Gröbner bases of toric ideals and their application

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Contents

Gröbner bases and toric ideals

Gröbner bases

- A "very good" set of polynomials
- keyword: division of a polynomial
  (by several polynomials in n variables.)
- invented by B. Buchberger in 1965.
  ("standard bases" H. Hironaka in 1964.)
- Elimination Theorem for systems of polynomial equations
- implemented in a lot of mathematical software
  `Mathematica`, `Maple`,
  `Macauley2`, `Singular`, `CoCoA`, `Risa/Asir`, ...

Gröbner bases and toric ideals

Toric ideals

- Prime ideals generated by binomials
- Gröbner bases of toric ideals have a lot of application
  - commutative algebra, algebraic geometry
  - triangulations of convex polytopes
  - integer programming
  - contingency tables (statistics)
  - · · ·

System of linear equations

Example

$$\begin{cases} f_1 = x_1 + x_3 + 3x_4 = 0 \\ f_2 = x_2 - x_3 - 2x_4 = 0 \\ f_3 = 2x_1 + 3x_2 - x_3 = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 2 & 3 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & -3 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f_3 = 2f_1 + 3f_2.$$

Division?

$$
\begin{array}{r}
x +1 \\
x - 1 \overline{\smash{)}\; x^2 \phantom{{}-x}} \\
\underline{x^2 \phantom{{}} -x} \\
x \\
\underline{x \phantom{{}} -1} \\
1
\end{array}
$$

For example, which monomial in

$$f = x_1^2 + 2x_1 x_2 x_3 - 3x_1 + x_3^5 + 5$$

shoud be the largest?

Monomial order

Definition

$\mathcal{M}_n$: set of all monomials in the variables $x_1, \ldots, x_n$
A total order $<$ on $\mathcal{M}_n$ is called a monomial order
if $<$ satisfies the following:

1. $u \in \mathcal{M}_n, \ u \neq 1 \implies 1 < u$.
2. $u, v, w \in \mathcal{M}_n, \ u < v \implies uw < vw$.

$$
x - 1 \enspace \overline{) \ \begin{array}{l} x \enspace +1 \\ x^2 \\ \underline{x^2 \enspace -x} \\ \quad x \\ \quad \underline{x \enspace -1} \\ \qquad 1 \end{array}}
$$

$$
-1 + x \enspace \overline{) \ \begin{array}{l} -x^2 \enspace -x^3 \\ x^2 \\ \underline{x^2 \enspace -x^3} \\ \quad x^3 \\ \underline{\quad x^3 \enspace -x^4} \\ \qquad x^4 \\ \qquad \vdots \end{array}}
$$

Lexicographic order

Example (Lexicographic order ($x_1 > \cdots > x_n$))

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} >_{\text{lex}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \overset{\text{def}}{\Longleftrightarrow}$$

$$a_1 > b_1$$
$$\text{or}$$
$$a_1 = b_1 \text{ and } a_2 > b_2$$
$$\text{or}$$
$$a_1 = b_1, a_2 = b_2, \text{ and } a_3 > b_3$$
$$\text{or}$$
$$\vdots$$

For example,

$$x_1 >_{\text{lex}} x_2^{100} x_3$$
$$x_1^2 x_2^2 x_5 >_{\text{lex}} x_1^2 x_2 x_3 x_4$$

(Degree) Reverse lexicographic order

> ### Example (Reverse lexicographic order ($x_1 > \cdots > x_n$))
>
> $$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} >_{\text{revlex}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \overset{\text{def}}{\iff}$$
>
> $$\sum_{i=1}^{n} a_i > \sum_{i=1}^{n} b_i$$
> $$\text{or}$$
> $$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \quad \text{and} \quad a_n < b_n$$
> $$\text{or}$$
> $$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i, \quad a_n = b_n \quad \text{and} \quad a_{n-1} < b_{n-1}$$
> $$\vdots$$

For example,

$$x_1 <_{\text{revlex}} x_2^{100} x_3$$
$$x_1^2 x_2^2 x_5 <_{\text{revlex}} x_1^2 x_2 x_3 x_4$$

Weight order

Example (Weight order $>_{\mathbf{w}}$)

$\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n_{\geq 0}$

$<$: a monomial order (for "tie break")

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} >_{\mathbf{w}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \overset{\mathrm{def}}{\Longleftrightarrow}$$

$$\sum_{i=1}^{n} a_i w_i > \sum_{i=1}^{n} b_i w_i$$
$$\text{or}$$
$$\sum_{i=1}^{n} a_i w_i = \sum_{i=1}^{n} b_i w_i \text{ and } x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} > x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$$

Remark

If $n = 1$, then a monomial order is unique.
In fact, if $<$ is a monomial order on monomials in x_1, then

$$x_1 > 1 \Longrightarrow x_1^2 > x_1 \Longrightarrow x_1^3 > x_1^2 \Longrightarrow x_1^4 > x_1^3 \Longrightarrow \cdots$$

Hence, we have

$$1 < x_1 < x_1^2 < x_1^3 < \cdots$$

If $n \geq 2$, then there exist infinitely many monomial orders.

Initial monomial

$K[x_1, \ldots, x_n]$: polynomial ring over a field K (e.g., $K = \mathbb{C}$)

- The set of all polynomials in variables $x_1, \ldots, x_n$
  with coefficients in K.

Fix a monomial order $<$ on $K[x_1, \ldots, x_n]$.

