Symmetric Indefinite Triangular Factorization Revealing the Rank Profile Matrix

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Context

Applications of symmetric Gaussian elimination

- Symmetric linear system solving
- Signature
- LLL: R factor of rational QR [Villard'12])

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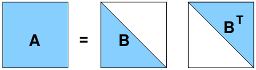
Motivation here

- `fsytrf`: finite field dense symmetric elimination in `fflas-ffpack`
- to be lifted for `LinBox` signature over $\mathbb{Z}$
- reduction to matrix product: $O(n^2 r^{\omega-2})$ and BLAS3
- investigate symmetric rank profile matrix and related pivoting

Outline

1. State of the art on symmetric factorizations
2. Rank profile and pivoting
3. Algorithms
4. The characteristic 2 case
5. Performance

Symmetric factorizations

Symmetric Decomposition	Exists for
	Field with sqrt & Generic rank profile

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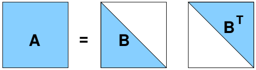
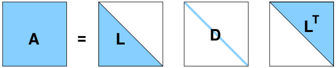
	Field with sqrt & Generic rank profile
	Generic rank profile

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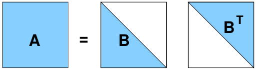
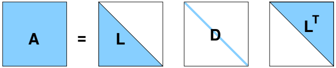
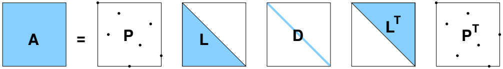
	Field with sqrt & Generic rank profile
	Generic rank profile
	No $\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$-like blocks

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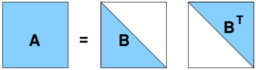
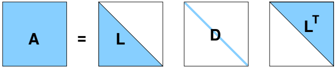
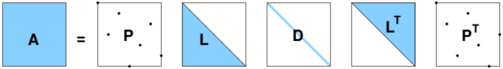
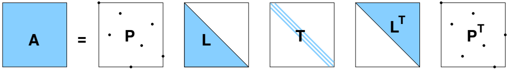
	Field with sqrt & Generic rank profile
	Generic rank profile
	No $\begin{bmatrix}0&1\\1&0\end{bmatrix}$-like blocks
	Any [Parlett-Reid 1970] T tridiagonal

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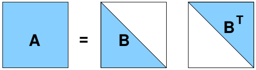
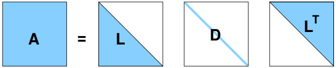
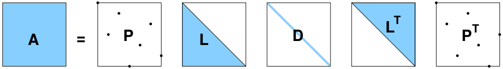
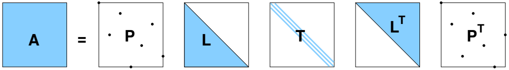
	Field with sqrt & Generic rank profile
	Generic rank profile
	No $\left[\begin{smallmatrix}0 & 1\\1 & 0\end{smallmatrix}\right]$-like blocks
	Any [Parlett-Reid 1970] T tridiagonal
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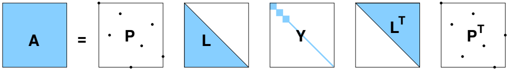
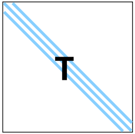
	Any [Bunch-Kaufmann 1977] Y with 1×1 and 2×2 blocks

State of the art

Form	Properties		
	[Parlett-Reid 1970] Diagonal Pivoting Iterative $\frac{2}{3}n^3$	[Bunch-Parlett 1971] Full pivoting Iterative $\frac{2}{3}n^3$	[Aasen 1971] Partial pivoting Iterative $\frac{1}{3}n^3$

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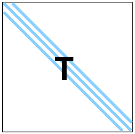
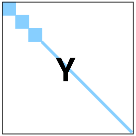
	[Parlett-Reid 1970] Diagonal Pivoting Iterative $\frac{2}{3}n^3$	[Bunch-Parlett 1971] Full pivoting Iterative $\frac{2}{3}n^3$	[Aasen 1971] Partial pivoting Iterative $\frac{1}{3}n^3$
	[Bunch-Kaufmann 1977] Partial pivoting Iterative $\frac{1}{3}n^3$		

