R}_i = \{z \in \mathbb{C}^n : d(\alpha_i, z) < \epsilon_0\}$, $\{\beta_1, \ldots, \beta_\nu\} = V_{\mathbb{C}}(I(\bar{b})) \cap \mathcal{R}_i$.

Main Theorem : Proof (Outline) : Step 5

Assume $\mu_i < \nu \quad \Rightarrow \quad \forall \sigma \in \mathfrak{S}_l \ \exists k \in \{1, \ldots, \nu\} \ \sigma(k) > \mu_i$

$\Rightarrow \quad d(\alpha_{\sigma(k)}, \beta_k) > \epsilon_0$ ($\because d(\alpha_{\sigma(k)}, \beta_k) > |h(\alpha_{\sigma(k)}) - h(\beta_k)| > 2\epsilon_0$)

$\Rightarrow \quad \exists j \in \{1, \ldots, n\} \ \alpha^{(j)}_{\sigma(k)} - \beta^{(j)}_k > \epsilon_0$

$\Rightarrow$ contradiction. ($\because D(\pi_j(\bar{a}), \pi_j(\bar{b})) < \epsilon_0$)

Assume $\mu_i > \nu \quad \Rightarrow \quad \exists \alpha_k \in V_{\mathbb{C}}(I(\bar{a})) \ \mu_k < \#(V_{\mathbb{C}}(I(\bar{b})) \cap \mathcal{R}_k)$

($\because \#(V_{\mathbb{C}}(I(\bar{a})) = \#(V_{\mathbb{C}}(I(\bar{b}))$, $\forall \beta \in V_{\mathbb{C}}(I(\bar{b})$ $[\beta \in \cup_{h=1}^{l} \mathcal{R}_h]$)

$\Rightarrow$ contradiction. ($\because$ similar with the case $\mu_i < \nu$)

Thus we obtain the claim. Q.E.D.

ISSAC 2015

Main parts of CGS-QE methods eliminate $\exists \bar{X} \in \mathbb{R}^n$ from

$$\psi \equiv \phi(\bar{A}) \wedge \exists \bar{X} \in \mathbb{R}^n (\bigwedge_{f \in F} f = 0 \wedge \bigwedge_{h \in H} h > 0),$$

where $F, H \subset \mathbb{Q}[\bar{A}, \bar{X}]$ and a quantifier free formula ϕ satisfy

$$\forall \bar{a} \in \{\bar{\alpha} \in \mathbb{R}^m : \phi(\bar{\alpha})\} \ \langle F(\bar{a}) \rangle \text{ is zero-dimensional.}$$

The main parts eliminate $\exists \bar{X} \in \mathbb{R}^n$ by the following theorem.

Th. Let $H = \{h_1, \ldots, h_s\}$. Then for any $\bar{a} \in \{\alpha \in \mathbb{R}^m : \phi(\alpha)\}$,

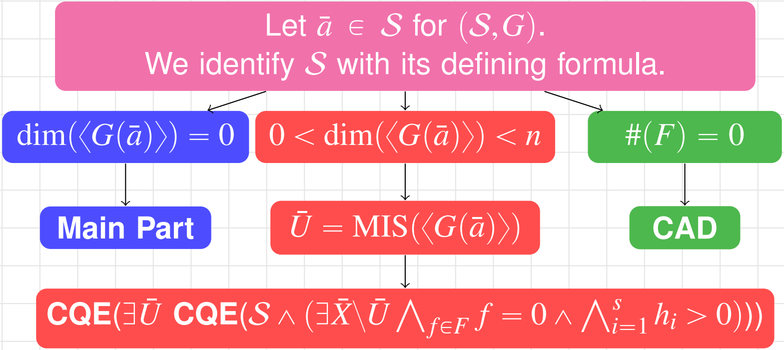
$$\psi(\bar{a}, \bar{X}) \Leftrightarrow \sum_{(e_1,\ldots,e_t) \in \{1,2\}^t} \mathrm{sign}(M^{\langle F(\bar{a}) \rangle}_{\prod_{i=1}^t h_i^{e_i}(\bar{a})}) \neq 0 \qquad \square$$

ISSAC 2015

$\mathrm{MIS}(*)$ denotes a maximal independent set.

$\mathrm{CQE}(*)$ denotes the output produced by a CGS-QE method.