$0 \neq f \in K[x_1, \ldots, x_n]$

$\mathrm{in}_<(f)$: the largest monomial among the monomials in f

the initial monomial of f

Example

$>_{\mathrm{lex}}$: lexicographic order $(x_1 > \cdots > x_5)$
$>_{\mathrm{revlex}}$: reverse lexicographic order $(x_1 > \cdots > x_5)$

$$f = x_1^2 + 2x_1x_2x_3 - 3x_1 + x_3^5 + 5$$

Then, $\mathrm{in}_{>_{\mathrm{lex}}}(f) = x_1^2$ and $\mathrm{in}_{>_{\mathrm{revlex}}}(f) = x_3^5$

Division algorithm

Theorem (Division algorithm)

$<$: *monomial order*
$0 \neq f, g_1, g_2, \ldots, g_s \in K[x_1, \ldots, x_n]$

Then, there exist $f_1, \ldots, f_s, r \in K[x_1, \ldots, x_n]$ such that

- $f = f_1 g_1 + f_2 g_2 + \cdots + f_s g_s + r.$
- *If $r \neq 0$, then any monomial in r is divided by none of* $\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s)$.
- *If $f_i \neq 0$, then $\mathrm{in}_<(f) \geq \mathrm{in}_<(f_i g_i)$.*

r is called a remainder of f w.r.t. $\{g_1, \ldots, g_s\}$.

Example

$n = 2$, lexicographic order ($x > y$)

$$
\begin{array}{r}
x \quad +y \\
 1 \\
\hline
xy-1 \quad) \quad x^2y \quad +xy^2 \quad +y^2 \\
y^2-1 \quad \underline{x^2y \quad -x} \\
\hline
xy^2 \quad +x \quad +y^2 \\
\underline{xy^2 \quad -y} \\
\hline
x \quad +y^2 \quad +y \\
\underline{y^2 \quad -1} \\
\hline
x \quad +y \quad +1
\end{array}
$$

$$x^2y + xy^2 + y^2 = (x+y)(xy-1) + 1 \cdot (y^2-1) + x + y + 1$$

Example

$n = 2$, lexicographic order ($x > y$)

$$
\begin{array}{r|rrrr}
& & x & & +1 \\
\hline
xy - 1 \quad) & x^2y & +xy^2 & +y^2 & \\
y^2 - 1 & x^2y & -x & & \\
\hline
& & xy^2 & +x & +y^2 \\
& & xy^2 & -x & \\
\hline
& & & 2x & +y^2 \\
& & & & y^2 & -1 \\
\hline
& & & 2x & & +1 \\
\end{array}
$$

$x^2y + xy^2 + y^2 = x \cdot (xy - 1) + (x + 1) \cdot (y^2 - 1) + 2x + 1$

Ideals of polynomial rings

Definition

Let $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$. Then, we define

$$\langle f_1, \ldots, f_s \rangle := \{h_1 f_1 + \cdots + h_s f_s \mid h_i \in K[x_1, \ldots, x_n]\}$$

ideal generated by $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$.

Proposition

$f_1, \ldots, f_s, g_1, \ldots, g_t \in K[x_1, \ldots, x_n]$
If $\langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$, then

$$\left\{ \begin{array}{c} f_1 = 0 \\ \vdots \\ f_s = 0 \end{array} \right. \quad and \quad \left\{ \begin{array}{c} g_1 = 0 \\ \vdots \\ g_t = 0 \end{array} \right. \quad have \ the \ same \ solutions.$$

Gröbner bases

Fix a monomial order $<$.

$f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$

$I = \langle f_1, \ldots, f_s \rangle \subset K[x_1, \ldots, x_n]$: ideal

Definition

$in_<(I) := \langle in_<(f) \mid 0 \neq f \in I \rangle$ the initial ideal of I

Definition

$\{g_1, \ldots, g_t\} \subset I$ is a Gröbner basis of I w.r.t. $<$

$\overset{\text{def}}{\Longleftrightarrow} in_<(I) = \langle in_<(g_1), \ldots, in_<(g_t) \rangle$

$\Longleftrightarrow$ For any nonzero element $f \in I$,

     $in_<(f)$ is divided by $in_<(g_i)$ for some i.

Example

Example

$f_1 = x^2 + y^2$, $f_2 = xy$
$I = \langle f_1, f_2 \rangle$, lexicographic order ($x > y$)

Then, $\{f_1, f_2\}$ is not a Gröbner basis of I since

- $f = yf_1 - xf_2 = y(x^2 + y^2) - x \cdot xy = y^3$ belongs to I.
- So, $\mathrm{in}_<(f) = y^3$ belongs to $\mathrm{in}_<(I)$.
- $\mathrm{in}_<(f) = y^3$ is divided by
  neither $\mathrm{in}_<(f_1) = x^2$ nor $\mathrm{in}_<(f_2) = xy$.

$\longrightarrow$ In this case, $\{f_1, f_2, f\}$ is a Gröbner basis of I.

Gröbner bases

Basic properties of a Gröbner basis $\mathcal{G}$ of an ideal I:

- Always exists.
- Not unique.
- $\mathcal{G}$ generates I.
- For any nonzero polynomial $f \in K[x_1, \ldots, x_n]$,
  the remainder of f with respect to $\mathcal{G}$ is unique.
- For any nonzero polynomial $f \in K[x_1, \ldots, x_n]$,

  $f \in I \Longleftrightarrow$ the remainder of f with respect to $\mathcal{G}$ is 0.

Special Gröbner bases

$\mathcal{G} = \{g_1, \ldots, g_t\}$: a Gröbner basis of an ideal I

Definition

$\mathcal{G}$ is called minimal if each g_i is monic and

- $\mathrm{in}_<(g_j)$ is not divided by $\mathrm{in}_<(g_i)$ if $i \neq j$.

$\mathcal{G}$ is minimal $\Longleftrightarrow \mathcal{G} \setminus \{g_i\}$ is not a Gröbner basis for $\forall i$.

Definition

$\mathcal{G}$ is called reduced if each g_i is monic and

- Any monomial in g_j is not divided by $\mathrm{in}_<(g_i)$ if $i \neq j$.

If we fix an ideal and a monomial order,
then the reduced Gröbner basis exists (and unique).