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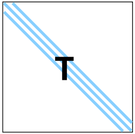
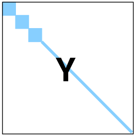
	[Parlett-Reid 1970] Diagonal Pivoting Iterative $\frac{2}{3}n^3$	[Bunch-Parlett 1971] Full pivoting Iterative $\frac{2}{3}n^3$	[Aasen 1971] Partial pivoting Iterative $\frac{1}{3}n^3$
	[Bunch-Kaufmann 1977] Partial pivoting Iterative $\frac{1}{3}n^3$	[Shklarski-Toledo 2011] Partial pivoting Recursive (GRP hyp.) $\frac{1}{3}n^3$	

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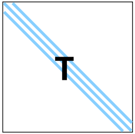
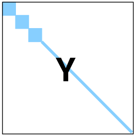
	[Parlett-Reid 1970] Diagonal Pivoting Iterative $\frac{2}{3}n^3$	[Bunch-Parlett 1971] Full pivoting Iterative $\frac{2}{3}n^3$	[Aasen 1971] Partial pivoting Iterative $\frac{1}{3}n^3$
	[Bunch-Kaufmann 1977] Partial pivoting Iterative $\frac{1}{3}n^3$	[Shklarski-Toledo 2011] Partial pivoting Recursive (GRP hyp.) $\frac{1}{3}n^3$	[Yamazaki-Dongarra 2017] Partial pivoting Block Iterative $\frac{1}{3}n^3$

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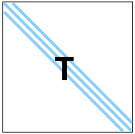
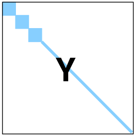
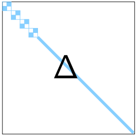
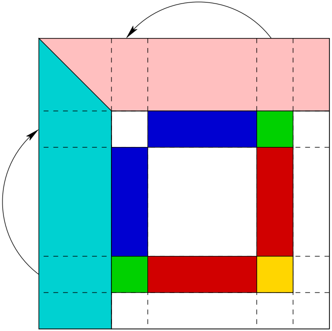
	[Parlett-Reid 1970] Diagonal Pivoting Iterative $\frac{2}{3}n^3$	[Bunch-Parlett 1971] Full pivoting Iterative $\frac{2}{3}n^3$	[Aasen 1971] Partial pivoting Iterative $\frac{1}{3}n^3$
	[Bunch-Kaufmann 1977] Partial pivoting Iterative $\frac{1}{3}n^3$	[Shklarski-Toledo 2011] Partial pivoting Recursive (GRP hyp.) $\frac{1}{3}n^3$	[Yamazaki-Dongarra 2017] Partial pivoting Block Iterative $\frac{1}{3}n^3$
	Here Pivoting revealing the rank profile matrix Recursive for any matrix $O(n^2 r^{\omega-2})$ (gives $\frac{1}{3}n^3$ when $rank{=}n$ & $\omega{=}3$)		

Symmetric pivoting



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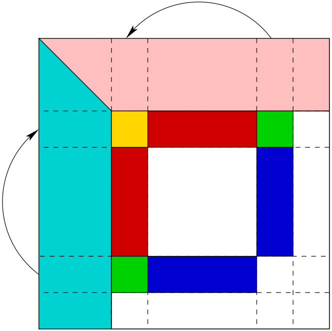

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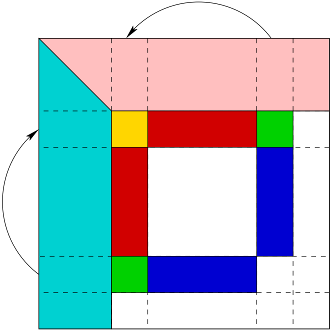

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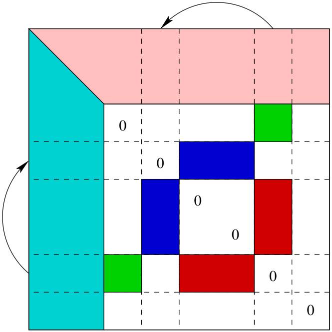

