Data-Discriminants of Likelihood Equations

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Motivation
First Example

Assume \( p_i \) is probability of observing side \( i \) (\( i = 1, 2, 3, 4 \)) the die is unfair (\( j \) such that \( p_j \) is not 25%)

Given Constraints on \( p_1 \); \( p_2 \); \( p_3 \) and \( p_4 \)

\( f(p_1; p_2; p_3; p_4) \geq 0 \)

\( i = 1 \) \( p_i = 1 \)

We artificially assume \( p_1 + 2p_2 + 3p_3 + 4p_4 = 0 \)

Data Record
We toss the die 100 times and record the times of getting each side e.g. \[ u_1 = 11; u_2 = 24; u_3 = 15; u_4 = 50 \]

Question
For given constraints and data, how to estimate \( p_1 \); \( p_2 \); \( p_3 \) and \( p_4 \) which BEST explains the data?

Answer
Maximize likelihood function \( p_1^{u_1} p_2^{u_2} p_3^{u_3} p_4^{u_4} \) subjected to given constraints
Assume

- $p_i$ is probability of observing side $i$ ($i = 1, 2, 3, 4$)
- the die is unfair ($\iff \exists j$ such that $p_j$ is not 25%)
Motivation
First Example

Assume
- \( p_i \) is probability of observing side \( i \) (\( i = 1, 2, 3, 4 \))
- the die is unfair (\( \leftrightarrow \exists j \) such that \( p_j \) is not 25%)

Given Constraints on \( p_1, p_2, p_3 \) and \( p_4 \)

- \( \{(p_1, p_2, p_3, p_4) \in \mathbb{R}_{>0}^4 | \sum_{i=1}^{4} p_i = 1\} \)
- We artificially assume \( p_1 + 2p_2 + 3p_3 - 4p_4 = 0 \)
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- $\{(p_1, p_2, p_3, p_4) \in \mathbb{R}_{>0}^4 | \sum_{i=1}^{4} p_i = 1\}$
- We artificially assume $p_1 + 2p_2 + 3p_3 - 4p4 = 0$

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We toss the die 100 times and record the times of getting each side e.g. \( [u_1 = 11, u_2 = 24, u_3 = 15, u_4 = 50] \)

Question
For given constraints and data, how to estimate \( p_1, p_2, p_3 \) and \( p_4 \) which BEST explains the data?

Answer
Maximize likelihood function \( \frac{11}{p_1} \frac{24}{p_2} \frac{15}{p_3} \frac{50}{p_4} \) subjected to given constraints
Question

How to maximize likelihood function $p_1^{11} p_2^{24} p_3^{15} p_4^{50}$ subjected to given constraints?
Motivation

First Example

Question

How to maximize likelihood function \( p_1^{11} p_2^{24} p_3^{15} p_4^{50} \) subjected to given constraints?

Answer.

It is equivalent to maximize \( \log(p_1^{11} p_2^{24} p_3^{15} p_4^{50}) \). By the Lagrange Multiplier Method, we solve

\[
\begin{align*}
p_1 \lambda_1 + p_1 \lambda_2 - 11 &= 0 \\
p_2 \lambda_1 + 2p_2 \lambda_2 - 24 &= 0 \\
p_3 \lambda_1 + 3p_3 \lambda_2 - 15 &= 0 \\
p_4 \lambda_1 - 4p_4 \lambda_2 - 50 &= 0 \\
p_1 + 2p_2 + 3p_3 - 4p_4 &= 0 \\
p_1 + p_2 + p_3 + p_4 - 1 &= 0
\end{align*}
\]
Motivation

First Example

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How to maximize likelihood function $p_1^{11} p_2^{24} p_3^{15} p_4^{50}$ subjected to given constraints?

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 p_4 \lambda_1 - 4p_4 \lambda_2 - 50 &= 0 \\
 p_1 + 2p_2 + 3p_3 - 4p_4 &= 0 \\
 p_1 + p_2 + p_3 + p_4 - 1 &= 0
\end{align*}
$$

and get 3 solutions

$[p_1 = 1.2691, p_2 = -0.2903, p_3 = -0.0862, p_4 = 0.1075, \lambda_1 = 100, \lambda_2 = -91.3324],$

$[p_1 = 0.1857, p_2 = 1.2980, p_3 = -0.6737, p_4 = 0.1901, \lambda_1 = 100, \lambda_2 = -40.7547],$

$[p_1 = 0.1232, p_2 = 0.3057, p_3 = 0.2214, p_4 = 0.3497, \lambda_1 = 100, \lambda_2 = -10.7463].$
Question

How to maximize likelihood function $p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} p_4^{\mu_4}$ subjected to given constraints?
Motivation

First Example

Question

How to maximize likelihood function $p_1^{u_1} p_2^{u_2} p_3^{u_3} p_4^{u_4}$ subjected to given constraints?

Answer.