Example

Example

$I = \langle x_1 - x_2, x_1 - x_3 \rangle$

$<_{\mathrm{lex}}$: lexicographic order

$\mathrm{in}_{<_{\mathrm{lex}}}(I) = \langle x_1, x_2 \rangle$

$\{x_1 - x_2, x_1 - x_3, x_2 - x_3\}$: Gröbner basis , not minimal

$\{x_1 - x_2, x_2 - x_3\}$: minimal Gröbner basis , not reduced

$\{x_1 - x_3, x_2 - x_3\}$: reduced Gröbner basis

S-polynomial

$(0 \neq) \ f, g \in K[x_1, \ldots, x_n]$
$m := \mathrm{LCM}(\mathrm{in}_<(f), \mathrm{in}_<(g))$
$f = c_f \cdot \mathrm{in}_<(f) + \cdots$
$g = c_g \cdot \mathrm{in}_<(g) + \cdots$
Then, we define *S*-polynomial of f and g by

$$S(f,g) := \frac{m}{c_f \cdot \mathrm{in}_<(f)} f - \frac{m}{c_g \cdot \mathrm{in}_<(g)} g.$$

Example

$f = x_1 x_4 - x_2 x_3, \ g = 2x_4 x_7 - x_5 x_6, \ <_{\mathrm{lex}}$: lexicographic order

$$
\begin{aligned}
S(f,g) &= \frac{x_1 x_4 x_7}{x_1 x_4}(x_1 x_4 - x_2 x_3) - \frac{x_1 x_4 x_7}{2x_4 x_7}(2x_4 x_7 - x_5 x_6) \\
&= -x_2 x_3 x_7 + \frac{1}{2} x_1 x_5 x_6
\end{aligned}
$$

Buchberger criterion

Theorem

$I = \langle g_1, \ldots, g_t \rangle \subset K[x_1, \ldots, x_n]$
Then, $\{g_1, \ldots, g_t\}$ is a Gröbner basis of I
$\iff$
The remainder of $S(g_i, g_j)$ with respect to $\{g_1, \ldots, g_t\}$ is 0
for all $i \neq j$.

Buchberger algorithm

Input: $g_1, \ldots, g_t \in K[x_1, \ldots, x_n]$, monomial order $<$

Output: A Gröbner basis $\mathcal{G}$ of $I = \langle g_1, \ldots, g_t \rangle \subset K[x_1, \ldots, x_n]$ w.r.t. $<$

Step 1. $\mathcal{G} = \{g_1, \ldots, g_t\}$

Step 2. Apply Buchberger criterion to $\mathcal{G}$.

Step 3. If it satisfies the condition of the criterion,
then $\mathcal{G}$ is a Gröbner basis .
If not, then there exists a nonzero remainder.
Add it to $\mathcal{G}$ and back to step 2.

Example

Example

$f = x_1 x_4 - x_2 x_3$, $g = x_4 x_7 - x_5 x_6$

$I = \langle f, g \rangle$

lexicographic order $(x_1 > \cdots > x_n)$

$\mathcal{G} = \{f, g\}$

$S(f, g) = x_7 f - x_1 g = x_1 x_5 x_6 - x_2 x_3 x_7$

A remainder of $S(f, g)$ w.r.t. $\{f, g\}$ is $x_1 x_5 x_6 - x_2 x_3 x_7 =: h$

$\mathcal{G} = \{f, g, h\}$

$S(f, h) = x_5 x_6 f - x_4 h = x_2 x_3 x_4 x_7 - x_2 x_3 x_5 x_6 = x_2 x_3 g$

$S(g, h) \longrightarrow$ remainder w.r.t. $\mathcal{G}$ is zero.

Thus, $\{f, g, h\}$ is a Gröbner basis of I.

Improving the efficiency of Buchberger algorithm

Proposition

$(0 \neq)\ f, g \in K[x_1, \ldots, x_n]$
$\mathrm{GCD}(\mathrm{in}_<(f), \mathrm{in}_<(g)) = 1$
$\implies$ *the remainder of $S(f, g)$ with respect to $\{f, g\}$ is 0.*

- Strategies for selecting S-polynomials
  Sugar Selection Strategy   (*in Proc. ISSAC 1991*)
  $\longrightarrow$ first implemented in CoCoA
- Homogenization
  (to avoid unnecessary intermediate coefficient swells)
- ...

Elimination theorem

Theorem

$0 < m < n$: integers

$<$: monomial order on $K[x_1, \ldots, x_n]$

$\{0\} \neq I \subset K[x_1, \ldots, x_n]$: ideal

$\mathcal{G}$: Gröbner basis of I w.r.t. $<$

If $<$ satisfies the condition

$$g \in \mathcal{G}, \mathrm{in}_<(g) \in K[x_1, \ldots, x_m] \Longrightarrow g \in K[x_1, \ldots, x_m],$$

then $\mathcal{G} \cap K[x_1, \ldots, x_m]$ is a Gröbner basis of
$I \cap K[x_1, \ldots, x_m]$.

Example

In order to solve the system of equations

$$\begin{cases} f_1 = x^2 + y + z - 1 = 0, \\ f_2 = x + y^2 + z - 1 = 0, \\ f_3 = x + y + z^2 - 1 = 0, \end{cases}$$

we compute a Gröbner basis of the ideal $\langle f_1, f_2, f_3 \rangle$ with respect to the lexicographic order $<_{\mathrm{lex}}$ ($x > y > z$):

$$\begin{cases} g_1 = x + y + z^2 - 1, \\ g_2 = y^2 - y - z^2 + z, \\ g_3 = 2yz^2 + z^4 - z^2, \\ g_4 = z^6 - 4z^4 + 4z^3 - z^2. \end{cases}$$

$<_{\mathrm{lex}}$ satisfies the condition in Elimination Theorem.
Since $\langle f_1, f_2, f_3 \rangle = \langle g_1, g_2, g_3, g_4 \rangle$ holds, $f_1 = f_2 = f_3 = 0$ and $g_1 = g_2 = g_3 = g_4 = 0$ have the same solutions.