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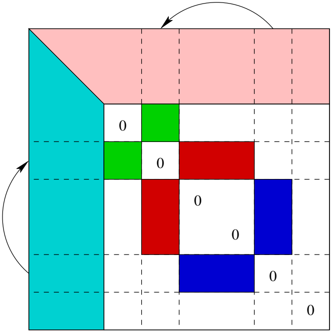

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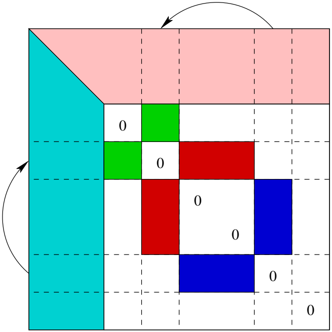

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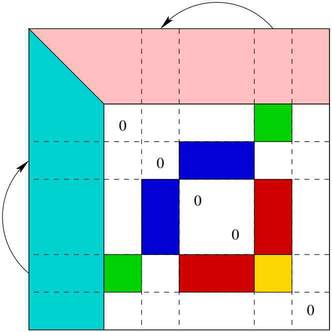

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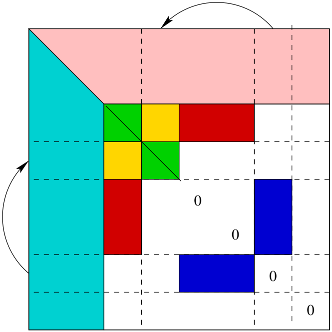

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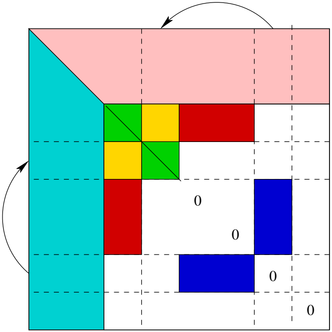

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  $\Rightarrow LDL^T$ with D diagonal
  $\Rightarrow$ requires division by 2

The rank profile matrix

Rank Profiles

Given a matrix A of rank r:

Example

$$A = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{pmatrix}$$

The rank profile matrix

Rank Profiles

Given a matrix A of rank r:

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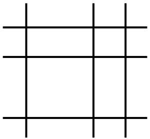

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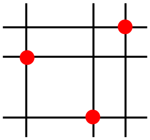

RPM (Rank Profile Matrix)

The unique $\mathcal{R}_A$ such that any pair of (i, j)-leading sub-matrix of $\mathcal{R}_A$ and of A have the same rank.

Pivoting revealing the rank profile matrix (unsymmetric)

[D., P., Sultan. ISSAC'15.] *Computing the Rank Profile Matrix.*

Definition

$A = PLUQ$ reveals the rank profile matrix if $P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q = \mathcal{R}_A$

Search	Row perm.	Col. perm.	RowRP	ColRP	$\mathcal{R}_A$	Instance
Row order	Transposition	Transposition	★			[IMH82] [JPS13]
Col. order	Transposition	Transposition		★		[KG85] [JPS13]
Lexico.	Transposition	Transposition	★			[Sto00]
Lexico.	Transposition	Rotation	★	★	★	[DPS15]
Lexico.	Rotation	Rotation	★	★	★	[DPS15]
Rev. lex.	Transposition	Transposition		★		[Sto00]
Rev. lex.	Rotation	Transposition	★	★	★	[DPS15]
Rev. lex.	Rotation	Rotation	★	★	★	[DPS15]
Product	Rotation	Transposition	★			[DPS15]
Product	Transposition	Rotation		★		[DPS15]
Product	Rotation	Rotation	★	★	★	[DPS13]

Pivoting revealing the rank profile matrix (symmetric case)