We solve

\begin{align*}
p_1 \lambda_1 + p_1 \lambda_2 - u_1 &= 0 \\
p_2 \lambda_1 + 2p_2 \lambda_2 - u_2 &= 0 \\
p_3 \lambda_1 + 3p_3 \lambda_2 - u_3 &= 0 \\
p_4 \lambda_1 - 4p_4 \lambda_2 - u_4 &= 0 \\
p_1 + 2p_2 + 3p_3 - 4p_4 &= 0 \\
p_1 + p_2 + p_3 + p_4 - 1 &= 0
\end{align*}

Remark

For general $[u_1; u_2; u_3; u_4]$, the system has 3 complex solutions.
**Motivation**

**First Example**

**Question**

How to maximize likelihood function $p_{u_1}p_{u_2}p_{u_3}p_{u_4}$ subjected to given constraints?

**Answer.**

We solve

\[
\begin{align*}
p_1 \lambda_1 + p_1 \lambda_2 - u_1 &= 0 \\
p_2 \lambda_1 + 2p_2 \lambda_2 - u_2 &= 0 \\
p_3 \lambda_1 + 3p_3 \lambda_2 - u_3 &= 0 \\
p_4 \lambda_1 - 4p_4 \lambda_2 - u_4 &= 0 \\
p_1 + 2p_2 + 3p_3 - 4p_4 &= 0 \\
p_1 + p_2 + p_3 + p_4 - 1 &= 0
\end{align*}
\]

**Remark**

For general $[u_1, u_2, u_3, u_4]$, the system has 3 complex solutions.
Motivation
First Example

Question
For which \( u_i \), the system has 0, 1, 2 and 3 REAL/POSITIVE solutions?

Answer. Use real quantifier elimination/real root classification tools. For example, by RealRootClassification in Maple2015 [C. Chen, J. H. Davenport, J. P. May, M. M. Maza, B. Xia and R. Xiao, 2010], for any \((u_1; u_2; u_3; u_4) \in \mathbb{R}^4 > 0\),

\[
D(u_1; u_2; u_3; u_4) = u_1 u_2 u_3 u_4 (u_1 + u_2 + u_3 + u_4)(441 u_1^4 + 4998 u_1^3 u_2 + 20041 u_1^2 u_2^2 + 33320 u_1 u_2^3 + 19600 u_2^4 + 756 u_1^3 u_3 + 20034 u_1^2 u_2 u_3 + 83370 u_1 u_2^2 u_3 + 79800 u_2^3 u_3 + 5346 u_1^2 u_3^2 + 55890 u_1 u_2 u_3^2 + 119025 u_2^2 u_3^2 + 4860 u_1 u_3^3 + 76950 u_2 u_3^3 + 18225 u_3^4) 
\]

where \( D(u_1; u_2; u_3; u_4) > 0 \) gives 3 distinct real solutions and 1 of them is positive; \( D(u_1; u_2; u_3; u_4) < 0 \) gives 1 real solution and it is positive.

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Data-Discriminants of Likelihood Equations
**Motivation**

**First Example**

**Question**

For which $u_i$, the system has 0, 1, 2 and 3 REAL/POSITIVE solutions?

**Answer.** Use real quantifier elimination/real root classification tools.

For example, by RealRootClassification in Maple2015 [C. Chen, J. H. Davenport, J. P. May, M. M. Maza, B. Xia and R. Xiao, 2010], for any $(u_1, u_2, u_3, u_4) \in \mathbb{R}_>^4$,

- $D(u_1, u_2, u_3, u_4) > 0 \Rightarrow 3$ distinct real solutions and 1 of them is positive;

- $D(u_1, u_2, u_3, u_4) < 0 \Rightarrow 1$ real solution and it is positive.

where

$$D = u_1 u_2 u_3 u_4 (u_1 + u_2 + u_3 + u_4)(441 u_1^4 + 4998 u_1^3 u_2 + 20041 u_1^2 u_2^2 + 33320 u_1 u_2^3 + 19600 u_2^4 - 756 u_1^3 u_3 + 20034 u_1^2 u_3^2 + 83370 u_1 u_2 u_3 + 79800 u_2 u_3^2 - 5346 u_1 u_3^3 + 55890 u_1 u_2^2 u_3 + 119025 u_2^2 u_3^2 + 4860 u_1 u_3^3 + 76950 u_2 u_3^3 + 18225 u_3^4 - 1596 u_1^3 u_4 - 11116 u_1^2 u_2 u_4 - 17808 u_1 u_2^2 u_4 + 4480 u_2^2 u_4^2 + 7452 u_1^2 u_3 u_4 - 7752 u_1 u_2 u_3 u_4 + 49680 u_2 u_3^2 u_4 - 17172 u_1 u_3^2 u_4 + 71460 u_2 u_3^2 u_4 + 27540 u_3^3 u_4 + 2116 u_1^2 u_4^2 + 6624 u_1 u_2 u_4^2 - 4224 u_2^2 u_4^2 - 9528 u_1 u_3 u_4^2 + 15264 u_2 u_3 u_4^2 + 14724 u_3^2 u_4^2 - 1216 u_1 u_4^3 - 512 u_2 u_4^3 + 3264 u_3 u_4^3 + 256 u_4^4)$$
Motivation
First Example

**Question**
For which \( u_i \), the system has 0, 1, 2 and 3 REAL/POSITIVE solutions?

**Answer.** Use real quantifier elimination/real root classification tools.

For example, by `RealRootClassification` in Maple2015 [C. Chen, J. H. Davenport, J. P. May, M. M. Maza, B. Xia and R. Xiao, 2010], for any \((u_1, u_2, u_3, u_4) \in \mathbb{R}_+^4\),