Toric ideals

$\mathbb{Z}^{d \times n}$: the set of all $d \times n$ integer matrices
$A = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbb{Z}^{d \times n}$

- A is called a configuration
  if $\exists \mathbf{w} \in \mathbb{R}^d$ s.t. $\mathbf{w} \cdot \mathbf{a}_1 = \cdots = \mathbf{w} \cdot \mathbf{a}_n = 1$.
- We usually assume that A is a configuration.

K: field (e.g., $K = \mathbb{C}$)
$K[X] = K[x_1, \ldots, x_n]$: poly. ring in n variables over K

- $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{Z}_{\geq 0}^n \Longrightarrow X^{\mathbf{u}} = x_1^{u_1} \cdots x_n^{u_n}$

$$
\begin{aligned}
I_A &= \left\langle X^{\mathbf{u}} - X^{\mathbf{v}} \in K[X] \mid \mathbf{u}, \mathbf{v} \in \mathbb{Z}_{\geq 0}^n, \ A\mathbf{u} = A\mathbf{v} \right\rangle \\
&= \left\langle X^{\mathbf{u}^+} - X^{\mathbf{u}^-} \in K[X] \mid \mathbf{u} \in \mathbb{Z}^n, \ A\mathbf{u} = \mathbf{0} \right\rangle
\end{aligned}
$$

Toric ideal of A

Example

Example

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$I_A = \langle x_1 x_5 - x_2 x_4, \ x_1 x_6 - x_3 x_4, \ x_2 x_6 - x_3 x_5 \rangle$$

(For example, $x_1 x_5 - x_2 x_4 \in I_A$ since $A \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \mathbf{0}$.)

$$K[A] = K[t_1 t_3, t_1 t_4, t_1 t_5, t_2 t_3, t_2 t_4, t_2 t_5] \cong K[X]/I_A$$

Basic properties

Properties of toric ideals:

- prime ideal
- The reduced Gröbner basis of I_A consists of binomials.
- A is a configuration
  $\iff I_A$ is homogeneous w.r.t. a usual grading
- $A \in \mathbb{Z}_{\geq 0}^n$
  $\implies I_A$ is homogeneous w.r.t. some positive grading
- If each $\mathbf{a}_i$ is a nonnegative integer vector, then

$$I_A = \langle x_1 - T^{\mathbf{a}_1}, \ldots, x_n - T^{\mathbf{a}_n} \rangle \cap K[X].$$

So, we can compute a Gröbner basis of I_A by Elimination theorem.
However, this method is not effective in practice.

Algorithm computing generators of I_A

Lemma

$J \subset K[X]$: *homogeneous ideal*
$<$: *reverse lexicographic order*
$\mathcal{G}$: *the reduced Gröbner basis of J w.r.t. $<$*
$(J : x_n^\infty) := \{f \in K[X] \mid \exists r \in \mathbb{N} \text{ s.t. } x_n^r f \in J\}$
Then, a GB of $(J : x_n^\infty)$ w.r.t. $<$ is obtained by dividing
each $g \in \mathcal{G}$ by the highest power of x_n that divides g.

Proposition

$A \in \mathbb{Z}^{d \times n}$: *configuration*
B: *lattice basis of $\{\mathbf{u} \in \mathbb{Z}^n \mid A\mathbf{u} = \mathbf{0}\}$*
$J := \left\langle X^{\mathbf{u}^+} - X^{\mathbf{u}^-} \mid \mathbf{u} \in B \right\rangle$
Then $I_A = (J : (x_1 \cdots x_n)^\infty) = ((\cdots (J : x_1^\infty) : x_2^\infty) \cdots) : x_n^\infty)$

Three breakthroughs

Toric ideals

0. Commutative algebra

- Toric ideals have been studied by commutative algebraists for a long time.

- For example,

  📄 J. Herzog
  Generators and relations of abelian semigroups and semigroup rings
  Manuscripta Math., **3** (1970), 175 – 193.

  is an early reference in commutative algebra.

Three breakthroughs

1. Integer programming

📄 P. Conti and C. Traverso
Buchberger algorithm and integer programming
in Proceedings of AAECC-9 (New Orleans)
Springer LNCS **539** (1991), 130 – 139.

Three breakthroughs

2. Triangulations of convex polytopes.

📄 I. M. Gel'fand, A. V. Zelevinskii and M. M. Kapranov
Hypergeometric functions and toral manifolds
Functional Analysis and Its Applications, **23** (1989),
94 – 106.

📄 B. Sturmfels
Gröbner bases of toric varieties
Tôhoku Math. J. **43** (1991), 249 – 261.

Three breakthroughs

3. Conditional test of contingency tables
   (Markov chain Monte Carlo method)

📄 P. Diaconis and B. Sturmfels
Algebraic algorithms for sampling from conditional
distributions
Annals of Statistics, **26** (1998), 363 – 397.
(Received June 1993; revised April 1997.)

Three breakthroughs

One can study three breakthroughs in

📕 B. Sturmfels
Gröbner bases and convex polytopes
Amer. Math. Soc., Providence, RI, 1995.

See also

📕 D. Cox, J. Little and D. O'Shea
Using algebraic geometry
GTM **185**, Springer, Berlin, 1998.

📕 T. Hibi (Ed.)
Gröbner bases –Statistics and Software Systems–
Springer, 2013.

B.1. Integer programming

Example (CLO, "Using algebraic geometry")

Each pallet from a customer A : 400 kg, 2 m^3
Each pallet from a customer B : 500 kg, 3 m^3

The customer A will pay \$ 11 for each pallet, and
the customer B will pay \$ 15 for each pallet.

We use trucks that can carry any load

up to 3700 kg, and up to 20 m^3.
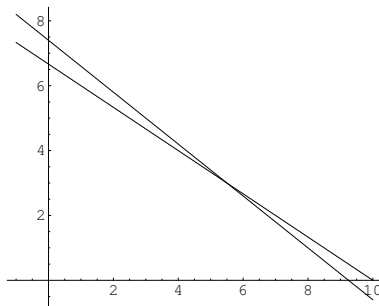
How to maximize the revenues generated?