Let $\mathcal{G}$ be a CGS of $\langle F \rangle$ with parameters $\bar{A}$.



ISSAC 2015

At ISSAC 2015, we gave an efficient algorithm by the QE of

$$\psi' \equiv \phi(\bar{A}) \wedge \exists \bar{X} \in \mathbb{R}^n (\bigwedge_{f \in F} f = 0 \wedge \bigwedge_{h \in H} h \geqslant 0).$$

ISSAC 2015 eliminates $\exists \bar{X} \in \mathbb{R}^n$ by the following theorem.

Th. Let $H = \{h_1, \ldots, h_s\}$. Then for any $\bar{a} \in \{\alpha \in \mathbb{R}^m : \phi(\alpha)\}$,

$$\psi'(\bar{a}, \bar{X}) \Leftrightarrow \sum_{(e_1,\ldots,e_t) \in \{0,1\}^t} \mathrm{sign}(M^{\langle F(\bar{a})\rangle}_{\prod_{i=1}^{t} h_i^{e_i}(\bar{a})}) \neq 0. \qquad \square$$

The QE of ψ' return more simple output than the QE of ψ. So

ISSAC 2015 is more efficient than other CGS-QE methods.

Benchmark Data

Input $\#(*)$ denotes the number of $*$.

I1	$\#(\exists) = 4$	, $\#(=) = 6$	, $\#(\neq) = 7$	, $\#(>) = 1$.
I2	$\#(\exists) = 2$	, $\#(=) = 1$	, $\#(\neq) = 1$	, $\#(>) = 0$.
I5	$\#(\exists) = 2$	, $\#(=) = 1$	, $\#(\neq) = 1$	, $\#(>) = 0$.
I8	$\#(\exists) = 1$	, $\#(=) = 1$	, $\#(\neq) = 3$	, $\#(>) = 0$.
I9	$\#(\exists) = 16$	, $\#(=) = 19$	, $\#(\neq) = 3$	, $\#(>) = 1$.

Time Computation time is written in second.
"N" means the computation doesn't terminate within 1h.

	new	old	syn	rc	red	rl	qep
I1	48	N	N	N	N	N	N
I2	2	N	N	N	N	N	1883
I5	1	N	482	N	N	N	N
I8	8	N	N	N	239	N	118
I9	27	N	N	53	N	N	N

Benchmark Data : Environment

- Computer Environment:
  - an Intel(R) Core(TM) i7-3635QM CPU @ 2.40GHz
  - 16 GB memory working on Ubuntu14.04
- Computer Algebra System:
  - Maple 2015
  - Mathematica 10.1.0
  - Reduce (Free CSL version), 04-Aug-11
  - QEPCAD (Version B 1.69, 16 Mar 2012)

Benchmark Data : I1 (Input & Output)

$$\exists(x_1,x_2,x_6,x_7)\ \bigwedge_{i=1}^{6} F_i = 0 \wedge \bigwedge_{i=1}^{7} P_i \neq 0 \wedge Q > 0, \text{where}$$

$$\max(\deg_{\bar{A},\bar{X}}(F_i):i)=4, \quad \max(\deg_{\bar{X}}(F_i):i)=4, \quad \max(\deg_{\bar{A}}(F_i):i)=2,$$
$$\max(\deg_{\bar{A},\bar{X}}(P_i):i)=21, \quad \max(\deg_{\bar{X}}(P_i):i)=21, \quad \max(\deg_{\bar{A}}(P_i):i)=5,$$
$$\deg_{\bar{A},\bar{X}}(Q)=8, \quad \deg_{\bar{X}}(Q)=3, \quad \deg_{\bar{A}}(Q)=7$$

for $\bar{A}=x_3,x_4,x_5,x_8$ and $\bar{X}=x_1,x_2,x_6,x_7$. Only new returns

$$(x_3=0 \wedge x_4-1\neq 0 \wedge 2x_4^2-4x_4+3=0)\vee$$
$$(x_3=0 \wedge x_4-1=0 \wedge x_5-1=0)\vee$$
$$(x_3=0 \wedge 2x_4-1=0 \wedge x_5-1=0)\vee$$
$$(x_3=0 \wedge x_5-1=0 \wedge 2x_4^2-4x_4+3=0)\vee$$
$$(x_3-1=0 \wedge x_4-1=0 \wedge x_5-1=0)\vee$$
$$(x_3=0 \wedge x_4+1\neq 0 \wedge 3x_4+1\neq 0 \wedge x_4^2+2x_4+3=0)\vee$$
$$(x_3-1=0 \wedge x_5-1=0 \wedge x_4+2\neq 0 \wedge x_4+3\neq 0 \wedge 5x_4^2+5x_4+14\neq 0 \wedge x_4^3-x_4^2+x_4-5=0).$$