- \( D(u_1, u_2, u_3, u_4) > 0 \Rightarrow 3 \) distinct real solutions and 1 of them is positive;

- \( D(u_1, u_2, u_3, u_4) < 0 \Rightarrow 1 \) real solution and it is positive.

where
\[
D = u_1 u_2 u_3 u_4 (u_1 + u_2 + u_3 + u_4) (441u_1^4 + 4998u_1^3u_2 + 20041u_1^2u_2^2 + 33320u_1u_2^3 + 19600u_2^4 - 756u_1^3u_3 + 20034u_1^2u_3u_4 + 83370u_1u_2^2u_3 + 79800u_2^3u_3 - 5346u_1^2u_3^2 + 55890u_1u_2u_3^2 + 119025u_2^2u_3^2 + 4860u_1u_3^3 + 76950u_2u_3^3 + 18225u_3^4 - 1596u_1^3u_4 - 11116u_1^2u_2u_4 - 17808u_1u_2^2u_4 + 4480u_2^3u_4 + 7452u_1^2u_3u_4 - 7752u_1u_2u_3u_4 + 49680u_2^2u_3u_4 - 17172u_1u_3^2u_4 + 71460u_2u_3^2u_4 + 27540u_3^3u_4 + 2116u_1^2u_4^2 + 6624u_1u_2u_4^2 - 4224u_2^2u_4^2 - 9528u_1u_3u_4^2 + 15264u_2u_3u_4^2 + 14724u_3^2u_4^2 - 1216u_1u_4^3 - 512u_2u_4^3 + 3264u_3u_4^3 + 256u_4^4)
\]

**Question**
How to compute \( D \) EFFICIENTLY?
Maximum Likelihood Estimation Problem

Algebraic Statistical Model

\[ X = \mathcal{V} \cap \Delta_n \]

where

- \( \mathcal{V} \): irreducible and generically reduced projective variety
  \[ \{(p_0, \ldots, p_n) \in \mathbb{C}^{n+1} | g_1(p_0, \ldots, p_n) = 0, \ldots, g_s(p_0, \ldots, p_n) = 0\} \]

- \( \Delta_n \): probability simplex
  \[ \{(p_0, \ldots, p_n) \in \mathbb{R}^{n+1}_{>0} | \Sigma_{i=0}^n p_i = 1\} \]
Maximum Likelihood Estimation Problem

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\( \{(p_0, \ldots, p_n) \in \mathbb{R}_{>0}^{n+1} | \Sigma_{i=0}^n p_i = 1 \} \)

Data Vector

\([u_0, u_1, \ldots, u_n]\)
### Maximum Likelihood Estimation Problem

#### Algebraic Statistical Model

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- \( \Delta_n \): probability simplex
  \[ \{ (p_0, \ldots, p_n) \in \mathbb{R}^{n+1}_{>0} | \sum_{i=0}^{n} p_i = 1 \} \]

#### Data Vector

\[ [u_0, u_1, \ldots, u_n] \]

#### Maximum Likelihood Estimation Problem

For given model and data, how to estimate \( p_0, \ldots, p_n \) which BEST explains the data?

#### Method

Maximize likelihood function \( \prod_{i=0}^{n} p_i^{u_i} \) subject to the algebraic statistical model.
Lagrange Likelihood Equations

Question

How to maximize likelihood function $\prod_{i=0}^{n} p_i^{u_i}$ subject to the algebraic statistical model $\mathcal{V}(g_1, \ldots, g_s) \cap \Delta_n$?

Answer

For every critical point $(p_0, \ldots, p_n)$ of the likelihood function, there exists $(\lambda_1, \ldots, \lambda_{s+1}) \in \mathbb{C}^{s+1}$ such that $(p_0, \ldots, p_n, \lambda_1, \ldots, \lambda_{s+1})$ is a solution to the Lagrange likelihood equations [S. Hosten, A. Khetan and B. Sturmfels, 2005; E. Gross and J. I. Rodriguez, 2014]:

\[
\begin{align*}
F_0 &= p_0(\lambda_1 + \frac{\partial g_1}{\partial p_0} \lambda_2 + \cdots + \frac{\partial g_s}{\partial p_0} \lambda_{s+1}) - u_0 = 0 \\
\vdots \\
F_n &= p_n(\lambda_1 + \frac{\partial g_1}{\partial p_n} \lambda_2 + \cdots + \frac{\partial g_s}{\partial p_n} \lambda_{s+1}) - u_n = 0 \\
F_{n+1} &= g_1(p_0, \ldots, p_n) = 0 \\
\vdots \\
F_{n+s} &= g_s(p_0, \ldots, p_n) = 0 \\
F_{n+s+1} &= p_0 + \cdots + p_n - 1 = 0
\end{align*}
\]

where

- $p_0, \ldots, p_n, \lambda_1, \ldots, \lambda_{s+1}$ are unknowns,
- $u_0, \ldots, u_n$ are parameters.
Real/Positive Root Classification Problem

Theorem 1 (System of Lagrange likelihood equations is generically zero-dimensional) [S. Hosten, A. Khetan and B. Sturmfels, 2005]

For a given algebraic statistical model, for general data \((u_0, \ldots, u_n)\), the number of complex solutions of Lagrange likelihood equations is a non-negative constant (ML-Degree).