Subject to $\begin{cases} 4a &+& 5b &\leq& 37 \\ 2a &+& 3b &\leq& 20 \\ && a,b &\geq& 0 \end{cases}$,

find integers a, b which maximize $11a + 15b$.

Example

Subject to $\begin{cases} 4a & + & 5b & \leq & 37 \\ 2a & + & 3b & \leq & 20 \\ & & a, b & \geq & 0 \end{cases}$,

find integers a, b which maximize $11a + 15b$.

Standard form

Subject to $\begin{cases} 4a &+& 5b &\leq& 37 \\ 2a &+& 3b &\leq& 20 \\ && a, b &\geq& 0 \end{cases}$,

find integers a, b which maximize $11a + 15b$.

$\downarrow$ the standard form

Subject to $\begin{cases} 4a &+& 5b &+& c &&&=& 37 \\ 2a &+& 3b &&&+& d &=& 20 \\ && a, b, && c, d &&\geq& 0 \end{cases}$,

find integers a, b, c, and d which minimize $-11a - 15b$.

Conti–Traverso algorithm

Subject to $A \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 37 \\ 20 \end{pmatrix}$ where $A = \begin{pmatrix} 4 & 5 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix} \cdots$

$\mathbf{w} := (-11, -15, 0, 0) + 2 \cdot (6, 8, 1, 1) = (1, 1, 2, 2)$.

$\mathcal{G} = \{x_3^4 x_4^2 - x_1, x_2 x_3^3 x_4 - x_1^2, x_1 x_3 x_4 - x_2, x_1^4 x_4 - x_2^3 x_3, x_2^2 x_3^2 - x_1^3\}$

is a Gröbner basis of I_A with respect to $<_{\mathbf{w}}$

$(a, b, c, d) = (0, 0, 37, 20)$ satisfies the constraints.

Therefore, we compute the remainder of $x_3^{37} x_4^{20}$ w.r.t. $\mathcal{G}$.

The remainder is $x_1^4 x_2^4 x_3$.

Hence, $(a, b, c, d) = (4, 4, 1, 0)$ is a solution.

Answer: Four pallets from A and four pallets from B.

B.2. Triangulations of convex polytopes

In this section, we always assume that
$A = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbb{Z}^{d \times n}$ is a configuration,
and often identify A with the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$.
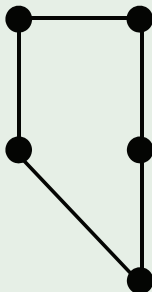
Definition

$$\mathrm{Conv}(A) := \left\{ \sum_{i=1}^{n} r_i \, \mathbf{a}_i \in \mathbb{Q}^d \ \middle|\ 0 \le r_i \in \mathbb{Q}, \ \sum_{i=1}^{n} r_i = 1 \right\}$$

the convex hull of A.

Example

Example

$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. Then, $\mathrm{Conv}(A)$ is

Triangulation

Definition

An integral convex polytope P is called simplex if the number of vertex of P is $1 + \dim P$.
(ex. line, triangle, tetrahedron.)

Definition

A covering Δ of A is a set of simplices whose vertices belong to A such that $\mathrm{Conv}(A) = \bigcup_{F \in \Delta} F$.

Definition

A covering Δ of A is called a triangulation if

1. F' is a face of $F \in \Delta \Longrightarrow F' \in \Delta$
2. $F, F' \in \Delta \Longrightarrow F \cap F'$ is a face of F and a face of F'.

Initial complex

Definition (initial complex)

$A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$

$<$: monomial order

$$\Delta(in_<(I_A)) := \left\{ \mathrm{Conv}(B) \;\middle|\; \begin{array}{c} B \subset A \\ \prod_{\mathbf{a}_i \in B} x_i \notin \sqrt{in_<(I_A)} \end{array} \right\}$$

Theorem

$\Delta(in_<(I_A))$ *is a triangulation of A.*

- (Gel'fand et al.) For $\mathbf{w} \in \mathbb{R}^n$, a triangulation $\Delta_\mathbf{w}$ is defined geometricaly. (regular triangulation)
- (Sturmfels) We have $\Delta(in_\mathbf{w}(I_A)) = \Delta_\mathbf{w}$.

Example

Example

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$I_A = \left\langle x_1 x_2 - x_3 x_5, \ x_1 x_4 - x_2 x_3, \ x_2^2 - x_4 x_5 \right\rangle$$

$<_1$: lexicographic order($x_2 > x_1 > x_3 > x_4 > x_5$)
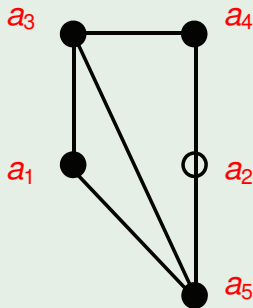
Gröbner bases of I_A with respect to $<_1$ is

$$\left\{ x_1 x_2 - x_3 x_5, \ x_2 x_3 - x_1 x_4, \ x_2^2 - x_4 x_5, x_1^2 x_4 - x_3^2 x_5 \right\}$$

$$in_{<_1}(I_A) = \left\langle x_1 x_2, x_2 x_3, x_2^2, x_1^2 x_4 \right\rangle, \ \ \sqrt{in_{<_1}(I_A)} = \left\langle x_1 x_4, x_2 \right\rangle$$

Examples

Example

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \sqrt{in_{<_1}(I_A)} = \langle x_1 x_4, x_2 \rangle$$

Examples

Example

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$I_A = \left\langle x_1 x_2 - x_3 x_5, \ x_1 x_4 - x_2 x_3, \ x_2^2 - x_4 x_5 \right\rangle$$

