Outline

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Example of four point Boundary Problem.

Given $f \in C^\infty[a, b]$, find $u \in C^\infty[a, b]$ such that

\[
-u'' = f,
\]

\[
u(0) + u(1/3) = u(1) + u(2/3) = 0.
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- Represented by pair $(-D^2, [0] + [1/3], [1] + [2/3])$.
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- Evaluation functionals: [ ]
Similarly: Regular boundary problem \((T, B)\) for LODE.
Meaning \(\exists\) unique solution \(u \in C^\infty[a, b]\) for all \(f \in C^\infty[a, b]\):

\[
Tu = f, \\
\beta(u) = 0 (\beta \in B).
\]

Algorithm to compute Green's operator \(G\):
\(f \mapsto u\).

However, sometimes we want Green's functions. Why?
- Nice intuition (see below).
- Standard form for solutions.
- Useful for communicating with engineers.

How to extract Green's function from Green's operators?

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Green's Functions

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Intuition behind Green’s Function.

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- Green’s function \( g_\xi \) is solution for \( f = \delta_\xi \).
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- Note that \( g(x, \xi) = g_\xi(x) \) and “\( \delta(x, \xi) = \delta_\xi(x) \)”. 

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- Note that \( g(x, \xi) = g_\xi(x) \) and “\( \delta(x, \xi) = \delta_\xi(x) \)”.

\[
\begin{align*}
\quad u &= \int g_\xi f(\xi) \, d\xi \\
\implies Tu &= \int Tg_\xi f(\xi) \, d\xi = \int \delta_\xi f(\xi) \, d\xi = f \\
\quad \beta(u) &= \int \beta(g_\xi) f(\xi) \, d\xi = 0
\end{align*}
\]
Extraction of Green’s functions: $G \leadsto g(x, \xi)$
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Well known for “classical case” [3].
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Sometimes one needs **Stieltjes boundary conditions**:

- More than two evaluation points \( \rightarrow \) **multipoint BP.**
- Derivatives of arbitrary order \( \rightarrow \) **ill-posed BP.**
- Global terms in the form of definite integrals \( \rightarrow \) **nonlocal BP.**
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Are there Green’s functions $g(x, \xi)$ for such Stieltjes BPs?
Can we extract it from $G$?
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  - Isomorphic rings (alternative normal forms).
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  - Standard integro-differential operator ring $\mathcal{F}_\Phi[\partial, \int]$
  - Equitable operator ring $\mathcal{F}[\partial, \int_\Phi]$.
  - Isomorphic rings (alternative normal forms).
- Later on will have $\mathcal{F} = C^\infty(\mathbb{R})$ again.
Stieltjes Boundary Conditions

**Definition**

The elements of right ideal $|\Phi\rangle = \Phi \cdot \mathcal{F}_\Phi[\partial, \int]$ are called Stieltjes boundary conditions.
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Normal forms:

\[
\beta(u) = \alpha_1(u^{k_1}) + \cdots + \alpha_r(u^{k_r}) + \int_{\beta_1}^\gamma + \cdots + \int_{\beta_s}^\gamma
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Boundary operators:

Two-sided ideal $(\Phi)$
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Boundary operators:
Two-sided ideal $(\Phi) =$ left $\mathcal{F}$-module generated by $|\Phi)$

Standard decomposition:
$$\mathcal{F}_\Phi[\partial, \int] = \mathcal{F}[\partial] + \mathcal{F}[\int] + (\Phi)$$
Equitable operator ring used for extracting Green’s function via

\[ \iota : \mathcal{F}_\Phi[\partial, \int] \to \mathcal{F}[\partial, \int_\Phi], \]

the translation isomorphism defined as follows:
Equitable Operators

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the translation isomorphism defined as follows:

- Fixes \( \mathcal{F} \) and \( \partial \).
Equitable operator ring used for extracting Green’s function via

$$\nu: \mathcal{F}_\Phi[\partial, \int] \rightarrow \mathcal{F}[\partial, \int_\Phi],$$

the translation isomorphism defined as follows:

- Fixes $\mathcal{F}$ and $\partial$.
- Translate characters by $\nu(\varphi) = \text{id} - \int_\varphi \partial$. 
Equitable operators ring used for extracting Green’s function via

\[ \iota : \mathcal{F}_\Phi [\partial, \int] \rightarrow \mathcal{F}[\partial, \int_\Phi], \]

the translation isomorphism defined as follows:

- Fixes \( \mathcal{F} \) and \( \partial \).
- Translate characters by \( \iota(\varphi) = \text{id} - \int \varphi \partial \).
- Translate back integrals by \( \iota^{-1}(\int \varphi) = (\text{id} - \varphi)\int \).
Interval $J \subset \mathbb{R}$ containing all evaluation points.
Extracting Green’s Function

Interval \( J \subseteq \mathbb{R} \) containing all evaluation points.