Standard Method for Real/Positive Root Classification


Step 1: Compute discriminant variety (REMARK: generally discriminant variety is not a hypersurface [D. Lazard and F. Rouillier, 2005])


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Data-Discriminants of Likelihood Equations
Real/Positive Root Classification Problem

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Real/Positive Root Classification Problem

Classify \( (u_0, \ldots, u_n) \) according to the number of real/positive solutions of Lagrange likelihood equations.
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Data-Discriminant and Problem Statement

Proposition (See propositions 1–2 in [J. I. Rodriguez and X. Tang, 2015].)

Discriminant varieties of Lagrange likelihood equations are projective varieties.

Proposition (Remark that we do need some extra assumptions for this definition. See Definition 5 in [J. I. Rodriguez and X. Tang, 2015].)

For a given algebraic statistics model $X$, the homogeneous polynomial that generates the reduced codimension 1 component of discriminant variety of Lagrange likelihood equations is said to be data-discriminant of Lagrange likelihood equations of $X$.

Problem Statement: Design Algorithm

Input: Lagrange likelihood equations

Output: Data-Discriminant
**Proposition** (See propositions 1-2 in [J. I. Rodriguez and X. Tang, 2015].)

Discriminant varieties of Lagrange likelihood equations are projective varieties.

**Data-Discriminant** (Remark that we do need some extra assumptions for this definition. See Definition 5 in [J. I. Rodriguez and X. Tang, 2015].)

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**Data-Discriminants of Likelihood Equations**
Proposition (See propositions 1–2 in [J. I. Rodriguez and X. Tang, 2015].)

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Problem Statement: Design Algorithm

- **Input:** Lagrange likelihood equations
- **Output:** Data-Discriminant
Algorithm 1 (Standard Algorithm)

Input.

\[ F = u_0^2 + u_1u_2, \quad J = 2u_0u_1 + u_1^2 + u_2 \]

Step 1.

Compute the generators of the elimination ideal \( \langle F; J \rangle \setminus Q[u_0; u_1; u_2] \)

Step 2.

Compute the codimension 1 component of the equidimensional radical decomposition of \( \langle u_1^2, 4u_0u_2 \rangle \)

Output.

\[ u_1^2 + 4u_0u_2 \]
Algorithm 1 (Standard Algorithm)

**Input.** \( F = u_0 p^2 + u_1 p + u_2, \)
Algorithm 1 (Standard Algorithm)

**Input.** \( F = u_0 p^2 + u_1 p + u_2, \)

\( J = 2u_0 p + u_1 \)
Algorithm 1 (Standard Algorithm)

Input. $F = u_0 p^2 + u_1 p + u_2,$
       $J = 2u_0 p + u_1$

Step 1. Compute the generators of the elimination ideal $\langle F, J \rangle \cap \mathbb{Q}[u_0, u_1, u_2]$

$$\{u_1^2 - 4u_0 u_2\}$$
Algorithm 1 (Standard Algorithm)

**Input.** \( F = u_0 p^2 + u_1 p + u_2, \)
\[ J = 2u_0 p + u_1 \]

**Step 1.** Compute the generators of the elimination ideal \( \langle F, J \rangle \cap \mathbb{Q}[u_0, u_1, u_2] \)

\[ \{ u_1^2 - 4u_0u_2 \} \]

**Step 2.** Compute the codimension 1 component of the equidimensional radical decomposition of \( \langle u_1^2 - 4u_0u_2 \rangle \)

\[ u_1^2 - 4u_0u_2 \]

**Output.** \( u_1^2 - 4u_0u_2 \)
Algorithm 2 (Probabilistic Algorithm)

Input. \( F = u_0 p^2 + u_1 p + u_2, \)
\[ J = 2u_0 p + u_1 \]
Algorithm 2 (Probabilistic Algorithm)

Input. \( F = u_0 p^2 + u_1 p + u_2, \)
\[ J = 2u_0 p + u_1 \]

Step 1 (Compute the degree and get the possible terms). We assume our output is \( D(u_0, u_1, u_2). \)
Algorithm 2 (Probabilistic Algorithm)

Input. \( F = u_0 p^2 + u_1 p + u_2, \)
\( J = 2u_0 p + u_1 \)

Step 1 (Compute the degree and get the possible terms). We assume our output is \( D(u_0, u_1, u_2) \). Substitute

\[
\begin{align*}
    u_0 &= 1 \cdot t + 11, \\
    u_1 &= 3 \cdot t + 2, \\
    u_2 &= 5 \cdot t + 6
\end{align*}
\]

(the red coefficients are “randomly” chosen)

into \( F, J \)
Algorithm 2 (Probabilistic Algorithm)

**Input.** \( F = u_0 p^2 + u_1 p + u_2, \)
\[ J = 2u_0 p + u_1 \]

**Step 1 (Compute the degree and get the possible terms).** We assume our output is \( D(u_0, u_1, u_2) \). Substitute
\[ u_0 = 1 \cdot t + 11, \]
\[ u_1 = 3 \cdot t + 2, \]
\[ u_2 = 5 \cdot t + 6 \]
(the red coefficients are “randomly” chosen)

into \( F, J \) and compute the radical of the elimination ideal \( \langle F(t, p), J(t, p) \rangle \cap \mathbb{Q}[t] \)
\[ \langle 11t^2 + 232t + 260 \rangle \]
(that means \( D(t + 11, 3t + 2, 5t + 6) = 11t^2 + 232t + 260 \))