$<_2$: lexicographic order($x_5 > x_3 > x_4 > x_2 > x_1$)
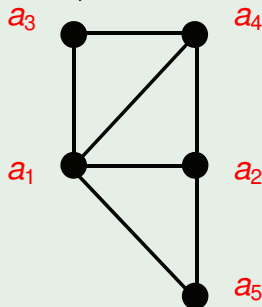
Gröbner bases of I_A with respect to $<_2$ is

$$\{ x_3 x_5 - x_1 x_2, \ x_2 x_3 - x_1 x_4, \ x_4 x_5 - x_2^2 \}$$

$$in_{<_2}(I_A) = \sqrt{in_{<_2}(I_A)} = \langle x_2 x_3, x_3 x_5, x_4 x_5 \rangle$$

Example

Example

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \sqrt{in_{<_2}(I_A)} = \langle x_2 x_3, x_3 x_5, x_4 x_5 \rangle$$

Unimodular triangulations

Definition

A covering (triangulation) Δ of A is called unimodular, if for the vertex set B of any maximal simplex in Δ, we have $\mathbb{Z}A = \mathbb{Z}B$,

(Here, $\mathbb{Z}A = \left\{ \displaystyle\sum_{i=1}^{n} z_i \mathbf{a}_i \ \middle| \ z_i \in \mathbb{Z} \right\}$.)

Theorem

$\Delta(in_<(I_A))$ is unimodular $\Longleftrightarrow \sqrt{in_<(I_A)} = in_<(I_A)$

Important properties

(i) *A* is unimodular (any triangulation of *A* is unimodular)
$(\Leftrightarrow \sqrt{in_<(I_A)} = in_<(I_A)$ for any $<)$

(ii) *A* is compressed
$(\Leftrightarrow \sqrt{in_<(I_A)} = in_<(I_A)$ for any reverse lex. order $<)$

(iii) *A* has a regular unimodular triangulation
$(\Leftrightarrow \sqrt{in_<(I_A)} = in_<(I_A)$ for some $<)$

(iv) *A* has a unimodular triangulation

(v) *A* has a unimodular covering

(vi) *K*[*A*] is normal $(\Leftrightarrow \mathbb{Z}_{\geq 0}A = \mathbb{Z}A \cap \mathbb{Q}_{\geq 0}A)$

Then, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) hold.
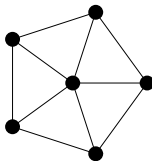However, the converse of them are false in general.

Edge polytopes

G: finite connected graph on the vertex set $\{1, 2, \ldots, d\}$
$E(G) = \{e_1, \ldots, e_n\}$: the edge set of G
(no loop, no multiple edges)

For each edge $e = \{i, j\} \in E(G)$, let $\rho(e) := \mathbf{e}_i + \mathbf{e}_j \in \mathbb{Z}^d$.

$$A_G := (\rho(e_1), \ldots, \rho(e_n)) \in \mathbb{Z}^{d \times n}$$

$\mathrm{Conv}(A_G)$ is called an edge polytope of G.

Edge polytopes

Theorem (O–Hibi (1998), Simis et al. (1998))

For a finite connected graph G, TFAE:

1. $K[A_G]$ *is normal;*
2. A_G *has a unimodular covering;*
3. *For any two odd cycles C and C' of G without common vertices, there exists an edge of G which joins a vertex of C with a vertex of C'.*

An interesting edge polytope

Example (O–Hibi (1999))

Let *G* be the following graph. Then,

- For any monomial order $<$, $\sqrt{in_<(I_{A_G})} \neq in_<(I_{A_G})$
- A_G has a unimodular triangulation.
  (Checked by the software PUNTOS by De Loera.)
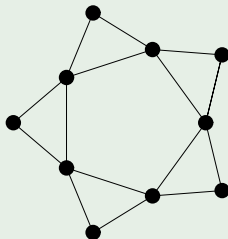
B.3. Contingency tables

5×5 contingency table:

algebra \ statistics	5	4	3	2	1	total
5	2	1	1	0	0	4
4	8	3	3	0	0	14
3	0	2	1	1	1	5
2	0	0	0	1	1	2
1	0	0	0	0	1	1
total	10	6	5	2	3	26

Is there a correlation between the two scores?

Markov chain Monte Carlo method

Markov chain Monte Carlo method

$$
F = \left\{ T = (t_{ij}) \; \middle| \; \begin{array}{ccccc|c} & & & & & 4 \\ & & & & & 14 \\ & & t_{ij} & & & 5 \\ & & & & & 2 \\ & & & & & 1 \\ \hline 10 & 6 & 5 & 2 & 3 & 26 \end{array} \;,\quad 0 \le t_{ij} \in \mathbb{Z} \right\}
$$

By a random walk on F, we sample elements of F and compare certain features (χ^2-statistics).
(In this example, $\sharp|F| = 229,174$.)

Markov chain Monte Carlo method

For example, fix α_i, β_j such that $\sum_i \alpha_i = \sum_j \beta_j$ and let

$$
F = \left\{ T = (t_{ij}) \;\middle|\;
\begin{array}{ccc|c}
t_{11} & t_{12} & t_{13} & \alpha_1 \\
t_{21} & t_{22} & t_{23} & \alpha_2 \\
\hline
\beta_1 & \beta_2 & \beta_3 &
\end{array}, \quad 0 \le t_{ij} \in \mathbb{Z} \right\}.
$$

Then by adding or subtracting one of the elements of

$$
M = \left\{ \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \right\},
$$

any of two elements T, T' of F are connected:

$$T = T_0 \longrightarrow T_1 \in F \longrightarrow T_2 \in F \longrightarrow \cdots \longrightarrow T_s = T'.$$

for any alpha_i & beta_j.
Such an M is called a Markov basis.

