Bilinear systems with two supports: Koszul resultant matrices, eigenvalues, and eigenvectors

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Solving 2-bilinear systems

Objective

Solve (symbolically) square **mixed sparse** bilinear systems where

- $f_1, \ldots, f_r \in \mathbb{K}[\boldsymbol{X}, \boldsymbol{Y}]$, bilinear in the blocks $\boldsymbol{X}$ and $\boldsymbol{Y}$, and
- $f_{r+1}, \ldots, f_n \in \mathbb{K}[\boldsymbol{X}, \boldsymbol{Z}]$, bilinear in the blocks $\boldsymbol{X}$ and $\boldsymbol{Z}$.

- Take into the account the sparseness
- Polynomial time wrt the number of solutions

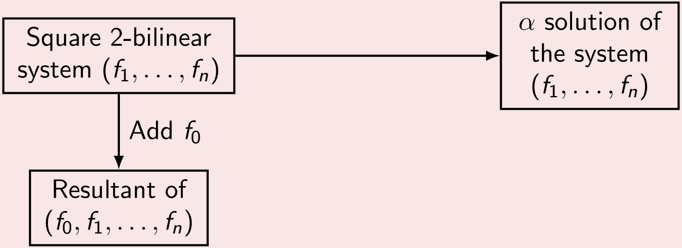
Results

- Koszul-like determinantal formula for the resultant
- Extension of the Eigenvalue criteria
- Extension of the Eigenvector criteria

Overview


Square 2-bilinear system $(f_1, \ldots, f_n)$
α solution of the system $(f_1, \ldots, f_n)$

Overview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Add f_0
Resultant of $(f_0, f_1, \ldots, f_n)$

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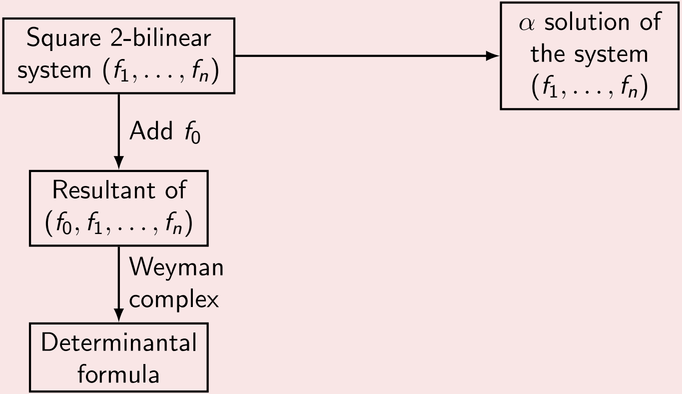

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α solution of the system $(f_1, \dots, f_n)$
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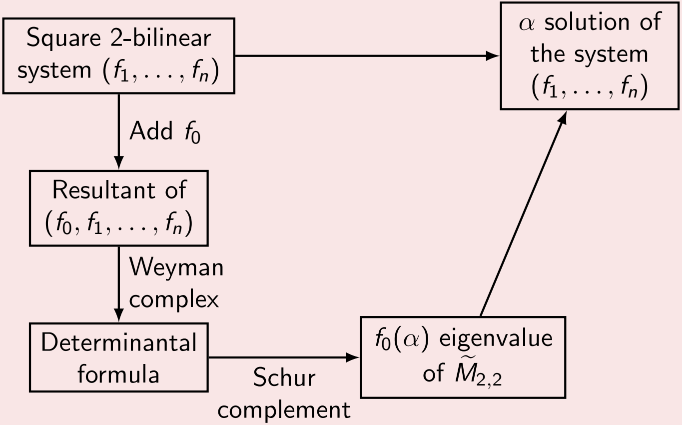
Resultant of $(f_0, f_1, \dots, f_n)$
Weyman complex
Determinantal formula

Overview



$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \rightarrow \begin{bmatrix} M_{1,1} & M_{1,2} \\ 0 & \widetilde{M}_{2,2} \end{bmatrix}$$