**Theorem**

The Green’s function of any regular Stieltjes boundary problem with \( m \) evaluations \( \alpha_1, \ldots, \alpha_m \) has the form \( g(x, \xi) = \tilde{g}(x, \xi) + \hat{g}(x, \xi) \), where the functional part \( \tilde{g} \in C(J^2) \) is defined by the \( 2(m - 1) \) case branches

\[
\xi \in [\alpha_i, \alpha_{i+1}] \quad x \leq \xi,
\]

\[
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\]

while the distributional part \( \hat{g}(x, \xi) \) is an \( \mathcal{F} \)-linear combination of the \( \delta(\xi - \alpha_i) \) and their derivatives.
Illustration of Proof

Consider an example:

- **Green’s operator**

\[
G = x \frac{d}{dx} - \int x + x[1] \int x - x[1] \int + e^x[1] \partial \in \mathcal{F}_\Phi[\partial, \int].
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Illustration of Proof

Consider an example:

- Green’s operator
  \[ G = x \int - \int x + x[1] \int x - x[1] \int + e^x [-1] \partial \in F_\Phi [\partial, \int]. \]
- Translate to equitable ring by \([\alpha] \int \rightarrow \int_0 - \int_\alpha.\)
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- **Yields** \[ G = x \int_0 x - x \int_1 x - \int_0 x + x \int_1 + e^x [-1] \partial \in \mathcal{F}[\partial, \int_\Phi]. \]
Consider an example:

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  \[ G = x \int - x \int x + x [1] \int x - x [1] \int + e^x [-1] \partial \in \mathcal{F}_\Phi[\partial, \int]. \]
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- Extract \( \tilde{g} \) by \( f \int_\alpha g = f(x) g(\xi) [\alpha \leq \xi][\xi \leq x] \]
  \[ - f(x) g(\xi) [\xi \leq \alpha][x \leq \xi]. \]
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- Extract \( \hat{g} \) by \[ f [\alpha] \partial^i = (-1)^i f(x) \delta^{(i)}(\xi - \alpha). \]
Illustration of Proof

Consider an example:

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  \[ G = x \int - \int x + x [1] \int x - x [1] \int + e^x [−1] \partial \in \mathcal{F}_\Phi[\partial, \int]. \]
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- Extract \(\hat{g}\) by \(f [\alpha] \partial^i = (-1)^i f(x) \delta^{(i)}(\xi - \alpha).\)
- Resulting Green’s function:
  \[ \tilde{g}(x, \xi) = \begin{cases} (x - 1)x & \text{for } 0 \leq \xi \leq x \leq 1, \\ x(\xi - 1) & \text{for } 0 \leq x \leq \xi \leq 1. \end{cases} \]
  \[ \hat{g}(x, \xi) = -e^x \delta'(\xi + 1) \]
Back to our first example:

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- Green’s operator (computed using \textsc{INTDIFFOP} by A. Korporal):

\[G = x \int - \int x + (-5/24 + x/4)[1/3] \int + (5/8 - 3x/4)[1/3] \int x + (1/8 - 3x/4)[1] \int x + (1/12 - x/2)[2/3] \int x + (-1/8 + 3x/4)[2/3] \int x\]
Four Point Boundary Problem

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\]

- Green’s function (computed using extra code)

\[
g(x, \xi) = \begin{cases} 
(3/4)x\xi - (5/8)\xi & : 0 \leq \xi \leq 1/3, \xi \leq x \\
(3/4)x\xi + (3/8)\xi - x & : 0 \leq \xi \leq 1/3, x \leq \xi \\
(3/2)x\xi - (5/4)\xi - (1/4)x + 5/24 & : 1/3 \leq \xi \leq 2/3, \xi \leq x \\
(3/2)x\xi - (1/4)\xi - (5/4)x + 5/24 & : 1/3 \leq \xi \leq 2/3, x \leq \xi \\
(3/4)x\xi - (9/8)\xi + (1/4)x + 1/8 & : 2/3 \leq \xi \leq 1, \xi \leq x \\
(3/4)x\xi - (1/8)\xi - (3/4)x + 1/8 & : 2/3 \leq \xi \leq 1, x \leq \xi 
\end{cases}
\]
Graph of its Green’s Function
Example with three evaluations, nonlocal part and higher-order derivative:

\[
\begin{align*}
    u'' - u &= f, \\
    u'''(-1) - \int_{0}^{1} u(\xi) \xi \, d\xi &= 0, \\
    u'(-1) - u''(1) + \int_{-1}^{1} u(\xi) \, d\xi &= 0,
\end{align*}
\]
Nonclassical Boundary Problem