Markov bases and toric ideals

Example (continued)

$$
A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, A \begin{pmatrix} t_{11} \\ t_{12} \\ t_{13} \\ t_{21} \\ t_{22} \\ t_{23} \end{pmatrix} = \begin{pmatrix} t_{11} + t_{12} + t_{13} \\ t_{21} + t_{22} + t_{23} \\ t_{11} + t_{21} \\ t_{12} + t_{22} \\ t_{13} + t_{23} \end{pmatrix}
$$

$$
I_A = \langle x_1 x_5 - x_2 x_4, \quad x_1 x_6 - x_3 x_4, \quad x_2 x_6 - x_3 x_5 \rangle
$$

$$
\left\{ \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \right\}
$$

Diaconis–Sturmfels

Theorem (Diaconis–Sturmfels)

Let M be a finite set of integer matrices.
Then, M is a Markov basis if and only if I_A is generated by the corresponding binomials.

- 2 way contingency tables:
  It is known that I_A has a quadratic Gröbner basis .

- $\geq$ 3 way contingency tables:
  Except for some classes, the set of generators of I_A is unknown and it is not easy to compute in general.
  (You should try to use powerful software 4ti2.)

No *n* way interaction models

For $r_1 \times r_2 \times \cdots \times r_\ell$ contingency
table($r_1 \geq r_2 \geq \cdots \geq r_\ell \geq 2$)

$$T = (t_{i_1 i_2 \cdots i_\ell})_{i_k = 1, 2, \ldots, r_k}, \quad 0 \leq t_{i_1 i_2 \cdots i_\ell} \in \mathbb{Z},$$

we associates a configuration $A_{r_1 r_2 \cdots r_\ell}$ consisting of the
following vectors:

$$\mathbf{e}^{(1)}_{i_2 i_3 \cdots i_\ell} \oplus \mathbf{e}^{(2)}_{i_1 i_3 \cdots i_\ell} \oplus \cdots \oplus \mathbf{e}^{(\ell)}_{i_1 i_2 \cdots i_{\ell-1}}$$

where each i_k belongs to $\{1, 2, \ldots, r_k\}$ and $\mathbf{e}^{(k)}_{i_1 \cdots i_{k-1} i_{k+1} \cdots i_\ell}$ is
a unit vector in $\mathbb{R}^{r_1 \cdots r_{k-1} r_{k+1} \cdots r_\ell}$.

	O	O	O	O	X	X	X	X
A_{222}	O	O	X	X	O	O	X	X
	O	X	O	X	O	X	O	X

OO	1	1						
OX			1	1				
XO					1	1		
XX							1	1
OO	1				1			
OX		1				1		
XO			1				1	
XX				1				1
OO	1		1					
OX		1		1				
XO					1		1	
XX						1		1

Classification

Classification

$r_1 \times r_2$ $r_1 \times r_2 \times 2 \times \cdots \times 2$	*unimodular*
$r_1 \times 3 \times 3$	*compressed,* *not unimodular*
$5 \times 5 \times 3$ $5 \times 4 \times 3$ $4 \times 4 \times 3$	*normal* (4ti2 & Normaliz) *not compressed*
otherwise, i.e., $\ell \geq 4$ *and* $r_3 \geq 3$ $\ell = 3$ *and* $r_3 \geq 4$ $\ell = 3,\ r_3 = 3,\ r_1 \geq 6$ *and* $r_2 \geq 4$	*not normal*

Decompositions/constructions

- Algebras of Veronese type
- Segre–Veronese configurations
  (O–Hibi 2000)
- Extended Segre–Veronese configurations
  (Aoki–Hibi–O–Takemura 2010)
- Nested configurations
  (Aoki–Hibi–O–Takemura 2008, O–Hibi 2010)

- Higher Lawrence configurations (Santos–Sturmfels 2003) $\longrightarrow$ N-fold configurations
- Toric fiber product (Sullivant 2007)

Segre–Veronese configurations

$\tau \geq 2$, n: integers

$\mathbf{b} = (b_1, \ldots, b_n)$, $\mathbf{c} = (c_1, \ldots, c_n)$, $\mathbf{p} = (p_1, \ldots, p_n)$,
$\mathbf{q} = (q_1, \ldots, q_n)$: integer vectors satisfying

1. $0 \leq c_i \leq b_i$ for all $1 \leq i \leq n$

2. $1 \leq p_i \leq q_i \leq d$ for all $1 \leq i \leq n$

Let A be the matrix whose columns are all vectors
$(f_1, \ldots, f_d) \in \mathbb{Z}_{\geq 0}^d$ s.t.

1. $\displaystyle\sum_{j=1}^{d} f_j = \tau$

2. $\displaystyle c_i \leq \sum_{j=p_i}^{q_i} f_j \leq b_i$ for all $1 \leq i \leq n$

A is called Segre–Veronese configuration.

Segre–Veronese configurations

Example

$\tau = 2$, $n = 2$, $d = 5$
$\mathbf{b} = (1, 1)$, $\mathbf{c} = (0, 0)$, $\mathbf{p} = (1, 3)$, $\mathbf{q} = (2, 5)$
Let A be the matrix whose columns are all vectors
$(f_1, \ldots, f_5) \in \mathbb{Z}_{\geq 0}^5$ s.t.

1. $f_1 + f_2 + f_3 + f_4 + f_5 = 2$
2. $0 \leq f_1 + f_2 \leq 1$
3. $0 \leq f_3 + f_4 + f_5 \leq 1$

Then $A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$.

Segre–Veronese configurations

Theorem

Suppose that A is a Segre–Veronese configuration.
Then the toric ideal I_A has a quadratic Gröbner basis.

📄 S. Aoki, T. Hibi, H. Ohsugi and A. Takemura
Markov basis and Gröbner basis of Segre-Veronese
configuration for testing independence in group-wise
selections
Annals of the Institute of Statistical Math. **62** (2010),
299–321.