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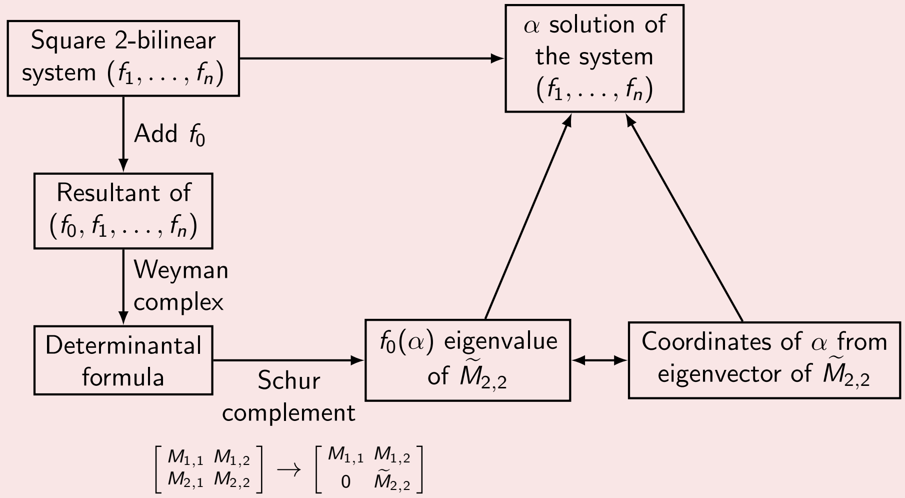
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$f_0(\alpha)$ eigenvalue of $\widetilde{M}_{2,2}$
Coordinates of α from eigenvector of $\widetilde{M}_{2,2}$
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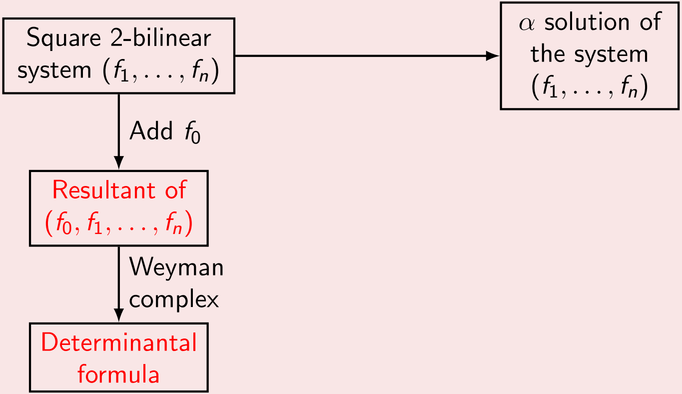
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Determinantal formulas and Weyman complex

Approach

Add a trilinear $f_0 \in \mathbb{K}[\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}] \rightarrow$ Sparse resultant of $(f_0, f_1, \ldots, f_n)$.

- The resultant of $(f_0, f_1, \ldots, f_n)$ vanishes $\iff$ the system has a solution over $\mathbb{P}^{n_x} \times \mathbb{P}^{n_y} \times \mathbb{P}^{n_z}$.

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- Several works in this direction: (Non-exhaustive!)
  - [Sturmfels, Zelevinsky, 1994], [Canny, Emiris, 1995] [Kapur, Saxena, 1997], [Chtcherba, Kapur, 2000], [D´Andrea, Dickenstein, 2001]

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- [Weyman, Zelevinsky, 1994] $\rightarrow$ determinantal formulas for **unmixed multihomogeneous** systems using **Weyman complexes**.
  - [Dickenstein, Emiris, 2003], [Emiris, Mantzaflaris, 2012],[Emiris, Mantzaflaris, Tsigaridas, 2016], [Busé, Mantzaflaris, Tsigaridas, 2017]

Determinantal formulas and Weyman complex

Approach

Add a trilinear $f_0 \in \mathbb{K}[\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}] \rightarrow$ Sparse resultant of $(f_0, f_1, \ldots, f_n)$.