Definition

$A = PL\Delta L^T P^T$ reveals the rank profile matrix $\mathcal{R}_A$ if

$$P\Delta P^T = \mathrm{Diag}(d_1, \ldots, d_n)\mathcal{R}_A$$

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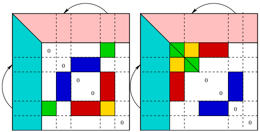
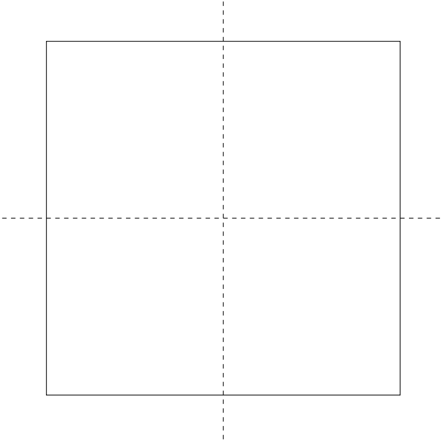
⇒ Off-diagonal pivoting with non-zero diagonal
⇒ Requires division by 2.

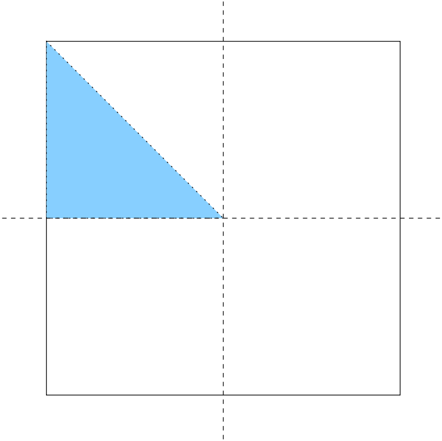


The recursive algorithm: full rank case



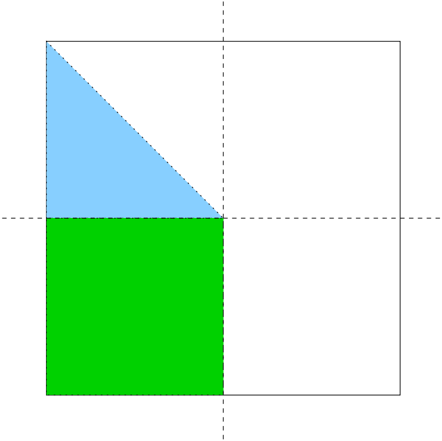
1. Split

The recursive algorithm: full rank case



1. Split
2. Recursive call: $A_{11} = L_1 L_1^T$

The recursive algorithm: full rank case



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3. TRSM: $G \leftarrow A_{21} L_1^{-1}$

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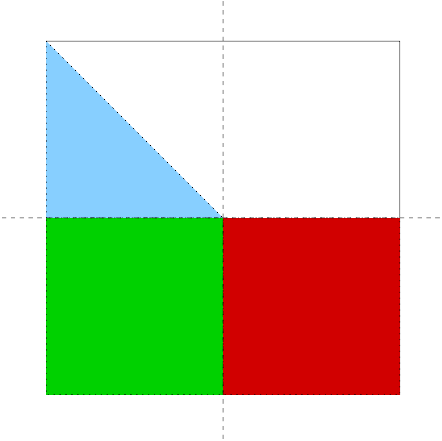

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4. SYRK: $H \leftarrow A_{22} - GG^T$

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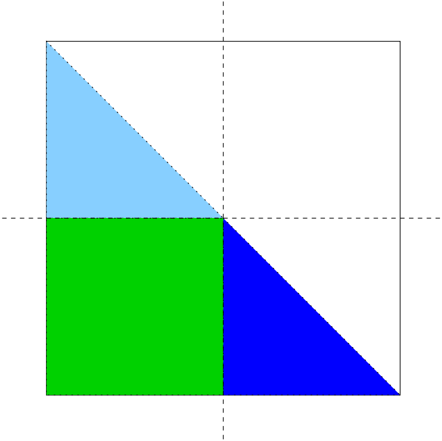