Benchmark Data : I2 (Input & Output)

$\exists(v_1,v_2)\ F=0 \wedge P\neq 0,$ where for $\bar{A}=a,b$ and $\bar{X}=v_1,v_2$

$$\begin{aligned} \deg_{\bar{A},\bar{X}}(F)&=4, & \deg_{\bar{X}}(F)&=4, & \deg_{\bar{A}}(F)&=1,\\ \deg_{\bar{A},\bar{X}}(P)&=13, & \deg_{\bar{X}}(P)&=12, & \deg_{\bar{A}}(P)&=3.\end{aligned}$$

- new returns $a\neq 0 \vee b\neq 0$ within 2 sec.
- qep return $a\neq 0 \vee b\neq 0$ within 1883 sec.

Benchmark Data : I5 (Input & Output)

$\exists (x,y)\ F = 0 \wedge P \neq 0,$ where for $\bar{A} = a, b$ and $\bar{X} = v_1, v_2$

$$\deg_{\bar{A},\bar{X}}(F) = 5, \quad \deg_{\bar{X}}(F) = 5, \quad \deg_{\bar{A}}(F) = 1,$$
$$\deg_{\bar{A},\bar{X}}(P) = 21, \quad \deg_{\bar{X}}(P) = 19, \quad \deg_{\bar{A}}(P) = 4.$$

new, syn returns True within 1 sec., 482 sec., respectively.

Benchmark Data : I8 (Input & Output)

$$\exists(s,t)\ F = 0 \wedge \bigwedge_{i=1}^{3} P_i \neq 0,\ \text{where for } \bar{A} = a,b \text{ and } \bar{X} = s,t$$

$$\begin{array}{ccc} \deg_{\bar{A},\bar{X}}(F) = 8, & \deg_{\bar{X}}(F) = 8, & \deg_{\bar{A}}(F) = 8, \\ \max(\deg_{\bar{A},\bar{X}}(P_i) : i) = 1, & \max(\deg_{\bar{X}}(P_i) : i) = 1, & \max(\deg_{\bar{A}}(P_i) : i) = 1. \end{array}$$

- new returns $(a \neq 0 \vee b \neq 0) \wedge a - b \neq 0$ within 8 sec.
- red returns $a < b \vee b < a$ within 239 sec.
- qep returns $a \neq b$ within 118 sec.

Benchmark Data : I9 (Input & Output)

$$\exists(b_1,b_2,c_1,c_2,d_1,d_2,e_1,e_2,f_1,f_2,h_1,h_2,k_1,k_2,o_1,o_2)$$
$$\bigwedge_{i=1}^{19} F_i = 0 \wedge \bigwedge_{i=1}^{3} P_i \neq 0 \wedge Q > 0, \text{ where}$$

$$\begin{array}{rrr} \max(\deg_{\bar{A},\bar{X}}(F_i) : i) = 2, & \max(\deg_{\bar{X}}(F_i) : i) = 2, & \max(\deg_{\bar{A}}(F_i) : i) = 2, \\ \max(\deg_{\bar{A},\bar{X}}(P_i) : i) = 6, & \max(\deg_{\bar{X}}(P_i) : i) = 6, & \max(\deg_{\bar{A}}(P_i) : i) = 0, \\ \deg_{\bar{A},\bar{X}}(Q) = 2, & \deg_{\bar{X}}(Q) = 0, & \deg_{\bar{A}}(Q) = 2 \end{array}$$

for $\bar{A} = a_1, a_2, r$ and $\bar{X} = b_i, c_i, d_i, e_i, f_i, h_i, k_i, o_i$ $(i = 1, 2)$.

- new returns the complicated output within 27 sec.
- rc returns the complicated output within 53 sec.