- Weyman complex $\rightarrow$ Complex associated to an overdetermined system, parameterized by a vector $\boldsymbol{m}$.

$$K_\bullet(\boldsymbol{m}) : 0 \rightarrow K_{n+1}(\boldsymbol{m}) \xrightarrow{\delta_{n+1}(\boldsymbol{m})} \cdots \rightarrow K_1(\boldsymbol{m}) \xrightarrow{\delta_1(\boldsymbol{m})} K_0(\boldsymbol{m}) \xrightarrow{\delta_0(\boldsymbol{m})} \cdots \rightarrow K_{-n}(\boldsymbol{m}) \rightarrow 0$$

- Determinant of $K_\bullet(\boldsymbol{m}) = 0 \iff$ the system has a solution $\rightarrow$ Resultant of the system.

Determinantal formulas and Weyman complex

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- Determinant of $K_\bullet(\boldsymbol{m}) = 0 \iff$ the system has a solution $\rightarrow$ Resultant of the system.
- If $(\forall i \notin \{0,1\})\ K_i(\boldsymbol{m}) = 0$, Determinant of $K_\bullet(\boldsymbol{m})$ = Determinant of $\delta_1(\boldsymbol{m})$ $\rightarrow$ Determinantal formula.

$$K_\bullet(\boldsymbol{m}) : 0 \rightarrow \cdots \rightarrow 0 \rightarrow K_1(\boldsymbol{m}) \xrightarrow{\delta_1(\boldsymbol{m})} K_0(\boldsymbol{m}) \rightarrow 0 \rightarrow \cdots \rightarrow 0$$

Determinantal formula for the Resultant

Results

- Weyman complex $\rightarrow$ determinantal formula for the resultant of square 2-bilinear system + trilinear polynomial.
- Koszul-like matrix:
  - The elements in the matrix are $\pm$ the coefficients of the polynomials.
  - Generalization of Sylvester-like matrices, i.e. $(g_0, \ldots, g_n) \mapsto \sum_{i=0}^{n} g_i f_i$
  - These matrices were used in previous works, [Weyman & Zelevinsky, 1994], [Dickenstein & Emiris, 2003], [Emiris & Mantzaflaris, 2012], [Emiris, Mantzaflaris & Tsigaridas, 2016], [Busé, Mantzaflaris & Tsigaridas, 2017]

Number of solutions over $\mathbb{P}^{n_x} \times \mathbb{P}^{n_y} \times \mathbb{P}^{n_z}$	Size of the Koszul-like matrix
$\binom{r}{n_y}\binom{n-r}{n_z}$	$(n_x+1)\binom{r}{n_y}\binom{n-r}{n_z}\frac{r\cdot(n-r)-n_y\cdot n_z+n+1}{(r-n_y+1)(n-r-n_z+1)}$

Solving 2-bilinear systems

Example

$$\left\{\begin{array}{l} \\ \\ f_1 := \mathbf{7}\, x_0 y_0 + \mathbf{-8}\, x_0 y_1 + \mathbf{-1}\, x_1 y_0 + \mathbf{2}\, x_1 y_1 \\ f_2 := \mathbf{-5}\, x_0 y_0 + \mathbf{7}\, x_0 y_1 + \mathbf{-1}\, x_1 y_0 + \mathbf{-1}\, x_1 y_1 \\ f_3 := \mathbf{-6}\, x_0 z_0 + \mathbf{9}\, x_0 z_1 + \mathbf{-1}\, x_1 z_0 + \mathbf{-2}\, x_1 z_1 \end{array}\right. \quad \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \in \mathbb{K}[\boldsymbol{X}, \boldsymbol{Y}] \\ \in \mathbb{K}[\boldsymbol{X}, \boldsymbol{Z}] \end{array}$$