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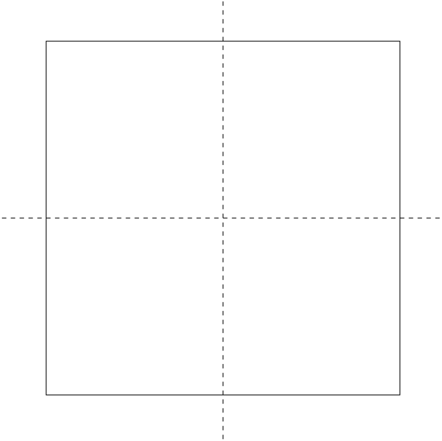

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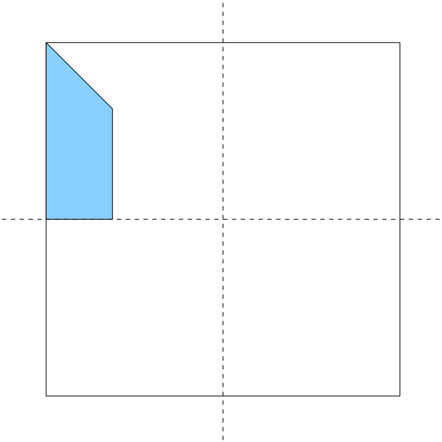
The recursive algorithm: arbitrary rank case



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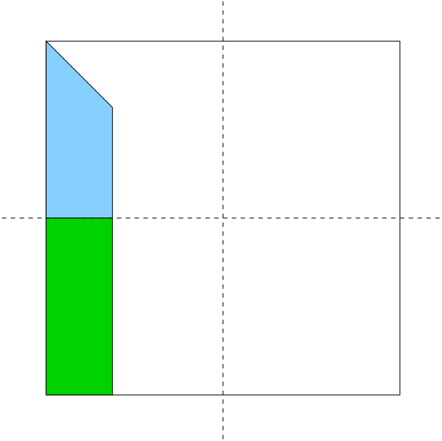

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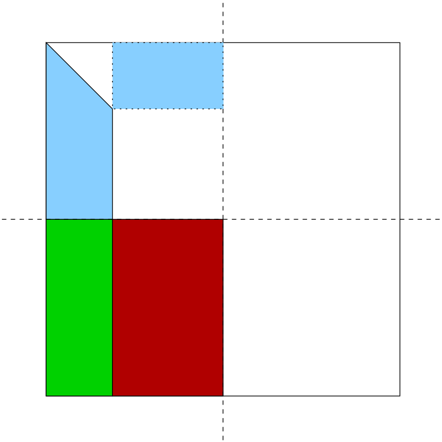
The recursive algorithm: arbitrary rank case



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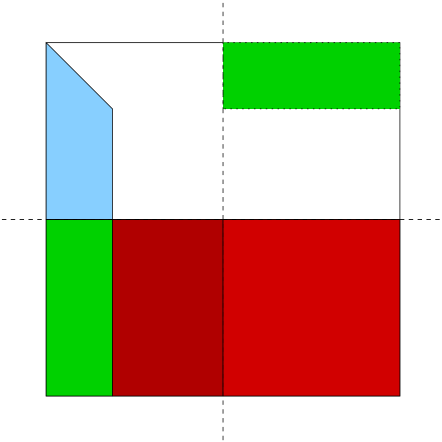

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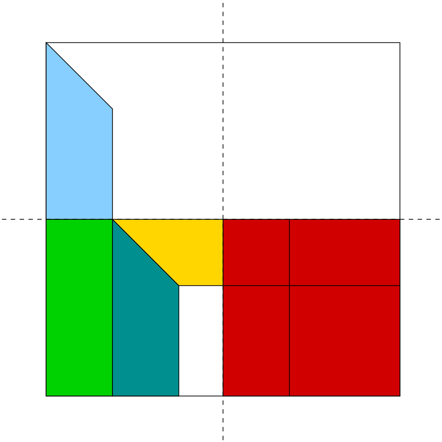