Example with three evaluations, nonlocal part and higher-order derivative:

\[
\begin{align*}
  u'' - u &= f, \\
  u'''(-1) - \int_0^1 u(\xi) \xi \, d\xi &= 0, \\
  u'(-1) - u''(1) + \int_{-1}^1 u(\xi) \, d\xi &= 0,
\end{align*}
\]

Green’s operator (with \( \sigma := 2(2e - 3)(e - 1) \) for brevity):

\[
\begin{align*}
  \sigma G &= \sigma/2 \left( e^x \int e^{-x} - e^{-x} \int e^x \right) \\
  &+ 2(-e^{x+3} + e^{x+2} - e^{x+1} + e^{-x+2} - e^{-x+1})([-1] \partial + [1] \int x) \\
  &+ (e - 1)(-e^{x+2} - 2e^{x+1} + e^{-x+1})([-1] \int + [1] \int) \\
  &+ (3e^{x+2} - e^{x+1} - 3e^{-x+1} + 3e^{-x})[1] \int e^x \\
  &+ (2e^{x+2} - 3e^{x+1})(e^{-1}[-1] \int e^{-x} + e [-1] \int e^x) \\
  &+ (-e^{x+3} - e^{x+2} + 2e^{x+1} + e^{-x+2} - e^{-x+1})[1]
\end{align*}
\]
Green’s Function for Nonclassical Problem

Green’s function

\[ \hat{g}(x, \xi) = \left( -e^x + 3 - e^x + 2 + 2e^x + 1 + e^{-x} + 2 - e^{-x} + 1 \right) \delta(\xi - 1) + 2 \left( -e^x + 3 + e^x + 2 - e^x + 1 + e^{-x} + 2 - e^{-x} + 1 \right) \delta'(\xi - 1) \]

Functional part

\[ \tilde{g}(x, \xi) = \begin{cases} -1 \leq \xi \leq 0 & \xi \leq x \\
3e^x + 2 + \xi + 3e^x - \xi - 2e^x + 1 - \xi - 2e^x + 2e^x + \xi + 3e^x + 2e^{-x} + 1 - \xi + e^{-x} + 2e^x - \xi + 2e^{-x} + 2e^x - \xi + e^{-x} + 3e^x + 3e^x - \xi + e^{-x} + 5e^x + 1 + e^{-x} + 2e^x + 2e^{-x} + 2e^x - \xi + e^{-x} + 2e^x + e^{-x} + 1 + e^x + 3e^x + 2e^{-x} + 1 - \xi + 3e^x + 3e^x - \xi + e^{-x} + 2e^x + 2e^x - \xi + e^{-x} + 2e^x + e^{-x} + 1 - \xi + e^{-x} + 3e^x + 3e^x - \xi + e^{-x} + 2e^x + 2e^{-x} + 2e^x - \xi + e^{-x} + e^x + 3e^x + 3e^x - \xi + e^{-x} + 2e^x + 2e^{-x} + 2e^x - \xi + e^{-x} + e^{-x} + 1 - \xi + e^x + 3e^x + 3e^x - \xi + e^{-x} + 2e^x + 3e^x - \xi + e^{-x} + 2e^x + 2e^{-x} + 2e^x - \xi + e^{-x} + 2e^x + e^{-x} + 1 - \xi + e^{-x} + 3e^x + 3e^x - \xi + e^{-x} + 2e^x + 2e^{-x} + 2e^x - \xi + e^{-x} + e^x + 3e^x + 3e^x - \xi + e^{-x} + 2e^x + 2e^{-x} + 2e^x - \xi + e^{-x} + e^{-x} + 1 - \xi + e^x + 3e^x + 3e^x - \xi + e^{-x} + 2e^x + 2e^{-x} + 2e^x - \xi + e^{-x} + e^{-x} + 1 - \xi + e^x + 3e^x + 3e^x - \xi + e^{-x} + 2e^x + 3e^x - \xi + e^{-x} + 2e^x + 2e^{-x} + 2e^x - \xi + e^{-x} + e^{-x} + 1 - \xi + e^x + 3e^x + 3e^x - \xi + e^{-x} + 2e^x + 2e^{-x} + 2e^x - \xi + e^{-x} + e^{-x} + 1 \end{cases} \]