So the total degree of \( D \) is 2.
Algorithm 2 (Probabilistic Algorithm)

Input. $F = u_0 p^2 + u_1 p + u_2,$  
$J = 2u_0 p + u_1$

Step 1 (Compute the degree and get the possible terms). We assume our output is $D(u_0, u_1, u_2)$. Substitute

$$u_0 = 1 \cdot t + 11,$$
$$u_1 = 3 \cdot t + 2,$$
$$u_2 = 5 \cdot t + 6$$

(the red coefficients are “randomly” chosen)

into $F, J$ and compute the radical of the elimination ideal $\langle F(t, p), J(t, p) \rangle \cap \mathbb{Q}[t]$

$$\langle 11t^2 + 232t + 260 \rangle$$

(that means $D(t + 11, 3t + 2, 5t + 6) = 11t^2 + 232t + 260$)

So the total degree of $D$ is 2. Similarly, we compute

$$\deg(D, u_0) = 1,$$  
$$\deg(D, u_1) = 2$$  
$$\deg(D, u_2) = 1$$

(so all the possible monomials in $D$ are $u_1^2, u_0 u_1, u_1 u_2, u_0 u_2$)
Algorithm 2 (Probabilistic Algorithm)

Step 2 (Evaluation/Interpolation). Assume

\[ D(u_0, u_1, u_2) = u_1^2 + (C_1 u_0 + C_2 u_2)u_1 + C_3 u_0 u_2 \]  

(1)
Algorithm 2 (Probabilistic Algorithm)

Step 2 (Evaluation/Interpolation). Assume

\[ D(u_0, u_1, u_2) = u_1^2 + (C_1 u_0 + C_2 u_2)u_1 + C_3 u_0 u_2 \]  

(1)

Step 2.1. Substitute \( u_0 = 13, u_2 = 4 \) into \( F, J \) and compute the radical of the elimination ideal \( \langle F(u_1, p), J(u_1, p) \rangle \cap \mathbb{Q}[u_1] \)

\[ \langle u_1^2 - 208 \rangle \]  

(2)

(that means \( D(13, u_1, 4) = u_1^2 - 208 \))

Comparing (1) and (2), we see

\[ 13C_1 + 4C_2 = 0 \]  

(3)

and \( 52C_3 = -208 \). Therefore, \( C_3 = -4 \).

(We need one more evaluation to solve \( C_1 \) and \( C_2 \))
**Step 2 (Evaluation/Interpolation).** Assume

\[
D(u_0, u_1, u_2) = u_1^2 + (C_1 u_0 + C_2 u_2) u_1 + C_3 u_0 u_2 \tag{1}
\]

**Step 2.1.** Substitute \( u_0 = 13, u_2 = 4 \) into \( F, J \) and compute the radical of the elimination ideal \( \langle F(u_1, p), J(u_1, p) \rangle \cap \mathbb{Q}[u_1] \)

\[
\langle u_1^2 - 208 \rangle \tag{2}
\]

(that means \( D(13, u_1, 4) = u_1^2 - 208 \))

Comparing (1) and (2), we see

\[
13C_1 + 4C_2 = 0 \tag{3}
\]

and \( 52C_3 = -208 \). Therefore, \( C_3 = -4 \).

(We need one more evaluation to solve \( C_1 \) and \( C_2 \))

**Step 2.2.** Substitute \( u_0 = 7 \) and \( u_2 = 3 \) into \( F \) and \( J \). Similarly, we get

\[
7C_1 + 3C_2 = 0 \tag{4}
\]

By (3) and (4), \( C_1 = C_2 = 0 \).

**Output.** \( u_1^2 - 4u_0u_2 \)
Algorithm 2 (Probabilistic Algorithm)

Step 2 (Evaluation/Interpolation). Assume

\[ D(u_0, u_1, u_2) = u_1^2 + (C_1 u_0 + C_2 u_2) u_1 + C_3 u_0 u_2 \]  

(1)

Step 2.1. Substitute \( u_0 = 13 \), \( u_2 = 4 \) into \( F \), \( J \) and compute the radical of the elimination ideal \( \langle F(u_1, p), J(u_1, p) \rangle \cap \mathbb{Q}[u_1] \)

\[ \langle u_1^2 - 208 \rangle \]  

(2)

(that means \( D(13, u_1, 4) = u_1^2 - 208 \))

Comparing (1) and (2), we see

\[ 13C_1 + 4C_2 = 0 \]  

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and \( 52C_3 = -208 \). Therefore, \( C_3 = -4 \).

(We need one more evaluation to solve \( C_1 \) and \( C_2 \))

Step 2.2. Substitute \( u_0 = 7 \) and \( u_2 = 3 \) into \( F \) and \( J \). Similarly, we get

\[ 7C_1 + 3C_2 = 0 \]  

(4)

By (3) and (4), \( C_1 = C_2 = 0 \).