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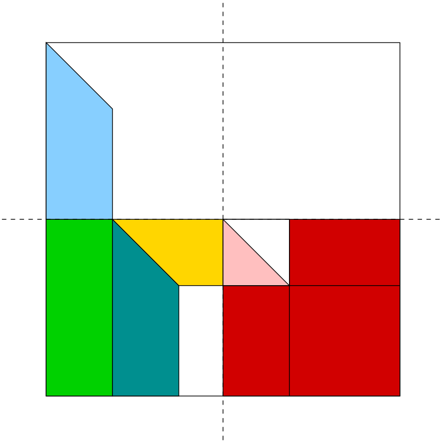

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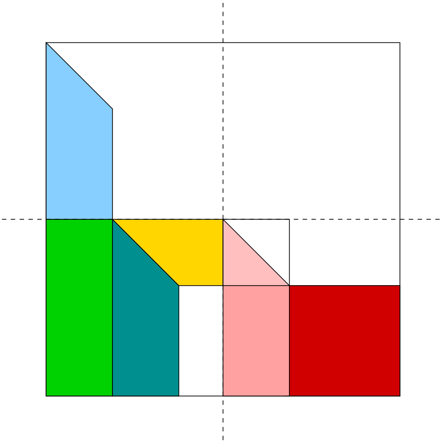

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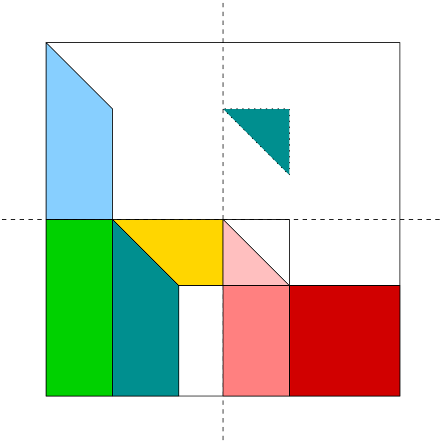

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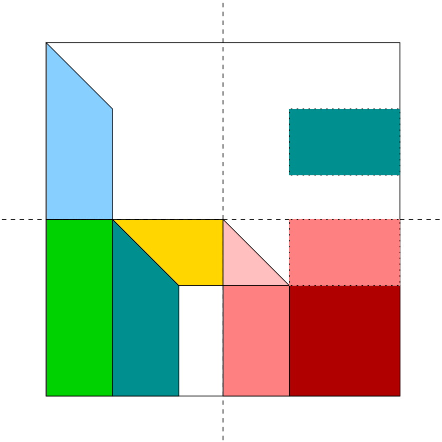

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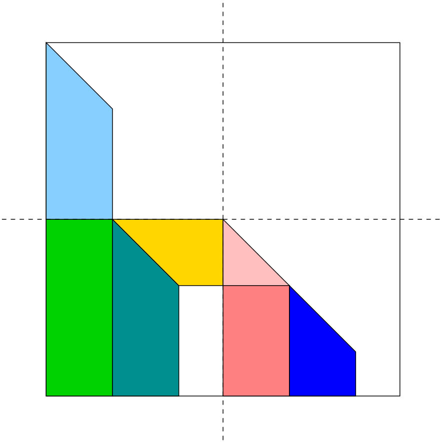

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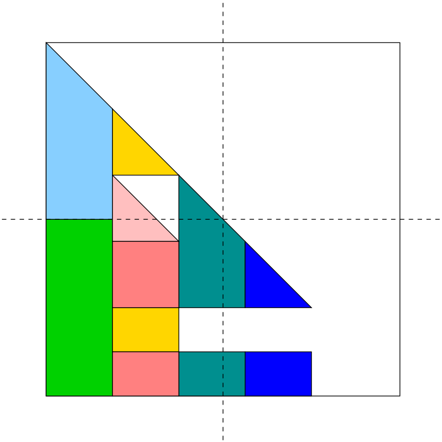