Solving 2-bilinear systems

Example

$$
\begin{cases}
f_0 := 3\,x_0y_0z_0 + -1\,x_0y_0z_1 + -4\,x_0y_1z_0 + 2\,x_0y_1z_1 \\
\qquad +1\,x_1y_0z_0 + 2\,x_1y_0z_1 + 2\,x_1y_1z_0 + -2\,x_1y_1z_1 & \in \mathbb{K}[\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}] \\
f_1 := 7\,x_0y_0 + -8\,x_0y_1 + -1\,x_1y_0 + 2\,x_1y_1 \\
f_2 := -5\,x_0y_0 + 7\,x_0y_1 + -1\,x_1y_0 + -1\,x_1y_1 & \in \mathbb{K}[\boldsymbol{X},\boldsymbol{Y}] \\
f_3 := -6\,x_0z_0 + 9\,x_0z_1 + -1\,x_1z_0 + -2\,x_1z_1 & \in \mathbb{K}[\boldsymbol{X},\boldsymbol{Z}]
\end{cases}
$$

Solving 2-bilinear systems

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$$
\begin{cases}
f_0 := \mathbf{3}\, x_0 y_0 z_0 + \mathbf{-1} x_0 y_0 z_1 + \mathbf{-4}\, x_0 y_1 z_0 + \mathbf{2}\, x_0 y_1 z_1 \\
\qquad + \mathbf{1} x_1 y_0 z_0 + \mathbf{2}\, x_1 y_0 z_1 + \mathbf{2}\, x_1 y_1 z_0 + \mathbf{-2}\, x_1 y_1 z_1
\end{cases} \in \mathbb{K}[\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}]
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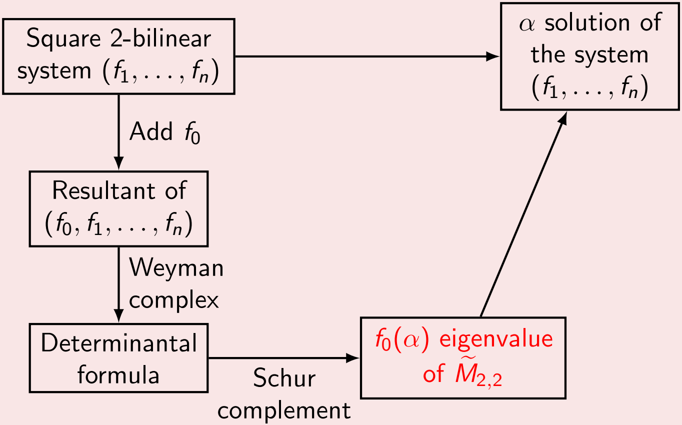
$$
f_3 := \mathbf{-6}\, x_0 z_0 + \mathbf{9}\, x_0 z_1 + \mathbf{-1}\, x_1 z_0 + \mathbf{-2}\, x_1 z_1 \in \mathbb{K}[\boldsymbol{X}, \boldsymbol{Z}]
$$

$$
M = \begin{bmatrix}
 & & & 5 & -7 & 1 & 1 & & & \\
 & & & 7 & -8 & -1 & 2 & & & \\
 & -1 & & & & & & -1 & -5 & 7 \\
7 & & -1 & & & & & -1 & & -5 \\
 & 1 & & & & & & -2 & -7 & 8 \\
8 & & -2 & & & & & 1 & & -7 \\
 & 2 & & 9 & & -2 & & -2 & -1 & 2 \\
2 & & -2 & & 9 & & -2 & 2 & & -1 \\
 & 1 & & -6 & & -1 & & 2 & 3 & -4 \\
-4 & & 2 & & -6 & & -1 & 1 & & 3
\end{bmatrix}
$$

Overview



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Motivation

- From Sylvester-like matrices $\rightarrow$ multiplication map of f_0 over $\mathbb{K}[\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}]/\langle f_1, \ldots, f_n \rangle$.
- Solve using eigenvalues and eigenvectors.
- We do not compute the resultant, we use the structure of the matrix.

But we do not have a Sylvester-like matrix...

Solving 2-bilinear systems

Eigenvalues - Main theorem

- Let M be a matrix such that $Res(f_0, f_1, \ldots, f_n)$ divides $det(M)$.
- Consider a $\mathfrak{m}$ monomial of f_0 such that
  - We can reorder M as $\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix}$,
  - $M_{1,1}$ is square and invertible.
  - The elements in diagonal of $M_{2,2}$ = coefficient of $\mathfrak{m}$.