Output. \( u_1^2 - 4u_0u_2 \)

Remark: The EVALUATION/INTERPOLATION idea is NOT the first time investigated. See [M. Giusti, G. Lecerf and B. Salvy, 2001; E. Schost, 2003].
## Experiment

Timings for Random Models (s: seconds; h: hours)

<table>
<thead>
<tr>
<th>Algorithm 1</th>
<th>Algorithm 2</th>
<th>Algorithm 1</th>
<th>Algorithm 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Algorithm 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Strategy 1</td>
<td>Strategy 2</td>
<td>Strategy 1</td>
</tr>
<tr>
<td>4.9s</td>
<td>0.8s</td>
<td>0.6s</td>
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<tr>
<td>3.0s</td>
<td>0.7s</td>
<td>0.6s</td>
<td></td>
</tr>
<tr>
<td>5.0s</td>
<td>0.8s</td>
<td>0.6s</td>
<td></td>
</tr>
<tr>
<td>5.4s</td>
<td>0.8s</td>
<td>0.7s</td>
<td></td>
</tr>
<tr>
<td>6.3s</td>
<td>0.8s</td>
<td>0.7s</td>
<td></td>
</tr>
<tr>
<td>3.9s</td>
<td>0.7s</td>
<td>0.6s</td>
<td></td>
</tr>
<tr>
<td>2.0s</td>
<td>0.7s</td>
<td>0.5s</td>
<td></td>
</tr>
<tr>
<td>1.7s</td>
<td>0.7s</td>
<td>0.5s</td>
<td></td>
</tr>
<tr>
<td>3.8s</td>
<td>0.8s</td>
<td>0.6s</td>
<td></td>
</tr>
<tr>
<td>5.8s</td>
<td>0.8s</td>
<td>0.7s</td>
<td></td>
</tr>
</tbody>
</table>

Algorithm System: Macaulay 2
Processor: 3.2 GHz Inter Core i5 (8GB total memory)
Computer System: Mac OS X 10.9.3
Experiment
Timings for Literature Models (s: seconds; h: hours; d: days)

<table>
<thead>
<tr>
<th>Models</th>
<th>Algorithm 1</th>
<th>Algorithm 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Algorithm 1</td>
<td>Algorithm 2</td>
</tr>
<tr>
<td></td>
<td>Strategy 1</td>
<td>Strategy 2</td>
</tr>
<tr>
<td>Example 3</td>
<td>11.1s</td>
<td>5.3s</td>
</tr>
<tr>
<td>Example 4</td>
<td>36446.4s</td>
<td>360.2s</td>
</tr>
<tr>
<td>Example 5</td>
<td>&gt;16h</td>
<td>&gt;16h</td>
</tr>
<tr>
<td>Example 6</td>
<td>&gt;12d</td>
<td>&gt;30d</td>
</tr>
<tr>
<td></td>
<td>2768.2s</td>
<td>56.3s</td>
</tr>
<tr>
<td></td>
<td>30d</td>
<td></td>
</tr>
</tbody>
</table>

Example 3 (Random Censoring [M. Drton, B. Sturmfels and S. Sullivant, 2009]).

\[2p_0p_1p_2 + p_1^2 p_2 + p_1p_2^2 - p_0^2 p_{12} + p_1p_2p_{12}\]

Example 4 (3 × 3 Zero-Diagonal Matrix [E. Gross and J. I. Rodriguez, 2014]).

\[
\begin{vmatrix}
0 & p_{12} & p_{13} \\
p_{21} & 0 & p_{23} \\
p_{31} & p_{32} & 0
\end{vmatrix}
\]

Example 5 (Grassmannian of 2-planes in \(\mathbb{C}^4\) [S. Hosten, A, Khetan and B. Sturmfels, 2005]).

\[p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}\]

Example 6 (3 × 3 Symmetric Matrix Model [J. I. Rodriguez, 2014]).
Experiment
Comparing Memory Pressure for Computing Example 6

Left: running standard algorithm after 3 days

Right: running probabilistic algorithm after 3 days
A gambler has a coin and two pairs of three-sided dice. All the coin and dice are unfair. The two dice in the first pair have the same weights. The two dice in the second pair have the same weights.
A gambler has a coin and two pairs of three-sided dice. All the coin and dice are unfair. The two dice in the first pair have the same weights. The two dice in the second pair have the same weights.

He plays the same game 1000 rounds

Toss the coin.
- If the coin lands on side 1, toss the first pair of dice.
- If the coin lands on side 2, toss the second pair of dice.
A gambler has a coin and two pairs of three-sided dice. All the coin and dice are unfair. The two dice in the first pair have the same weights. The two dice in the second pair have the same weights.

He plays the same game 1000 rounds

Toss the coin.
- If the coin lands on side 1, toss the first pair of dice.
- If the coin lands on side 2, toss the second pair of dice.

After the 1000 rounds, he records the times of getting side i and side j with respect to the two dice every time he tosses

$$[u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33}]$$
A gambler has a coin and two pairs of three-sided dice. All the coin and dice are unfair. The two dice in the first pair have the same weights. The two dice in the second pair have the same weights.

He plays the same game 1000 rounds

Toss the coin.
- If the coin lands on side 1, toss the first pair of dice.
- If the coin lands on side 2, toss the second pair of dice.