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12. Permutation

The `trsyr2k` routine

Problem

Given C symmetric and L lower triangular
Find X lower triangular such that

$$LX^T + XL^T = C$$

$$\begin{bmatrix} L_1 & \\ L_2 & L_3 \end{bmatrix} \begin{bmatrix} X_1^T & X_2^T \\ & X_3^T \end{bmatrix} + \begin{bmatrix} X_1 & \\ X_2 & X_3 \end{bmatrix} \begin{bmatrix} L_1^T & L_2^T \\ & L_3^T \end{bmatrix} = \begin{bmatrix} C_1 & C_2^T \\ C_2 & C_3 \end{bmatrix}.$$

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$$L_1 {X_1}^T + X_1 L_1^T = C_1$$

1. Find X_1 s.t. $L_1 X_1^T + X_1 L_1^T = C_1$ `recurse`

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$$L_2X_1^T + X_2L_1^T = C_2$$

1. Find X_1 s.t. $L_1X_1^T + X_1L_1^T = C_1$ — `recurse`
2. $Y \leftarrow C_2 - L_2X_1^T$ — `trmm`
3. $X_2 \leftarrow (L_1^T)^{-1}$ — `trsm`

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Problem in characteristic 2

Consider $\begin{bmatrix} 0 & c \\ c & d \end{bmatrix}$ (with $\mathcal{R} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$)

Diagonal pivoting: $\begin{bmatrix} 0 & c \\ c & d \end{bmatrix} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{c}{d} & 1 \end{bmatrix} \begin{bmatrix} d & \\ & -\frac{c^2}{d} \end{bmatrix} \begin{bmatrix} 1 & \frac{c}{d} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$

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In characteristic 2

$\Rightarrow$ In general there is no RPM-revealing $P \cdot L \cdot \Delta \cdot L^T \cdot P^T$!

Characteristic 2: Preserve $\left[\begin{smallmatrix}0 & 1\\ 1 & 1\end{smallmatrix}\right]$ factors

- Compute instead $A = P \cdot L \cdot D \cdot \Psi \cdot L^T \cdot P^T$:

$\Rightarrow$ D diagonal, Ψ block diagonal with 1×1, $\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0 & 1\\ 1 & 1\end{smallmatrix}\right]$ blocks

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- From this intermediate form:
  - Either recover $\mathcal{R}_A = P \cdot \mathcal{R}_\Psi \cdot P^T$;
  - Or use:

$$\begin{bmatrix} & c \\ c & d \end{bmatrix} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \cdot \begin{bmatrix} 1 & \\ c/d & 1 \end{bmatrix} \cdot \begin{bmatrix} d & \\ & -c^2/d \end{bmatrix} \cdot \begin{bmatrix} 1 & c/d \\ & 1 \end{bmatrix} \cdot \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$$

☺ + commuting

$\Rightarrow$ $\tilde{P} \cdot \tilde{L} \cdot \Delta \cdot \tilde{L}^T \cdot \tilde{P}^T$ symmetric factorization (but $\mathcal{R}_A$ is lost)

Iterative base case

Practical performance:

⚠ Stop recursion (less mod reductions/data move on small matrices)

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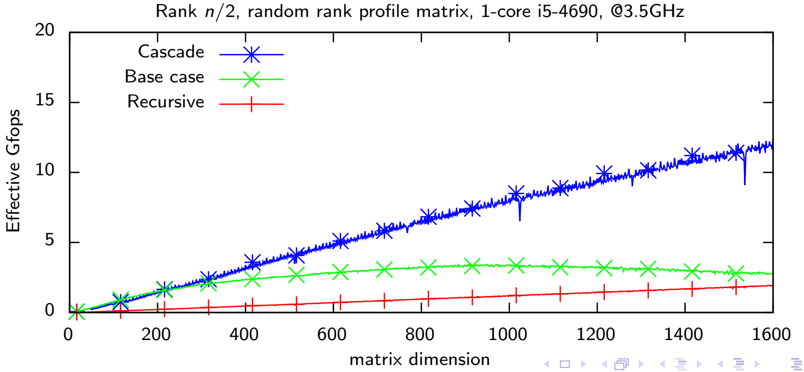

LAPACK vs FFPACK modulo 8 388 593

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