Then, for each α solutions of $(f_1, \ldots, f_n)$ s.t. $\mathfrak{m}(\alpha) \neq 0$,
$\rightarrow \frac{f_0}{\mathfrak{m}}(\alpha)$ eigenvalue of $\left(M_{2,2} - M_{2,1} \cdot M_{1,1}^{-1} \cdot M_{1,2}\right)$ *(Schur complement)*

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If M determinantal formula $\rightarrow$ every eigenvalue is of this form.

Solving 2-bilinear systems
Example

$$\begin{cases} f_0 := & \boxed{3}\, \boldsymbol{x_0 y_0 z_0} + -1 x_0 y_0 z_1 + -4\, x_0 y_1 z_0 + 2\, x_0 y_1 z_1 \\ & +1 x_1 y_0 z_0 + 2\, x_1 y_0 z_1 + 2\, x_1 y_1 z_0 + -2\, x_1 y_1 z_1 \\ f_1 := & 7\, x_0 y_0 + -8\, x_0 y_1 + -1\, x_1 y_0 + 2\, x_1 y_1 \\ f_2 := & -5\, x_0 y_0 + 7\, x_0 y_1 + -1\, x_1 y_0 + -1\, x_1 y_1 \\ f_3 := & -6\, x_0 z_0 + 9\, x_0 z_1 + -1\, x_1 z_0 + -2\, x_1 z_1 \end{cases}$$

$$\left[\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & \boldsymbol{M_{2,2}} \end{array}\right] = \left[\begin{array}{cccccccc|cc} 0 & 0 & 0 & 5 & -7 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & -8 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & 7 \\ 7 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -7 & 8 \\ 8 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & -7 \\ 0 & 2 & 0 & 9 & 0 & -2 & 0 & -2 & -1 & 2 \\ 2 & 0 & -2 & 0 & 9 & 0 & -2 & 2 & 0 & -1 \\ \hline 0 & 1 & 0 & -6 & 0 & -1 & 0 & 2 & \boxed{3} & -4 \\ -4 & 0 & 2 & 0 & -6 & 0 & -1 & 1 & 0 & \boxed{3} \end{array}\right]$$

Solving 2-bilinear systems

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$$

$$
\widetilde{M}_{2,2} := \left(M_{2,2} - M_{2,1} \cdot M_{1,1}^{-1} \cdot M_{1,2} \right) = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}
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(f_1, f_2, f_3) has 2 solutions

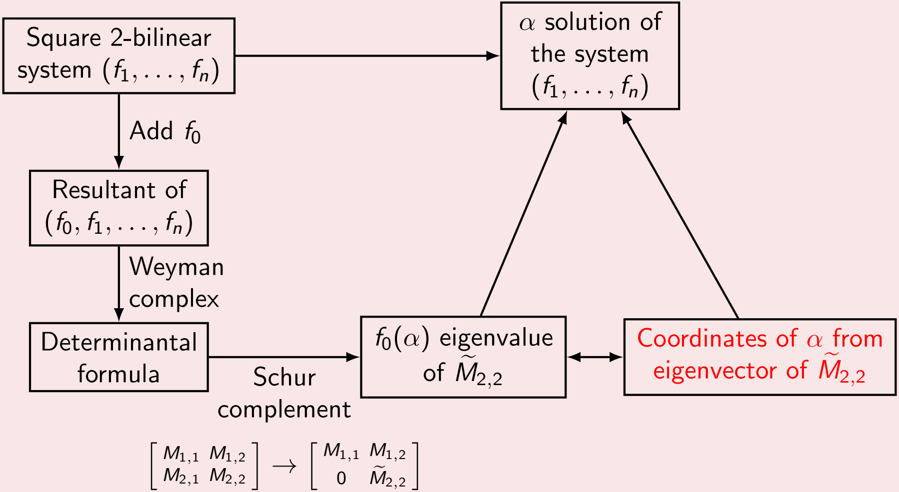
$$\left.\begin{array}{l} (1:1\ ;\ 1:1\ ;\ 1:1) \\ (1:3\ ;\ 1:2\ ;\ 1:3) \end{array}\right\} \in \mathbb{P}^{\alpha} \times \mathbb{P}^{\beta} \times \mathbb{P}^{\gamma}$$