After the 1000 rounds, he records the times of getting side $i$ and side $j$ with respect to the two dice every time he tosses

$$[u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33}]$$

Question

How to estimate the probability $p_{ij}$ of getting the sides $i$ and $j$ with respect to the two dice?
Assume that the probabilities of observing the sides 1 and 2 of the coin are $c_1$ and $c_2$, and the probabilities of observing the sides 1, 2 and 3 of one die in the first and second pair are $[b_1, b_2, b_3]$ and $[r_1, r_2, r_3]$, respectively. We know

\[
\begin{bmatrix}
\frac{p_{11}}{2} & \frac{p_{12}}{2} & \frac{p_{13}}{2} \\
\frac{p_{12}}{2} & \frac{p_{22}}{2} & \frac{p_{23}}{2} \\
\frac{p_{13}}{2} & \frac{p_{23}}{2} & \frac{p_{33}}{2}
\end{bmatrix} = c_1 \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix} [b_1, b_2, b_3] + c_2 \begin{bmatrix}
r_1 \\
r_2 \\
r_3
\end{bmatrix} [r_1, r_2, r_3].
\] (5)

Therefore, the matrix on the left side has at most rank 2.
Assume that the probabilities of observing the sides 1 and 2 of the coin are $c_1$ and $c_2$, and the probabilities of observing the sides 1, 2 and 3 of one die in the first and second pair are $[b_1, b_2, b_3]$ and $[r_1, r_2, r_3]$, respectively. We know

\[
\begin{bmatrix}
  \frac{p_{11}}{2} & \frac{p_{12}}{2} & \frac{p_{13}}{2} \\
  \frac{p_{12}}{2} & \frac{p_{22}}{2} & \frac{p_{23}}{2} \\
  \frac{p_{13}}{2} & \frac{p_{23}}{2} & \frac{p_{33}}{2}
\end{bmatrix}
= c_1 \begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{bmatrix}
+ c_2 \begin{bmatrix}
  r_1 \\
  r_2 \\
  r_3
\end{bmatrix}.
\]

Therefore, the matrix on the left side has at most rank 2. We have an algebraic statistical model below.

\[
\mathcal{V}(g(p_{11}, p_{12}, p_{13}, p_{22}, p_{23}, p_{33})) \cap \Delta_5,
\]

where

\[
g = \det \begin{bmatrix}
  2p_{11} & p_{12} & p_{13} \\
  p_{12} & 2p_{22} & p_{23} \\
  p_{13} & p_{23} & 2p_{33}
\end{bmatrix},
\]

\[
\Delta_5 = \{(p_{11}, \ldots, p_{33}) \in \mathbb{R}^6_+ | p_{11} + p_{12} + p_{13} + p_{22} + p_{23} + p_{33} = 1\}.\]
Symmetric Matrix Model

**Question**

How to maximize likelihood function

\[ p_{11} \ p_{12} \ p_{13} \ p_{22} \ p_{23} \ p_{33} \]

subject to the algebraic statistical model \( \mathcal{V}(g) \cap \Delta_5 \)?

**Answer.**

Solve the Lagrange likelihood equations

\[
\begin{align*}
F_0 &= p_{11} \lambda_1 + p_{11} \lambda_2 (8p_{22}p_{33} - 2p_{23}^2) - u_{11} = 0 \\
F_1 &= p_{12} \lambda_1 + p_{12} \lambda_2 (2p_{13}p_{23} - 4p_{12}p_{33}) - u_{12} = 0 \\
F_2 &= p_{13} \lambda_1 + p_{13} \lambda_2 (2p_{12}p_{23} - 4p_{13}p_{22}) - u_{13} = 0 \\
F_3 &= p_{22} \lambda_1 + p_{22} \lambda_2 (8p_{11}p_{33} - 2p_{13}^2) - u_{22} = 0 \\
F_4 &= p_{23} \lambda_1 + p_{23} \lambda_2 (2p_{12}p_{13} - 4p_{11}p_{23}) - u_{23} = 0 \\
F_5 &= p_{33} \lambda_1 + p_{33} \lambda_2 (8p_{11}p_{22} - 2p_{12}^2) - u_{33} = 0 \\
F_6 &= g(p_{11}, p_{12}, p_{13}, p_{22}, p_{23}, p_{33}) = 0 \\
F_7 &= p_{11} + p_{12} + p_{13} + p_{22} + p_{23} + p_{33} - 1 = 0
\end{align*}
\]

where

- \( p_{11}, p_{12}, p_{13}, p_{22}, p_{23}, p_{33}, \lambda_1 \) and \( \lambda_2 \) are unknowns
- \( u_{11}, u_{12}, u_{13}, u_{22}, u_{23} \) and \( u_{33} \) are parameters.
Data-Discriminant (By Probabilistic Algorithm)

\[- \mathcal{D}_X \rho = u_{11} u_{12} u_{13} u_{22} u_{23} u_{33} \]

\[- \mathcal{D}_X \infty = (u_{11} + u_{22} + u_{33} + u_{12} + u_{13} + u_{23})(u_{11} + u_{22} + u_{12})(u_{11} + u_{33} + u_{13})(u_{22} + u_{33} + u_{23})(u_{12} + 2u_{22} + u_{23})(u_{13} + 2u_{33} + u_{23})(u_{13} + 2u_{11} + u_{12})(8u_{11} u_{22} u_{33} - 2u_{11} u_{23}^2 - 2u_{12}^2 u_{33} + 2u_{12} u_{13} u_{23} - 2u_{13}^2 u_{22}). \]

\[- \mathcal{D}_X J = -64 u_{11}^5 u_{22}^3 u_{23}^4 + \ldots + u_{13}^4 u_{22}^2 u_{23}^6 \]