📄 S. Aoki, T. Otsu, A. Takemura and Y. Numata
Statistical Analysis of Subject Selection Data in
NCUEE Examination
Oyo Tokeigaku, **39**, (2)–(3), 2010, 71–100. (Japanese)

Nested configurations

$A = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbb{Z}_{\geq 0}^{d \times n}$: configuration

For each $i = 1, 2, \ldots, d$:

$B_i \in \mathbb{Z}_{\geq 0}^{\mu_i \times \lambda_i}$: configuration

$K \left[u_1^{(i)}, \ldots, u_{\mu_i}^{(i)} \right]$: polynomial ring in μ_i variables

$K[B_i] = K \left[m_1^{(i)}, \ldots, m_{\lambda_i}^{(i)} \right] \subset K \left[u_1^{(i)}, \ldots, u_{\mu_i}^{(i)} \right]$

$$K[A(B_1, \ldots, B_d)] := K \left[\left. m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)} \right| \begin{array}{c} r \in \mathbb{N} \\ \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_r} \in A \\ 1 \leq j_k \leq \lambda_{i_k} \text{ for } \forall k \end{array} \right]$$

The configuration $A(B_1, \ldots, B_d)$ is called
the nested configuration of A, $B_1, ..., B_d$.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A(B_1, B_2) = \left(\begin{array}{ccccc|cccccc} 4 & 3 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right)$$

- You have 2 coupons.
- Shop α accepts at most 2 coupons.
- Shop β accepts at most 1 coupon.
- Each shop has 2 different items.
- A coupon allows you to buy 2 items at a discount at α.
- A coupon allows you to buy 1 item at a discount at β.

Nested configurations

Theorem

Let $n \geq 2$.

- $I_A, I_{B_1}, \ldots, I_{B_d}$ *have Gröbner bases of degree $\leq n$*
  $\implies I_{A(B_1,\ldots,B_d)}$ *has a Gröbner basis of degree $\leq n$.*
- $I_A, I_{B_1}, \ldots, I_{B_d}$ *have squarefree initial ideals*
  $\implies I_{A(B_1,\ldots,B_d)}$ *has a squarefree initial ideal.*

Theorem

$K[A], K[B_1], ..., K[B_d]$ *are normal*
$\implies K[A(B_1, \ldots, B_d)]$ *is normal.*
(The converse is not true in general.)

References

📄 S. Aoki, T. Hibi, H. Ohsugi and A. Takemura
Gröbner bases of nested configurations
J. Algebra, **320** (2008) no. 6, 2583 – 2593.

📄 H. Ohsugi and T. Hibi
Toric rings and ideals of nested configurations
J. commutative algebra, **2** (2010), 187 – 208.

📄 T. Shibuta
Gröbner bases of contraction ideals
Journal of Algebraic Combinatorics, **36** (2012), 1 – 19.

0. Quadratic Gröbner bases

The following properties of Gröbner bases of toric ideals are studied by many researchers:

(i) There exists a monomial order such that a Gröbner basis of I_A consists of quadratic binomials.

(ii) $K[A]$ is "Koszul algebra."

(iii) I_A is generated by quadratic binomials.

- (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold.
- But neither (ii) $\Rightarrow$ (i) nor (iii) $\Rightarrow$ (ii) holds in general.

Quadratic Gröbner bases

Example (O–Hibi (1999))

Let G be the following graph. Then,

- I_{A_G} is generated by quadratic binomials.
- $K[A_G]$ is not Koszul.
- Hence, for any monomial order $<$,
  the reduced Gröbner basis of I_{A_G} is not quadratic.
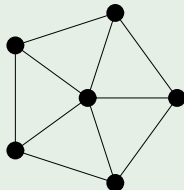
Infinite family of counterexamples

Hibi–Nishiyama–Ohsugi–Shikama (~~2008~~ 2014)

Using software `Risa/Asir, Macauley2, CaTS, ...`,
we chacked that there are a lot of graphs of ≤ 8 vertices
whose edge polytope is a counterexample.

Moreover, we proved that

Theorem

$n \geq 5$
C_n: cycle of length n
K_{n+1}: the complete graph with $n + 1$ vertices
$G := K_{n+1} - E(C_n)$
Then,

- I_{A_G} is generated by quadratic binomials
- I_{A_G} has no quadratic Gröbner basis .

Configurations arising from root systems

Gel'fand–Graev–Postnikov (1997)

$A_{d-1} = \{\mathbf{e}_i - \mathbf{e}_j \mid 1 \leq i < j \leq d\}$ ($\mathbf{e}_i \in \mathbb{Z}^d$ is a unit vector)

$$\widetilde{A_{d-1}} = \begin{pmatrix} 0 & \\ \vdots & A_{d-1} \\ 0 & \\ \hline 1 & 1 \quad \cdots \quad 1 \end{pmatrix}$$

There exists a monomial order such that
a Gröbner basis of $I_{\widetilde{A_{d-1}}}$ consists of quadratic binomials.
(They constructed a "regular unimodular triangulation.")

O–Hibi (2002)

$D_d = \{\mathbf{e}_i + \mathbf{e}_j \mid 1 \leq i < j \leq d\} \cup A_{d-1}$
$B_d = \{\mathbf{e}_1, \ldots, \mathbf{e}_d\} \cup D_d$
$C_d = \{2\mathbf{e}_1, \ldots, 2\mathbf{e}_d\} \cup D_d$

Toric ideals arising from matroids

Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ where

- Each B_i is an r-subset of $\{1, 2, \ldots, d\}$;
- (Basis Exchange Axiom)

    For each $1 \le i, j \le n$, for $\forall x \in B_i \setminus B_j$,

    $\exists y \in B_j \setminus B_i$ s. t. $(B_i \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

Let $A_{\mathcal{B}} = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbb{Z}^{d \times n}$ where $\mathbf{a}_i = \displaystyle\sum_{j \in B_i} \mathbf{e}_j \in \mathbb{R}^d$.

Conjecture (White (1980))

$I_{A_{\mathcal{B}}}$ *is generated by quadratic binomials.*

Conjecture

$I_{A_{\mathcal{B}}}$ *has a quadratic Gröbner basis .*