Eigenvalues of $\widetilde{M}_{2,2}$

$$\frac{f_0}{x_0 y_0 z_0}((1:1\ ;\ 1:1\ ;\ 1:1)) = 3$$

$$\frac{f_0}{x_0 y_0 z_0}((1:3\ ;\ 1:2\ ;\ 1:3)) = 1$$

Overview



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Coordinates of α from eigenvector of $\widetilde{M}_{2,2}$
$\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \rightarrow \begin{bmatrix} M_{1,1} & M_{1,2} \\ 0 & \widetilde{M}_{2,2} \end{bmatrix}$

Problem

The eigenvalues are not enough to recover $V_{\mathcal{P}}(f_1, \ldots, f_n)$.

Eigenvectors

From the eigenvectors, we can recover the coordinates of each root $\alpha \in V_{\mathcal{P}}(f_1, \ldots, f_n)$. (**We can not recover them directly!**)

Solving 2-bilinear systems

$$\frac{f_0}{x_0 y_0 z_0}((\mathbf{1}:\mathbf{3}\ ;\ \mathbf{1}:\mathbf{2}\ ;\ 1:3)) = \mathbf{1} \qquad \bar{\boldsymbol{v}} := \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \cdot \bar{\boldsymbol{v}} = \mathbf{1} \cdot \bar{\boldsymbol{v}}$$

We can not recover $(\mathbf{1}:\mathbf{3}\ ;\ \mathbf{1}:\mathbf{2}\ ;\ 1:3)$ from $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solving 2-bilinear systems

$$\frac{f_0}{x_0 y_0 z_0}((1:3\ ;\ 1:2\ ;\ 1:3)) = 1 \qquad \bar{\boldsymbol{v}} := \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \cdot \bar{\boldsymbol{v}} = 1 \cdot \bar{\boldsymbol{v}}$$

We can not recover $(1:3\ ;\ 1:2\ ;\ 1:3)$ from $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

We extend $\bar{\boldsymbol{v}} \to \boldsymbol{v}$ s.t.

$$\left[\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} \end{array}\right] \cdot \boldsymbol{v} = \frac{f_0}{\mathfrak{m}}(\alpha) \cdot \left[\begin{array}{c} 0 \\ \hline \bar{v} \end{array}\right] \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} 4 = 1\cdot 2\cdot 2 \\ 3 = 3\cdot 1\cdot 1 \\ 12 = 3\cdot 2\cdot 2 \\ 1 = 1\cdot 1\cdot 1 \\ 2 = 1\cdot 1\cdot 2 \\ 3 = 3\cdot 1 \\ 6 = 3\cdot 1\cdot 2 \\ 6 = 3\cdot 2 \\ 1 = 1\cdot 1 \\ 2 = 1\cdot 2 \end{bmatrix}$$

$$\begin{cases} (1 \cdot \partial x_0 + 3 \cdot \partial x_1) \otimes \left(1 \cdot 1 \cdot \partial y_0^2 + 1 \cdot 2 \cdot \partial y_0 \partial y_1 + 2 \cdot 2 \cdot \partial {y_1}^2\right) \otimes 1 \\ (1 \cdot \partial x_0 + 3 \cdot \partial x_1) \otimes (1 \cdot \partial y_0 + 2 \cdot \partial y_1) \otimes 1 \end{cases}$$

Summing-up

Tools

- Weyman complex $\rightarrow$ Determinantal formula
- Koszul-like matrices
- Eigenvalues/Eigenvectors
  (Evaluation of the solutions/coordinates of the solutions)

Results

- Koszul-like determinantal formula for the resultant
- Extension of the Eigenvalue criteria (General extension!)
- Extension of the Eigenvector criteria

Perspectives

- Koszul-like matrices for multilinear mixed systems.
- Studying the eigenvectors of Koszul-like matrices.

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Thank you!