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Green’s function

- Distributional part

\[
\sigma \hat{g}(x, \xi) = \left( -e^{x+3} - e^{x+2} + 2e^{x+1} + e^{-x+2} - e^{-x+1} \right) \delta(\xi - 1) \\
+ 2 \left( -e^{x+3} + e^{x+2} - e^{x+1} + e^{-x+2} - e^{-x+1} \right) \delta'(\xi - 1)
\]
Green’s Function for Nonclassical Problem

Green’s function

- **Distributional part**

\[
\sigma \hat{g}(x, \xi) = (-e^{x+3} - e^{x+2} + 2e^{x+1} + e^{-x+2} - e^{-x+1}) \delta(\xi - 1)
+ 2 (-e^{x+3} + e^{x+2} - e^{x+1} + e^{-x+2} - e^{-x+1}) \delta'(\xi - 1)
\]

- **Functional part**

\[
\tilde{g}(x, \xi) = \begin{cases} 
-1 \leq \xi \leq 0 \\
\xi \leq x \\
-1 \leq \xi \leq 0 \\
x \leq \xi \\
0 \leq \xi \leq 1 \\
\xi \leq x \\
0 \leq \xi \leq 1 \\
x \leq \xi 
\end{cases}
\begin{align*}
&= 3e^{x+2+\xi} + 3e^{x-\xi} - 2e^{x+1-\xi} - 2e^{3+x+\xi} \\
&\quad + e^{3+x} + e^{-x+1} + e^{x+2} - e^{-x+2} - 2e^{x+1} \\
&= -2e^{x+1} + 2e^{-x+2+\xi} - 5e^{-x+1+\xi} - 2e^{x+2-\xi} \\
&\quad - 2e^{3+x+\xi} + 3e^{-x+\xi} + e^{-x+1} + e^{x+2} \\
&\quad + e^{3+x} + 3e^{x+1-\xi} + 3e^{x+2+\xi} - e^{-x+2} \\
&= -2e^{3+x+\xi} - 2e^{-x+1+\xi} + 2e^{x+2+\xi} + 2e^{-x+2+\xi} \\
&\quad - 2e^{x+1+\xi} + 3e^{x+2+\xi} + 3e^{-x-\xi} - 5e^{x+1-\xi} \\
&\quad + 2e^{-x+1+\xi} - e^{x+1+\xi} - 2e^{-x+2+\xi} + 2e^{x+2-\xi} \\
&\quad - e^{3+x} - e^{-x+1} - e^{x+2} + e^{-x+2} + 2e^{x+1} \\
&= -2e^{3+x+\xi} - 2e^{-x+1+\xi} + 2e^{x+2+\xi} + 2e^{-x+2+\xi} \\
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&\quad - e^{-x+1} - e^{x+2} + e^{-x+2} + 2e^{x+1} \\
&\quad - 3e^{-x+1+\xi} - e^{x+1+\xi}
\end{align*}
\]
Graph of Functional Part of its Green’s Function
From Green’s operator to Green’s function:

Form of Green’s functions:

\[ g(x, \xi) = \tilde{g}(x, \xi) + \hat{g}(x, \xi). \]

Functional part \( \tilde{g}(x, \xi) \) defined by \( 2(m-1) \) case branches.

Distributional part \( \hat{g}(x, \xi) \) is \( F \)-linear combination of \( \delta(\xi - \alpha_i) \) and their derivatives.
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In this paper: $\mathcal{F} = \mathcal{C}^\infty(\mathbb{R})$. 

Need algebraic structures where these Green's functions "live":

- **Functional part** $g(x, \xi)$: Ring $\mathbb{F} \otimes \mathbb{F}$ is sufficient.
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A. Korporal, G. Regensburger, and M. Rosenkranz.
Also presented as a poster at ISSAC '10.

M. Rosenkranz and G. Regensburger.
Integro-differential polynomials and operators.

M. Rosenkranz and G. Regensburger.
Solving and factoring boundary problems for linear ordinary differential equations in differential algebras.
M. Rosenkranz. 
A new symbolic method for solving linear two-point boundary value problems on the level of operators. 

I. Stakgold and M. Holst. 
*Green’s functions and boundary value problems.* 