1307 terms
3 × 3 Symmetric Matrix Model

Data-Discriminant (By Probabilistic Algorithm)

- $\mathcal{D}_X p = u_{11} u_{12} u_{13} u_{22} u_{23} u_{33}$
- $\mathcal{D}_X \infty = (u_{11} + u_{22} + u_{33} + u_{12} + u_{13} + u_{23})(u_{11} + u_{22} + u_{12})(u_{11} + u_{33} + u_{13})(u_{22} + u_{33} + u_{23})(u_{12} + 2u_{22} + u_{23})(u_{13} + 2u_{33} + u_{23})(u_{13} + 2u_{11} + u_{12})(8u_{11} u_{22} u_{33} - 2u_{11} u_{23}^2 - 2u_{12}^2 u_{33} + 2u_{12} u_{13} u_{23} - 2u_{13}^2 u_{22})$.
- $\mathcal{D}_X J = -64 u_{11}^5 u_{22}^3 u_{23}^4 + \ldots + u_{13}^4 u_{22}^2 u_{23}^6$

1307 terms

Real Root Classification

(Sample points of data-discriminant are computed by RAClib
(M. Safey El Din and E. Schost, 2003; H. Hong and M. Safey El Din, 2012; A. Greuet and M. Safey El Din, 2014))

For $(u_{11}, \ldots, u_{33}) \in \mathbb{R}_0^6$, if $\mathcal{D}_X \infty(u_{11}, \ldots, u_{33}) \neq 0$, then

- $\mathcal{D}_X J(u_{11}, \ldots, u_{33}) > 0 \Rightarrow$ 6 distinct real solutions
- $\mathcal{D}_X J(u_{11}, \ldots, u_{33}) < 0 \Rightarrow$ 2 distinct real (positive) solutions.

Remark. Sign of data-discriminant is NOT enough for classifying positive solutions.

For data $(1; 1; 1; 280264116870825; 295147905179352825856; 1; 1; 34089009205592922038535; 141080698675730650759168; 32898355113670387769001; 141080698675730650759168; 1)$, the system has 6 distinct positive solutions.

For data $(1; 1; 1; 199008; 30; 2022; 1)$, the system has also 6 real solutions but only 2 positive solutions.
3 × 3 Symmetric Matrix Model

Data-Discriminant (By Probabilistic Algorithm)

\[ D_{X_p} = u_{11} u_{12} u_{13} u_{22} u_{23} u_{33} \]
\[ D_{X_{\infty}} = (u_{11} + u_{22} + u_{33} + u_{12} + u_{13} + u_{23})(u_{11} + u_{22} + u_{12})(u_{11} + u_{33} + u_{13})(u_{22} + u_{33} + u_{23})(u_{12} + 2u_{22} + u_{23})(u_{13} + 2u_{33} + u_{23})(u_{13} + 2u_{11} + u_{12})(8u_{11} u_{22} u_{33} - 2u_{11} u_{23}^2 - 2u_{12}^2 u_{33} + 2u_{12} u_{13} u_{23} - 2u_{13}^2 u_{22}). \]
\[ D_{X_{J}} = -64u_{11}^5 u_{22}^3 u_{23}^4 + \ldots + u_{13}^4 u_{22}^2 u_{23}^6 \]

1307 terms

Real Root Classification

(Sample points of data-discriminant are computed by RAGlib
[M. Safey El Din and E. Schost, 2003; H. Hong and M. Safey El Din, 2012; A. Greuet and M. Safey El Din, 2014])

For \((u_{11}, \ldots, u_{33}) \in \mathbb{R}^6_{>0}\), if \(D_{X_{\infty}}(u_{11}, \ldots, u_{33}) \neq 0\), then
- \(D_{X_{J}}(u_{11}, \ldots, u_{33}) > 0 \Rightarrow 6 \) distinct real solutions
- \(D_{X_{J}}(u_{11}, \ldots, u_{33}) < 0 \Rightarrow 2 \) distinct real (positive) solutions.

Remark. Sign of data-discriminant is NOT enough for classifying positive solutions.

- For data \((1, 1, 280264116870825, 295147905179352825856, 1, 3408909205592922038535, 32898355113670387769001, 141080698675730650759168, 141080698675730650759168)\), the system has 6 distinct positive solutions.
- For data \((1, 1, 199008, 30, 2022, 1)\), the system has also 6 real solutions but only 2 positive solutions.
Review

Maximun Likelihood Estimation Problem

Solving Likelihood Equations

Real/Positive Root Classification

Discriminant Variety

Data-Discriminant

Computing Cells of Discriminant Variety

Future work

Main contribution

Future work

Elimination Ideal

Evaluation/Interpolation

More Efficient Algorithm

Data-Discriminants of Likelihood Equations

Jose Israel Rodriguez and Xiaoxian Tang
Thank You for Your Attention!